Photoproduction and Electroproduction of Pions in the Region of the $N^*(1238)$†

N. ZAGURY*
University of California, San Diego, La Jolla, California
(Received 30 December 1965)

The photoproduction and electroproduction of pions is treated relativistically. A new solution for the 3-3 amplitudes is obtained which includes recoil and is insensitive to a cutoff. Corrections to the E_{0+} and M_{1-} amplitudes are evaluated by using fixed-t dispersion relations. Good results are obtained in the region around the 3-3 resonance up to π-N center-of-mass energies about 1300 MeV and electron momentum transfer up to 3.8 BeV^2.

I. INTRODUCTION

DISPERSION-relation techniques in the treatment of the related problems of photoproduction and electroproduction were first introduced, within the framework of the static model, by Chew, Goldberger, Low, and Nambu[1] and by Fubini, Nambu, and Wataghin,[2] respectively. Theories based on the Mandelstam representation for the scattering amplitudes were worked out by Ball[3] in photoproduction and Dennery[4] in electroproduction. Numerical calculations in photoproduction have been performed[3,5–10] in both the static limit and the relativistic case, and the FNW theory has also been reduced to numerical form.[11,12]

The CGLN and FNW theories assume that the most important effects in the region of low π-N center-of-mass energies can be taken into account by the Born terms and the $J=\frac{3}{2}$, $T=\frac{3}{2}$ π-N final-state interaction. In particular, the part of the amplitude proportional to the total magnetic moment, $M_{1+,G}$ (in CGLN this amplitude is called $M_{1+,\mu}$) is responsible for the most significant contribution for the 3-3 state. In the CGLN and FNW theories this amplitude is taken to be proportional to the π-N scattering amplitude, f_{1+}, the constant of proportionality being the ratio of the Born terms taken in the static limit.

In this paper we consider a new solution for this amplitude. In the low-energy region photoproduction[13] and electroproduction[2] amplitudes have the same phase as the correspondent π-N scattering amplitudes. Therefore, we can choose identical D functions (of the N/D method) for those amplitudes, and the ratio M_{1+}/f_{1+} is equal to the ratio of the associated N functions. These N functions satisfy integral equations with a cutoff. Our method consists in directing our attention to the ratio of the N functions. Although the integrals depend on the cutoff, the ratio of the N functions is insensitive to this quantity. Also, we expect that uncertainties in the "input" N function for π-N scattering, which appears in both integral equations, will not have much effect on this ratio. Solutions for the electric and scalar amplitudes in the 3-3 state are also found. By using fixed-t dispersion relations, the effect of those amplitudes on the other partial waves is calculated, and a fully relativistic treatment of the Born terms is made. Our solution has the following properties:

(1) It reduces to the CGLN and FNW theories in the static limit.

(2) The results are insensitive to a cutoff.

(3) It includes recoil corrections. In the FNW theory the dependence on λ^2, the electron momentum transfer, is limited to that coming from the form factors and from the threshold behavior of the amplitudes. In our treatment appears an extra λ^2 dependence from the N functions and from the fully relativistic treatment of all terms. This extra λ^2 dependence will be important for the explanation of the data at high momentum transfer which has recently become available.[12,14] At these values of λ^2 the cross sections will be reduced considerably in relation to the ones given by the FNW theory.

(4) Corrections to the E_{0+} and M_{1-} amplitudes are introduced by dispersion integrals.

Numerical calculations were performed both in photoproduction and in electroproduction. Good agreement was obtained around the 3-3 resonance up to a π-N center-of-mass energy ≈ 1.35 BeV and up to momentum transfers $\lambda^2 \approx 3.8$ BeV^2.

The basic work in determining the invariant amplitudes that satisfy the Mandelstam representation, their

---

† Work supported in part by the U. S. Atomic Energy Commission and Conselho Nacional de Pesquisas, Brazil.

* On leave of absence from Centro Brasileiro de Pesquisas Fisicas and Faculdade Nacional de Filosofia, Rio de Janeiro, Brazil. Present address: Cyclotron Laboratory, Harvard University, Cambridge, Massachusetts.

[1] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957). Hereafter referred to as CGLN.

[2] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958). Hereafter referred to as FNW.

[3] J. Stutsman Ball, Phys. Rev. **124**, 2014 (1961).

[4] P. Dennery, Phys. Rev. **124**, 2000 (1961).

[5] C. J. Robinson, University of Illinois report, 1959 (unpublished).

[6] A. J. Lazarus, W. K. H. Panofsky, and F. R. Tangherlini, Phys. Rev. **113**, 1330 (1959).

[7] J. McKinley, University of Illinois report, 1962 (unpublished).

[8] A. M. Baldin and A. I. Lebedev, Nucl. Phys. **40**, 44 (1963).

[9] W. Schmidt, Z. Physik **182**, 76 (1964).

[10] G. Höhler and W. Schmidt, Ann. Phys. **28**, 34 (1964).

[11] S. Gartenhaus and C. N. Lindner, Phys. Rev. **113**, 917 (1959).

[12] L. N. Hand, Phys. Rev. **129**, 1834 (1963).

[13] K. Watson, Phys. Rev. **95**, 228 (1954); K. Aizu, in *Proceedings of the International Conference of Theoretical Physics, Kyoto and Tokyo, 1953* (Science Council of Japan, Tokyo, 1953); E. Fermi, Nuovo Cimento **2**, Suppl. 58 (1955).

[14] A. A. Cone, K. W. Chen, J. R. Dunning, Jr., C. Hartwig, N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. Letters **14**, 326 (1965).

decomposition into partial waves and the inversion formulas are well known.[1–4] For completeness we have reproduced it with slight modifications in Appendixes I and II. General considerations of the problem are discussed in Sec. II. Section III deals with the partial-wave approach, and in Sec. IV we construct explicit solutions for the 3-3 amplitudes. Section V refers to the reaction of those amplitudes into the other states, and Sec. VI deals with the results and makes comparisons with the experiment.

II. GENERAL CONSIDERATIONS

Let k_1, k_2, p_1, p_2, and q denote the initial and final electron 4-momenta, the initial and final nucleon 4-momenta and the pion 4-momentum, respectively. The masses of the nucleon, the pion and the electron will be called m, μ and m_e. (We are using the metric $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$ and $\hbar=c=1$ units.)

The invariant T matrix is defined by

$$S_{fi}=\delta_{fi}+(2\pi)^4 i\delta^4(k+p_1-p_2-q)\times\left[\frac{m^2}{2q_0p_{10}p_{20}}\right]^{1/2}\left[\frac{m_e^2}{k_{10}k_{20}}\right]^{1/2}\bar{u}(p_2)Tu(p_1)\,, \quad (2.1)$$

where $k=k_1-k_2$. The second parenthesis should be replaced by $(2k_0)^{-1/2}$ in the case of photoproduction.

To first order in e^2 the electroproduction is described by the diagram in Fig. 1. Therefore, we can write

$$T=\epsilon_\mu j^\mu\,, \quad (2.2)$$

where j^μ describes the current produced by the strong interacting particles and ϵ_μ is the photon polarization vector. For electroproduction:

$$\epsilon_\mu=(1/k^2)\bar{u}(k_2)\gamma_\mu u(k_1)\,. \quad (2.3)$$

Current conservation is expressed by

$$k_\mu j^\mu=0\,. \quad (2.4)$$

It will be convenient to introduce the invariants:

$$\begin{aligned} s&=(k+p_1)^2=W^2;\quad \lambda^2=k^2,\\ t&=(k-q)^2=(p_2-p_1)^2=2m^2-2E_1E_2+2qk\cos\theta, \quad (2.5)\\ u&=(p_1-q)^2=(p_2-k)^2=m^2+\lambda^2-2k_0E_1-2qk\cos\theta, \end{aligned}$$

where W is the energy of the π-N system in its center of mass, E_1 and k and E_2 and q are the energy and momentum of the initial and final nucleons, $k_0=W-E_1$, and θ is the angle between the nucleons in the center-of-mass system.

The T matrix can be written in terms of invariant amplitudes $A_i(s,t,u,\lambda^2)$ that satisfy certain dispersion relations. This procedure is shown in Appendix I. Also shown is the relation of the T matrix to the scattering amplitudes in the center-of-mass system, $\mathfrak{F}_i$, and its decomposition into partial-wave amplitudes.

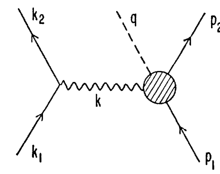

FIG. 1. Electroproduction in first order in e^2.

III. PARTIAL-WAVE APPROACH

Let l_γ be the orbital angular momentum of the photon-nucleon system. For each final π-N state $l_\pm$ with angular momentum $J=l\pm\frac{1}{2}$ we can construct two independent amplitudes $M_{l\pm}$ and $E_{l\pm}$ that appear in both photoproduction and electroproduction and the scalar amplitude $S_{l\pm}$ that appears only in electroproduction. [Many authors have been using a longitudinal amplitude L instead of the scalar one. They are related by $Sk_0-Lk=0$. We avoid using the longitudinal amplitude since the substitution $S=Lk/k_0$ may introduce spurious singularities in the cross section (k_0 may vanish in the physical region).] The magnetic, $M_{l\pm}$ and electric $E_{l\pm}$ amplitudes have $J=l\pm\frac{1}{2}$, parities $(-)^{l_\gamma+1}$ and $(-)^{l_\gamma}$, respectively, and correspond to transverse polarizations. The scalar amplitude has $J=l\pm\frac{1}{2}$, parity $(-)^{l_\gamma}$ and corresponds to virtual photons polarized in the time direction. The decomposition of the center-of-mass amplitudes $\mathfrak{F}_i$ into the partial waves and the inversions formulas are given in Appendix II.

Ball[3] pointed out that the magnetic and electric amplitudes satisfy the reflection relations:

$$\begin{aligned} M_{l+}(-W)&=[1/(l+1)]\\ &\quad\times[(l+2)M_{(l+1)-}(W)+E_{(l+1)-}(W)]\,, \quad (3.1)\\ E_{l+}(-W)&=[1/(l+1)][M_{(l+1)-}(W)-lE_{(l+1)-}(W)]\,. \end{aligned}$$

Analogously, using Eq. (I12) of Appendix I and Eqs. (II11) and (II12) of Appendix II, we can obtain reflection relations for the scalar amplitudes:

$$S_{l+}(-W)=S_{(l+1)-}(W)\,. \quad (3.2)$$

Starting from (I5) and (I6) of Appendix I, using the partial wave projection formulas in Appendix II and the reflection relations (3.1) and (3.2), we can derive the partial wave dispersion relations:

$$G_{l\pm}(W,\lambda^2)=B_{l\pm}(W,\lambda^2)+\frac{1}{\pi}\int_{m+\mu}^{\infty}dW'\frac{\mathrm{Im}G_{l\pm}(W',\lambda^2)}{W'-W}+\frac{1}{\pi}\int_{-\infty}^{-(m+\mu)}dW'\frac{\mathrm{Im}G_{l\pm}(W',\lambda^2)}{W'-W}\,, \quad (3.3)$$

where $B_{l\pm}(W,\lambda^2)$ represents all the singularities of $G_{l\pm}(W,\lambda^2)$ except those imposed by unitarity, and

$G_{l\pm}(W,\lambda^2)$ stands for any one of the amplitudes:

$$\mathfrak{M}_{l\pm}=\frac{M_{l\pm}}{(qk)^l}\left(\frac{E_1+m}{E_2+m}\right)^{1/2},$$
$$\mathcal{E}_{l\pm}=\frac{E_{l\pm}}{(qk)^l}\left(\frac{E_1+m}{E_2+m}\right)^{1/2}, \tag{3.4}$$
$$\mathcal{S}_{l\pm}=\frac{S_{l\pm}}{(qk)^l}\left(\frac{E_1+m}{E_2+m}\right)^{1/2}\frac{W}{k}.$$

The kinematical factors assure that $M_{l\pm}$, $E_{l\pm}$, $S_{l\pm}$ will have the proper threshold behavior on the right and left cuts. The factor W in the scalar amplitude was introduced to assure the same asymptotic behavior as the other amplitudes.

The π-N scattering amplitude with the kinematical factors as defined by Frazer and Fulco[15] is

$$h_{l\pm}(W)=\frac{f_{l\pm}(W)}{q^{2l}}\frac{W}{E_2+m}=\frac{e^{i\delta_{l\pm}}\sin\delta_{l\pm}}{\rho_l(W)}, \tag{3.5}$$

with

$$\rho_l(W)=q^{2l+1}(E_2+m)/W. \tag{3.6}$$

In the one-meson approximation for intermediate states, $G_{l\pm}$ and $h_{l\pm}$ have the same phase[2,13] in the energy region $m+\mu<W<\infty$. This phase relationship will be also valid in the region $-\infty<W<-(m+\mu)$ as a consequence of the reflection relations (3.1) and (3.2) for electroproduction and for π-N scattering[16]:

$$f_{l+}(W)=-f_{(l+1)-}(-W). \tag{3.7}$$

As was emphasized by Omnès,[17] the solution of (3.3) can be obtained if we know the phase of G_l on the real axis. We will express this solution in terms of the N and D functions of the well-known N/D method.[18] Let $G_{l\pm}=N_{l\pm}{}^\gamma/D_{l\pm}{}^\gamma$ and $h_{l\pm}=N_{l\pm}/D_{l\pm}$ where $D_{l\pm}{}^\gamma$ and $D_{l\pm}$ carry only the singularities due to the unitarity condition. Since G and h have the same phase on the real axis, we can choose $D_{l\pm}{}^\gamma=D_{l\pm}$. The nonphysical singularities of $G_{l\pm}$ are given by the term $B_{l\pm}$ in Eq. (3.3). Therefore, we can write the following solution for $N_{l\pm}{}^\gamma=G_{l\pm}D_{l\pm}$:

$$N_{l\pm}{}^\gamma(W,\lambda^2)=D_{l\pm}(W)B_{l\pm}(W,\lambda^2)-\frac{1}{\pi}\left(\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\right)dW'\frac{\mathrm{Im}D_{l\pm}(W')B_{l\pm}(W',\lambda^2)}{W'-W}. \tag{3.8}$$

The solution (3.8) has the right analytic properties for $N_{l\pm}{}^\gamma$. The first term has the proper nonphysical singularities of $N_{l\pm}{}^\gamma$, and the second term subtracts off the physical singularities introduced by the first term.

If we separate out the δ function in the integral of (3.8) and use $\mathrm{Im}D_{l\pm}=-\rho_l N_{l\pm}$, valid in the region of integration, we get

$$N_{l\pm}{}^\gamma(W,\lambda^2)=\mathrm{Re}D_{l\pm}(W)B_{l\pm}(W,\lambda^2)+\frac{P}{\pi}\left(\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\right)dW'\frac{\rho_l(W')N_{l\pm}(W')B_{l\pm}(W',\lambda^2)}{W'-W}. \tag{3.9}$$

Noticing that $(\mathrm{Re}D_{l\pm})/D_{l\pm}=e^{i\delta_{l\pm}}\cos\delta_{l\pm}$ we can write the equation for the full amplitude as

$$G_{l\pm}(W,\lambda^2)=B_{l\pm}(W,\lambda^2)e^{i\delta_{l\pm}}\cos\delta_{l\pm}+\frac{1}{D_{l\pm}(W)}\frac{P}{\pi}\left(\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\right)\frac{\rho_l(W')B_{l\pm}(W',\lambda^2)N_{l\pm}(W')dW'}{W'-W}. \tag{3.10}$$

It is well known that the $J=\frac{3}{2}$, $T=\frac{3}{2}$ scattering amplitude dominates the final-state interaction in the region of energies that we are concerned with. Therefore, we will assume, as a first approximation, that the solution of Eq. (3.3) is given by the B term alone except for the 3-3 state. Furthermore, we will approximate these terms by the Born terms corresponding to the Feynman diagrams of Fig. 2. In the next section we will calculate the solution for the 3-3 amplitude and in Sec. IV we will calculate corrections to the first solution given to the other amplitudes.

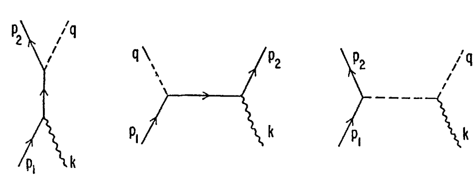

FIG. 2. Pole contributions to the current J_μ.

[15] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).
[16] S. W. MacDowell, Phys. Rev. **116**, 774 (1960).
[17] R. Omnès, Nuovo Cimento 8, 316 (1958).
[18] J. D. Bjorken, Phys. Rev. Letters 4, 473 (1960).

IV. THE $J=\frac{3}{2}$, $T=\frac{3}{2}$ AMPLITUDES

In Appendix II we have calculated the 3-3 projections of the Born terms corresponding to the Feynman diagrams in Fig. 2. These amplitudes can be written as a sum of terms proportional to the pion form factor $F_\pi(\lambda^2)$, to the isovector parts of the charge form factor $F_{1V}(\lambda^2)$ and to the magnetic form factor $G_{MV}(\lambda^2)=F_{2V}(\lambda^2)+F_{1V}(\lambda^2)/2m$. Denoting the Born terms by the superscript "B", we have

$$\begin{aligned}\mathcal{M}_{1+}{}^B(W,\lambda^2)&=\mathcal{M}_{1+,G}{}^B(W,\lambda^2)G_{MV}(\lambda^2)+\mathcal{M}_{1+,e}{}^B(W,\lambda^2)F_{1V}(\lambda^2)+\mathcal{M}_{1+,\pi}{}^B(W,\lambda^2)F_\pi(\lambda^2)\,,\\ \mathcal{E}_{1+}{}^B(W,\lambda^2)&=\mathcal{E}_{1+,G}{}^B(W,\lambda^2)G_{MV}(\lambda^2)+\mathcal{E}_{1+,e}{}^B(W,\lambda^2)F_{1V}(\lambda^2)+\mathcal{E}_{1+,\pi}{}^B(W,\lambda^2)F_\pi(\lambda^2)\,,\\ \mathcal{S}_{1+}{}^B(W,\lambda^2)&=\mathcal{S}_{1+,G}{}^B(W,\lambda^2)G_{MV}(\lambda^2)+\mathcal{S}_{1+,e}{}^B(W,\lambda^2)F_{1V}(\lambda^2)+\mathcal{S}_{1+,\pi}{}^B(W,\lambda^2)F_\pi(\lambda^2)\,.\end{aligned}\tag{4.1}$$

Since the Born terms are linear in the form factors, the solution of the integral equations in the last chapter will also be. We will denote by $\mathcal{M}_{1+,G}$, $\mathcal{M}_{1+,e}$, etc., the solutions of (3-3) correspondent to the Born terms $\mathcal{M}_{1+,G}{}^B$, $\mathcal{M}_{1+,e}{}^B$, etc.

We have studied the Born terms numerically in the regions $m+\mu<W<1.6$ BeV and -4 BeV$^2<\lambda^2<0$ and found that only $\mathcal{M}_{1+,G}{}^B$, $\mathcal{M}_{1+,\pi}{}^B$, $\mathcal{E}_{1+,\pi}{}^B$, and $\mathcal{S}_{1+,\pi}{}^B$ are not negligible. Therefore, we only consider the solutions for the corresponding amplitudes $\mathcal{M}_{1+,G}$, $\mathcal{M}_{1+,\pi}$, $\mathcal{E}_{1+,\pi}$, and $\mathcal{S}_{1+,\pi}$.

As we shall see, our final result will not be very dependent on scale factors or on fine details of the pion-nucleon N function. Therefore, it is reasonable to use the one given in the static model[15]:

$$N_{1+}=\gamma/(W-m)\,,\tag{4.2}$$

where γ is a constant.

We will first consider the solution for $\mathcal{M}_{1+,G}$. It is easy to verify that the principal value integral in Eq. (3.9) diverges for $B=\mathcal{M}_{1+,G}{}^B$ and N_{1+} given by Eq. (4.2). This problem also arises in the analogous equation for π-N scattering:

$$N_{1+}(W)=h_{1+}{}^B(W)\ \mathrm{Re}D_{1+}(W)+\frac{P}{\pi}\left(\int_{m+\mu}^{\infty}+\int_{-\infty}^{-(m+\mu)}\right)dW'\frac{\rho_1(W')h_{1+}{}^B(W')N_{1+}(W')}{W'-W}\tag{4.3}$$

when one takes $h_{1+}{}^B$ identical to the 3-3 projection of the nucleon exchange diagram of Fig. 3 and uses Eq. (4.2) as the input N function in the integrand.

In general a cutoff is introduced for evaluating the principal value integrals. Another problem is that (4.2) is certainly a bad approximation for the N function at very high energies. To minimize both effects we propose the following solution for $\mathcal{M}_{1+,G}$:

$$\mathcal{M}_{1+,G}(W,\lambda^2)=R(W,\lambda^2,\Lambda)\frac{e^{i\delta_{1+}}\sin\delta_{1+}}{q^3}\frac{W}{E_2+m}\tag{4.4}$$

with

$$R(W,\lambda^2,\Lambda)=\frac{N_{1+}{}^\gamma}{N_{1+}}=\frac{\mathcal{M}_{1+,G}{}^B(W,\lambda^2)\ \mathrm{Re}D_{1+}(W)+\dfrac{P}{\pi}\displaystyle\int_{m+\mu}^{\Lambda}\frac{\mathcal{M}_{1+,G}{}^B(W',\lambda^2)N_{1+}(W')\rho_1(W')}{W'-W}dW'}{h_{1+}{}^B(W)\ \mathrm{Re}D_{1+}(W)+\dfrac{P}{\pi}\displaystyle\int_{m+\mu}^{\Lambda}\frac{h_{1+}{}^B(W')N_{1+}(W')\rho_1(W')}{W'-W}dW'}\,.\tag{4.5}$$

(At this point we have dropped the left-hand cut where our N function is doubtful. Numerical calculation shows that its contribution amouts to less than 2%.)

In the static limit $\mathcal{M}_{1+,G}{}^B$ is proportional to $h_{1+}{}^B$ and (4.5) reduces to the CGLN results (apart from phase-space factors) independently of W or Λ. For the full Born terms we expect that R remains approximately constant in W since it is the ratio of N functions and also because $\mathcal{M}_{1+,G}$ is nearly proportional to $h_{1+}{}^B$. The important region

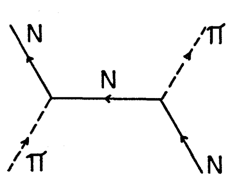

FIG. 3. Exchange of a nucleon in pion-nucleon scattering.

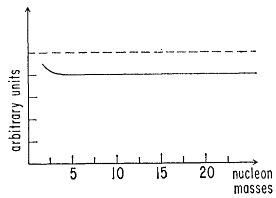

FIG. 4. Behavior of $M_{1+}^{3/2}$ as a function of the cutoff Λ for $\lambda^2=0$, $W=m^*$. The dashed line represents the CGLN value.

of energies in (4.4) is for $W=m^*$, for which $\delta=\pi/2$. Therefore, we will approximate

$$R(W,\lambda^2,\Lambda)\approx R(m^*,\lambda^2,\Lambda)=\frac{P}{\pi}\int_{m+\mu}^{\Lambda} dW'\frac{\mathfrak{M}_{1+,G}^B(W',\lambda^2)N_{1+}(W')\rho(W')}{W'-m^*}\Big/\frac{P}{\pi}\int_{m+\mu}^{\Lambda} dW'\frac{h_{1+}^B(W')N_{1+}(W')\rho(W')}{W'-m^*}. \quad (4.6)$$

In Fig. 4 we have plotted $\mathfrak{M}_{1+,G}$ at $W=m^*$, $\lambda^2=0$ for several values of the cutoff parameter, starting at $\Lambda=1.5m$. The dashed line represents the CGLN value at the same point. As we expected, the results are not very sensitive to Λ. From $\Lambda=1.5m$ to $\Lambda\approx 4m$, they vary from approximately 90% to approximately 80% of the CGLN value and for $\Lambda\gtrsim 5m$ remain nearly independent of Λ.

Although we factored the form factors and the threshold behavior out of M_{1+}, the amplitude $\mathfrak{M}_{1+,G}$ is still a function of λ^2. Having fixed Λ, we studied numerically this λ^2 dependence for -4 BeV$^2<\lambda^2<0$. It can be very well fitted by the expression

$$C(\lambda^2)\approx\frac{1}{1-\lambda^2/M^2} \quad (4.7)$$

with $M=1.96$, 2.19, 2.37, 2.41, and 2.47 BeV for $\Lambda=1.5m$, $3m$, $5m$, $6m$, and $8m$, respectively. The difference between these values will be important only at very high momentum transfer. If one chooses $\Lambda\gtrsim 5m$, which is the region where the solution is independent of Λ for $\lambda^2=0$, the difference between the values of C can be neglected (they are at most about 3% at $\lambda^2=-4$ BeV^2). From now on we will fix $\Lambda=5m$ in all calculations.

Dennery[4] and Adler[19] have assumed that $C(\lambda^2)$ is given by the λ^2 dependence in the Born term $\mathfrak{M}_{1+}^B$. We would like to point out that in the region around the resonance this λ^2 dependence is steeper than the one obtained by our method. Important numerical differences can appear only for very high momentum transfer. At $W=m^*$ and $\lambda^2\approx-3.80$ BeV^2 (highest momentum transfer in the experiments we will try to fit) our result gives $C\approx 0.59$, while from the λ^2 dependence of $\mathfrak{M}_{1+}^B$ we get $C\approx 0.4$.

The final result for $\mathfrak{M}_{1+,G}$ is

$$\mathfrak{M}_{1+,G}=0.61\frac{e}{(f/\mu)}C(\lambda^2)\frac{e^{i\delta_{1+}}\sin\delta_{1+}}{q^3}\frac{W}{E_2+m}. \quad (4.8)$$

For the multipoles proportional to F_π the integrals in (3.9) converge. Also, the Born terms are no longer approximately proportional to $h_{1+}^B(W)$, and their contribution to (3.9) is relatively important. Therefore, the solution (4.4)–(4.6) does not seem to be appropriate, as it stands, for the multipoles proportional to F_π. The integral in (3.9) is not a rapidly varying function of W, and it is only important for $W\approx m^*$ when $\mathrm{Re}D_{1+}$ vanishes. Therefore we will approximate

$$N_{1+}^{\gamma}(W,\lambda^2)\approx g_{1+}^B(W,\lambda^2)\,\mathrm{Re}D_{1+}(W)+\frac{P}{\pi}\int_{m+\mu}^{\infty}\frac{g_{1+}^B(W',\lambda^2)N_{1+}(W')\rho_1(W')}{W'-m^*}, \quad (4.9)$$

where N_{1+}^{γ} stands for any of the N functions and g_{1+} for any of the Born terms of the amplitudes proportional to F_π. Equation (4.9) can be written as

$$N_{1+}^{\gamma}(W,\lambda^2)\approx g_{1+}^B(W,\lambda^2)\,\mathrm{Re}D_{1+}(W)+N_{1+}^{\gamma}(m^*,\lambda^2). \quad (4.10)$$

Now, in the spirit of the solution for $\mathfrak{M}_{1+,G}$, to minimize the effects of the high energy region and to "normalize" N_{1+}^{γ}, we use (4.6) which is exact at $W=m^*$

$$N_{1+}^{\gamma}(m^*,\lambda^2)=R'(\lambda^2)N_{1+}(m^*), \quad (4.11)$$

[19] Stephen L. Adler, International Conference on Weak Interactions, Argonne National Laboratory, Argonne, Illinois, 1965 (unpublished).

with

$$R'(\lambda^2)=\frac{P}{\pi}\int_{m+\mu}^{\Lambda}d'W\frac{g_{1+}(W',\lambda^2)N_{1+}(W')\rho(W')}{W'-m^*}\Big/\frac{P}{\pi}\int_{m+\mu}^{\Lambda}dW'\frac{h_1^B(W')N_{1+}(W')\rho(W')}{W'-m^*}. \tag{4.12}$$

From (4.10), (4.11), and (4.12) we can write the solution for the whole amplitude, $G_{1+}=N_{1+}^{\gamma}/D_{1+}$:

$$G_{1+}(W,\lambda^2)=g_{1+}(W,\lambda^2)e^{i\delta_{1+}}\cos\delta_{1+}+R'(\lambda^2)[N_{1+}(m^*)/D_{1+}(W)]. \tag{4.13}$$

In order to introduce the scattering amplitude h_{1+}, we make the reasonable approximation

$$N_{1+}(m^*)/D_{1+}(W)\approx N_{1+}(W)/D_{1+}(W)=h_{1+}(W). \tag{4.14}$$

We have studied R' numerically as a function of λ^2 for -4 BeV$^2<\lambda^2<0$ and fitted the results to analytic curves. The final results for the multipoles are

$$\mathfrak{M}_{1+,\pi}=\mathfrak{M}_{1+,\pi}^{B}e^{i\delta_{1+}}\cos\delta_{1+}+0.036\frac{e}{f/\mu}C_1(\lambda^2)\frac{e^{i\delta_{1+}}\sin\delta_{1+}}{q^2}\frac{W}{E_2+m}, \tag{4.15}$$

$$\mathcal{E}_{1+,\pi}=\mathcal{E}_{1+,\pi}^{B}e^{i\delta_{1+}}\cos\delta_{1+}+0.013\frac{e}{f/\mu}C_2(\lambda^2)\frac{e^{i\delta_{1+}}\sin\delta_{1+}}{q^3}\frac{W}{E_2+m}, \tag{4.16}$$

$$\mathcal{S}_{1+,\pi}=\mathcal{S}_{1+,\pi}^{B}e^{i\delta_{1+}}\cos\delta_{1+}+0.084\frac{e}{f/\mu}C_3(\lambda^2)\frac{e^{i\delta_{1+}}\sin\delta_{1+}}{q^3}\frac{W}{E_2+m}, \tag{4.17}$$

with

$$\begin{aligned}
C_1(\lambda^2)&\approx 1 &&\text{for}\quad -0.1<\lambda^2<0,\\
&\approx(1.11-0.16\lambda^2)/(1-1.51\lambda^2) &&\text{for}\quad \lambda^2<-0.1,\\
C_2(\lambda^2)&\approx 1-2.55\lambda^2-5.2\lambda^4 &&\text{for}\quad -0.3<\lambda^2<0,\\
&\approx(1.59-0.12\lambda^2)/(1-0.91\lambda^2) &&\text{for}\quad \lambda^2<-0.3,\\
C_3(\lambda^2)&\approx 1-7.47\lambda^2-13.0\lambda^4 &&\text{for}\quad -0.16<\lambda^2<0,\\
&\approx(1.98-0.26\lambda^2)/(1-2.83\lambda^2) &&\text{for}\quad \lambda^2<-0.16,
\end{aligned} \tag{4.18}$$

where λ^2 is given in BeV^2.

V. THE DISPERSION INTEGRAL

Having calculated the $J=\frac{3}{2}$, $T=\frac{3}{2}$ amplitudes we will try to evaluate their effect on the other partial waves. We can write[3,4] fixed-t dispersion relations for the invariant amplitudes $A_i^{\pm}(s,t)$ defined in Appendix I:

$$\mathrm{Re}A_i^{\pm}(s,t)=A_i^{\pm}(s,t)_{\text{poles}}+\frac{P}{\pi}\int_{(m+\mu)^2}^{\infty}ds'\,\mathrm{Im}A_i^{\pm}(s',t)\left[\frac{1}{s'-s}\pm\xi_i^{\pm}\frac{1}{s'-u}\right], \tag{5.1}$$

where $\xi_i^{\pm}$ are defined in Eq. (I7) of Appendix I and will be ±1 if A_i^{+} is even or odd, respectively, under crossing of the nucleon lines. Following a method proposed by Ball[3] and Dennery[4] and applied in photoproduction by Ball,[3] Schmidt[9] and Schmidt and Holler,[10] we will approximate the imaginary part of $A_i^{\pm}$ by the contributions coming from the 3-3 final state:

$$\mathrm{Im}A_i^{\pm}(s,t)=\pm\frac{4\pi}{3}\binom{2}{1}\frac{f_i(s,t)}{qk[(E_1+m)(E_2+m)]^{1/2}}, \tag{5.2}$$

where

$$f_1(s,t)=\tfrac{3}{2}(2k_0q_0-\mu^2-\lambda^2+t)\,\mathrm{Im}(M_{1+}^{3/2}+E_{1+}^{3/2})-2(E_1+m)(E_2+m)\,\mathrm{Im}M_{1+}^{3/2}+2mf_4(s,t), \tag{5.3}$$

$$f_2(s,t)=\frac{1}{\lambda^2/2+m^2-s}[\lambda^2f_5(s,t)-3(W-m)(E_1+m)\,\mathrm{Im}(M_{1+}^{3/2}-E_{1+}^{3/2})], \tag{5.4}$$

$$f_3(s,t)=3(E_1+m)\,\mathrm{Im}(E_{1+}^{3/2}-M_{1+}^{3/2})+f_4(s,t), \tag{5.5}$$

$$f_4(s,t)=\frac{3}{4W}\frac{(-2q_0k_0+\mu^2+\lambda^2-t)}{E_1-m}\left[-(E_1-m)\,\mathrm{Im}M_{1+}^{3/2}-(W+k_0-m)\,\mathrm{Im}E_{1+}^{3/2}-\frac{\lambda^2}{k}\,\mathrm{Im}S_{1+}^{1/2}\right]$$
$$-\frac{\lambda^2}{2W}\frac{E_2+m}{k}\,\mathrm{Im}S_{1+}^{3/2}+\frac{(W+m)}{W}(E_2+m)\,\mathrm{Im}M_{1+}^{3/2}+\frac{3q_0(E_1+m)}{2W}\,\mathrm{Im}(M_{1+}^{3/2}-E_{1+}^{3/2}),\quad (5.6)$$

$$f_5(s,t)=-\frac{1}{2W(t-\mu^2)}\left\{\frac{3}{2}\frac{(t-\mu^2-\lambda^2+2k_0q_0)}{E_1-m}\left[[2(s-m^2)-\lambda^2]\left(-\frac{(W-m)}{k}\,\mathrm{Im}S_{1+}^{3/2}-\mathrm{Im}(E_{1+}^{3/2}+M_{1+}^{3/2})\right)\right.\right.$$
$$\left.+(4W-3k_0)(W-m)\,\mathrm{Im}(M_{1+}^{3/2}-E_{1+}^{3/2})\right]+(E_2+m)[2(s-m^2)-\lambda^2]\left[-\frac{(W+m)}{k}\,\mathrm{Im}S_{1+}^{3/2}+2\,\mathrm{Im}M_{1+}^{3/2}\right]$$
$$\left.+3(3q_0-2W)(W-m)(E_1+m)\,\mathrm{Im}(M_{1+}^{3/2}-E_{1+}^{3/2})\right\},\quad (5.7)$$

$$f_6(s,t)=\frac{3}{2}\frac{(t-\mu^2-\lambda^2+2q_0k_0)}{E_1-m}[(k_0/k)\,\mathrm{Im}S_{1+}^{3/2}+2\,\mathrm{Im}E_{1+}^{3/2}]+(E_2+m)[2\,\mathrm{Im}M_{1+}^{3/2}-(k_0/k)\,\mathrm{Im}S_{1+}^{3/2}]-f_4(s,t).\quad (5.8)$$

From Eqs. (5.1) to (5.8) the relationship among the center-of-mass amplitudes, $\mathfrak{F}_i$, and the invariant amplitudes A_i and the projection formulas given in Appendixes I and II, we can project the partial wave amplitudes.

We have calculated the contributions to E_{0+}, M_{1-}, E_{1+}, and M_{1+} using the experimental phase shifts proposed by Roper, Wright and Feld,[17] which are valid up to pion energies ≈ 700 MeV. Consistently we use a cutoff at $W\approx 1.57$ BeV on the integrals.

Equation (5.1) should be considered an iterative solution for $M_{1+}^{3/2}$ and $E_{1+}^{3/2}$. However, the results obtained in the first iteration reproduce quite well the solution given in Chap. IV, and the small differences will be neglected.

At $\lambda^2=0$ (photoproduction) the important contributions of the dispersion integral are to E_{0+}^{+} and M_{1-}^{+}. There is also a small contribution to E_{0+}^{-} at higher energies. The results are shown in Fig. 5 together with the Born multipoles for π^0 and π^+ photoproduction. One sees that the E_{0+} contribution is very important and cancels about 50% of the large E_{0+}^{+} Born term.

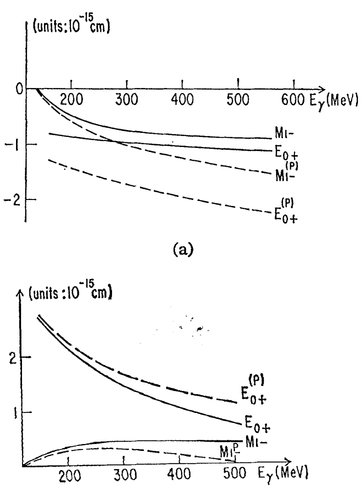

FIG. 5. E_{0+} and M_{1-} amplitudes for (a) π^0 production and (b) π^+ production. The dashed lines are the contributions from the Born terms alone. The solid lines represent the total multipoles.

To show how our results vary with λ^2, we have plotted E_{0+} and M_{1-} at fixed energy $W=1200$ MeV in Fig. 6. The corrections to E_{0+}^{+} and M_{1-}^{+} are still very important. However, the relative contribution of higher partial waves increases with λ^2, and the effect of the corrections in the cross sections will be reduced.

VI. RESULTS

The center-of-mass amplitudes, $\mathfrak{F}_i$, will be the sum of four terms

$$\mathfrak{F}=\mathfrak{F}_B-\mathfrak{F}_B^{33}+\mathfrak{F}^{33}+\mathfrak{F}_C,\quad (6.1)$$

where $\mathfrak{F}_B$ and $\mathfrak{F}_B^{33}$ are, respectively, the full Born term and its projection into the 3-3 state, $\mathfrak{F}^{33}$ is the total 3-3 amplitude and $\mathfrak{F}_C$ is the correction introduced by the principal part integral in Sec. V. As we saw in Sec. V, this correction will be only important for π^0 production. In Appendix III we have derived formulas for the cross sections in terms of the invariant amplitudes A_i. The numerical results are given below.

(a) Photoproduction—In Figs. 7 to 12 we have com-

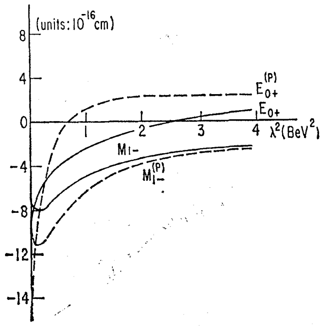

FIG. 6. E_{0+} and M_{1-} amplitudes for π^0 production as a function of λ^2 and for $W=1200$ MeV. The dashed lines are the contributions from the Born terms alone. The solid lines represent the total multipoles.

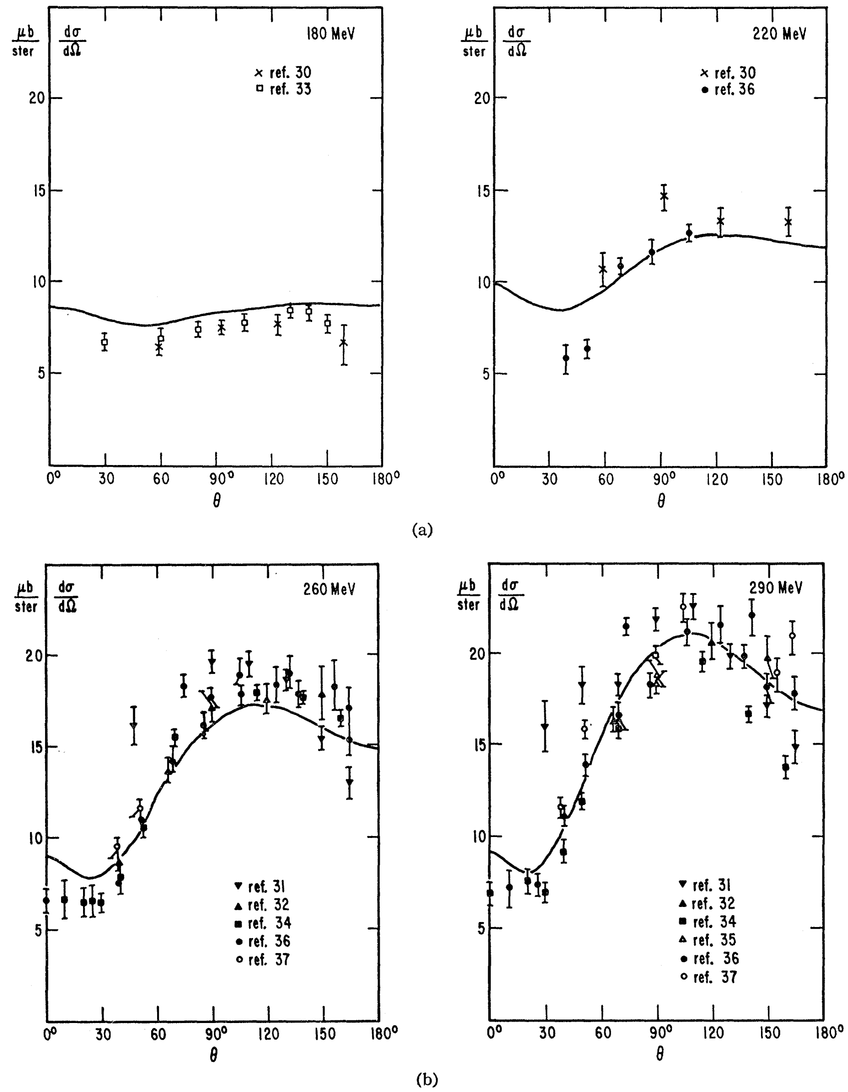

FIG. 7. Angular distribution for π^+ photoproduction in the center-of-mass system. The energies are in the lab. system.

pared our calculations with the experimental results for π^0 production[21–29] and π^+ production[30–37] from threshold to photon energies in the Lab system $k^L \approx 450$ MeV.

[20] L. David Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).
[21] Y. Goldschmidt-Clermont, L. S. Osborne, and M. Scott, Phys. Rev. **97**, 188 (1955).
[22] R. L. Walker and V. A. Tollestrup, Phys. Rev. **97**, 1279 (1955).
[23] D. C. Oakley and R. L. Walker, Phys. Rev. **97**, 1283 (1955).
[24] W. S. McDonald, V. Z. Peterson, and D. R. Carson, Phys. Rev. **107**, 577 (1957).
[25] P. C. Stein and K. C. Rogers, Phys. Rev. **110**, 1209 (1958).
[26] J. W. de Wire, H. E. Jackson, and R. M. Littauer, Phys. Rev. **110**, 1208 (1958).
[27] J. Y. Vette, Phys. Rev. **111**, 622 (1958).
[28] R. G. Vasil'kov, B. B. Govorkov, and V. I. Gol'danskii, Zh. Eksperim. i Teor. Fiz. **37**, 11 (1959) [English transl.: Soviet Phys.—JETP **10**, 7 (1960)].
[29] K. Berkelman and J. A. Waggoner, Phys. Rev. **117**, 1364 (1960).
[30] M. Beneventano, G. Bernardini, D. Carlson-Lee, G. Stoppini, and L. Tau, Nuovo Cimento **4**, 323 (1956).
[31] A. V. Tollestrup, T. C. Keck, and R. M. Worlock, Phys. Rev. **99**, 220 (1955).

The differential cross sections [(the data from Refs. 28 and 29 have been raised 7% (see Beneventano *et al.*[30] and Moravcsik[38])] for π^+ production are shown in Figs. 7 and 8 and the agreement with the experiment seems to be very good.

In Figs. 9 and 10 we show the differential cross sections for π^0 production and in Fig. 11 the A, B, and C coefficients of the expansion $d\sigma/d\Omega = A + Bz + Cz^2$. For energies $k^L > 300$ MeV we have obtained a reasonable agreement. Our results are not as good in the low-energy region. In particular, the coefficient B is relatively large

[32] R. L. Walker, J. G. Teesdale, V. Z. Patterson, and J. I. Vette, Phys. Rev. **99**, 210 (1955).
[33] M. Bazim and J. Pine, Phys. Rev. **132**, 830 (1963).
[34] E. A. Knapp, R. W. Kenney, and V. Perez-Mendez, Phys. Rev. **114**, 605 (1959).
[35] M. Heinberg, W. M. McClelland, F. Turkot, W. M. Woodward, R. R. Wilson, and D. M. Zijoy, Phys. Rev. **110**, 1211 (1958).
[36] K. Althoff, H. Fischer, and W. Paul, Z. Physik **175**, 1 (1963).
[37] K. Althoff, H. Fischer, and W. Paul, Z. Physik **175**, 19 (1963).
[38] M. J. Moravcsik, Phys. Rev. **105**, 267 (1957).

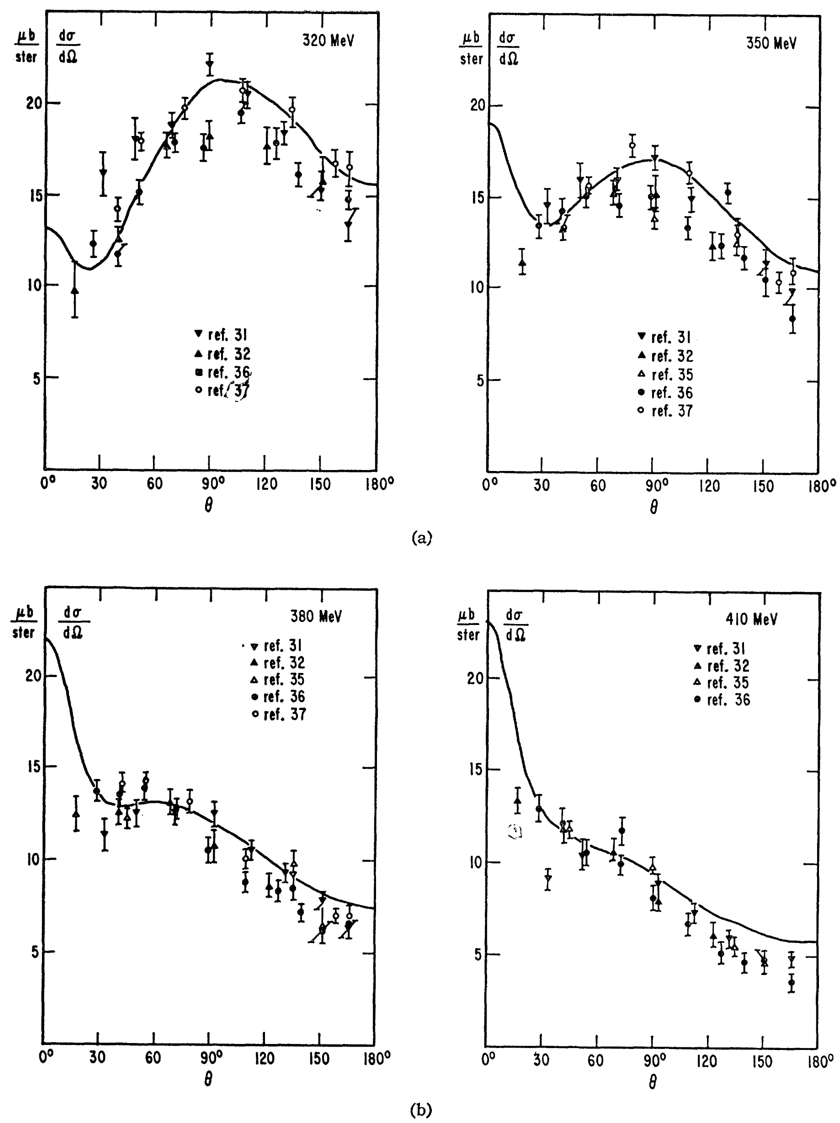

FIG. 8. Angular distribution for π^+ photoproduction in the center-of-mass system. The energies are in the lab. system.

and differs appreciably from the one given by the experimental results. This coefficient is given essentially by the interference of s and p waves

$$B \approx 2\,\mathrm{Re}E_{0+}^*(M_{1+}+3E_{1+}-M_{1-})\,. \tag{6.2}$$

This shows that a better determination of the $J=\frac{1}{2}$, E_{0+} and M_{1-}, amplitudes is necessary. In Fig. 11 we have shown in dashed line the values for B when the corrections obtained in Sec. V are not included. One can see that, although the corrections act in the right sense, they are still insufficient to provide a good fit for B.

Finally in Fig. 12 we show the total cross sections.

(b) Electroproduction—We have compared our results with the observed differential cross section in the laboratory system for the reactions

$$e^-+p \begin{array}{l} \nearrow e^-+p+\pi^0, \\ \searrow e^-+n+\pi^+. \end{array} \tag{6.3}$$

The cross section results depend on the pion and nucleon form factors. The pion form factor was set equal to the isovector part of the charge form factor F_{1V}. This agrees with the approximation of considering the ρ meson as the main contribution for both these form factors. Also, with this choice the sum of the three diagrams in Fig. 2 conserves charge. For the low mo-

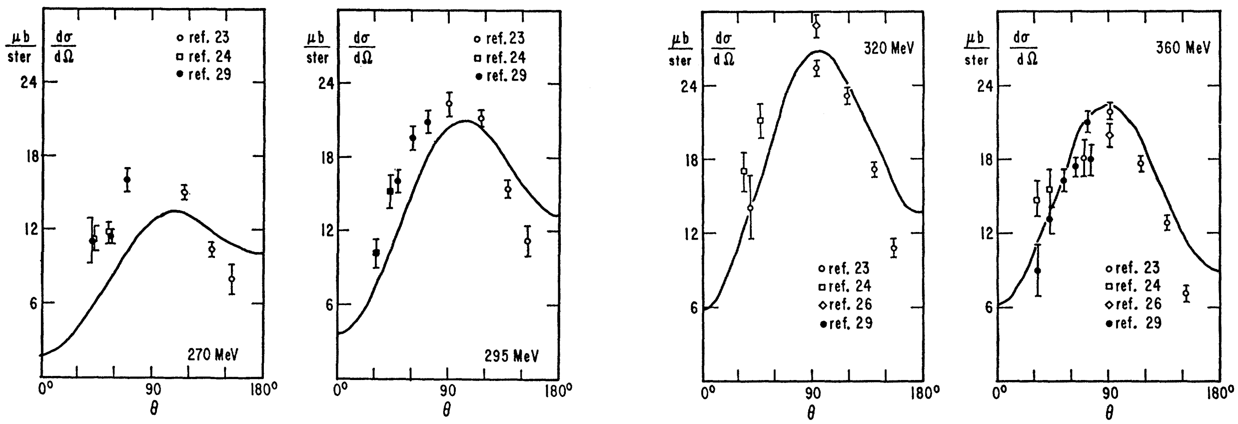

FIG. 9. Angular distribution for π^0 photoproduction in the center- of-mass system. The energies are in the lab. system.

mentum transfer (-0.467 BeV$^2<\lambda^2<0$) data,[12,39,40] we have set the nucleon form factors equal to the value (fit "b") given in de Vries *et al.*[41] for

$$G_E^{(S,V)}=F_1^{(S,V)}+(\lambda^2/2m)F_2^{(S,V)}\,,\qquad G_M^{(S,V)}=F_2^{(S,V)}+(1/2m)F_1^{(S,V)}\,. \tag{6.4}$$

As the neutron charge form factor is not known with accuracy, we have also used a different value $G_{EN}=G_E^S-G_E^V$ for testing the sensitivity of our calculation to this quantity.

For the high-energy momentum-transfer data[14] (-3.8 BeV$^2\lesssim\lambda^2\lesssim-0.88$ BeV^2) the nucleon form factors are

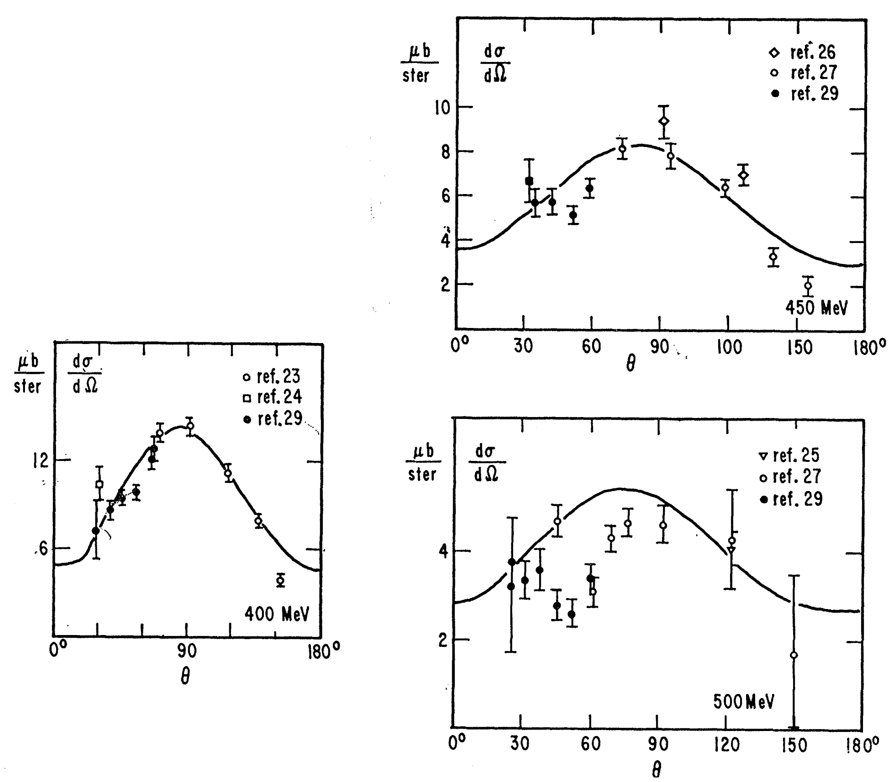

FIG. 10. Angular distribution for π^0 photoproduction in the center-of-mass system. The energies are in the lab. system.

[39] K. H. Panofsky and E. A. Allton, Phys. Rev. **110**, 1155 (1958).
[40] Gerald G. Olsen, Phys. Rev. **120**, 584 (1960).
[41] C. de Vries, R. Hofstadter, and A. Johnson, Phys. Rev. **134**, B848 (1964).

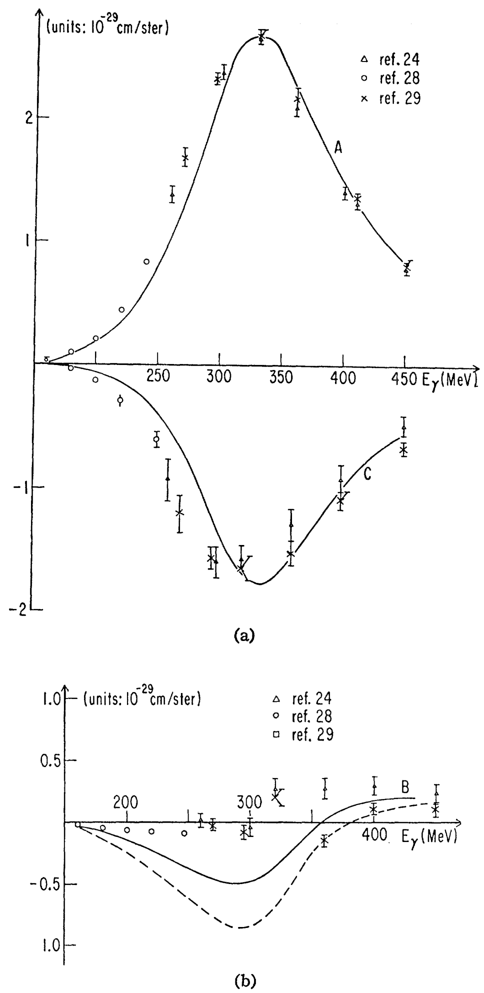

FIG. 11. (a) Coefficients A and C. (b) Coefficient B. The dashed line is the solution without the corrections to E_{0+} and M_{1-}. E_γ is the photon lab. energy.

not very well known. We have used the fit[42]:

$$G_{EN}\approx 0\,,\quad G_{EP}\approx\frac{G_{MP}}{\mu_P}\approx\frac{G_{MN}}{\mu_N}\approx\frac{1}{(1-\lambda^2/0.72)^2}\,. \qquad (6.5)$$

In Figs. 13 to 15 we show the results for the low-momentum transfer data. We consider two values of G_{EN}: (a) from the fit of de Vries *et al.*[41] and (b) a fixed negative value $G_{EN}=-0.20$. Our results seem to favor slightly negative values of G_{EN}. This agrees with previous calculations by Hand[12] using the FNW theory and by Salin[43] using an isobaric model. The results indicate good agreement up to a center of mass energy around $W\approx 1300$ MeV. However, our cross sections are much lower than the experimental results for energies above this value. The same result was obtained by Hand.[12]

The high λ^2 data is shown in Fig. 16 for fixed incident electron energies $k_{10}{}^L\approx 2.4$, 3.0, and 4.9 BeV. The curves for 2.4 and 4.9 BeV seem to reproduce the data reasonably well. For $k_{10}{}^L\approx 3.0$ BeV our results are larger than the experimental ones mainly in the region below the resonance. However, if we change (6.5) by about 20% (the standard deviations of the experimental values[42] are about 10% in this region), we obtain better agreement.

In Appendix III we have written the differential cross section as:

$$\frac{d^2\sigma}{dk_2{}^L d\Omega_2}=-\frac{e^2}{8\pi^2}\frac{k_2{}^L}{mk_1{}^L}\frac{Wk}{\lambda^2}\left\{\sigma_T(\lambda^2,S)\left[1-\frac{m^2\lambda^2}{2k^2W^2}\cot^2\left(\frac{\alpha}{2}\right)\right]-\frac{\lambda^2m^2}{2k^2W^2}\cot^2\left(\frac{\alpha}{2}\right)\sigma_L(\lambda^2,S)\right\}, \qquad (6.6)$$

where σ_T and σ_L are the differential cross sections for photoproduction by virtual photons of "mass" λ^2 polarized transversely and in the longitudinal direction,

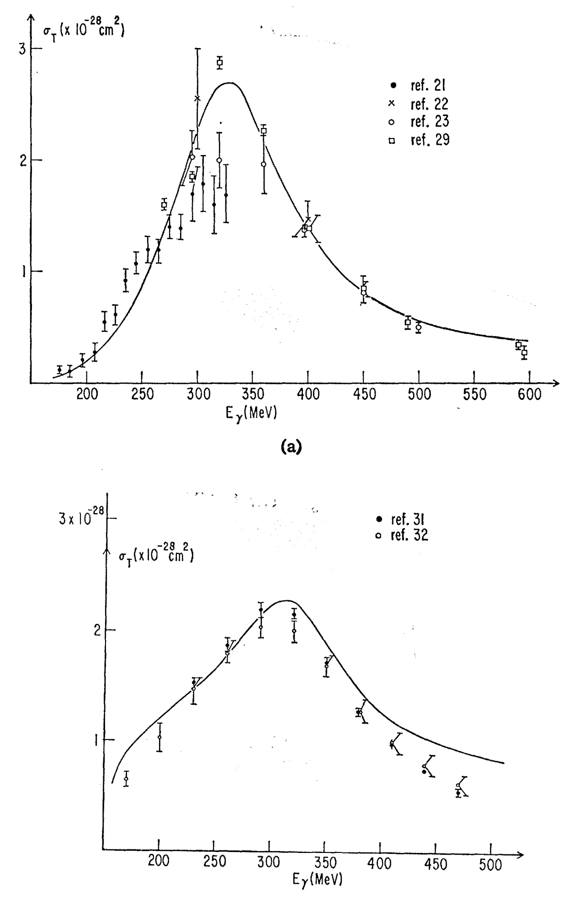

FIG. 12. Total cross section for (a) π_ν production and (b) π^+ production. E_γ is the photon lab. energy.

[42] J. R. Dunning, Jr., K. W. Chen, A. A. Cone, G. Hartwig, N. F. Ramsey, J. K. Walker, and R. Wilson, Phys. Rev. Letters **13**, 631 (1964).

[43] Ph. Salin, Nuovo Cimento **32**, 521 (1964).

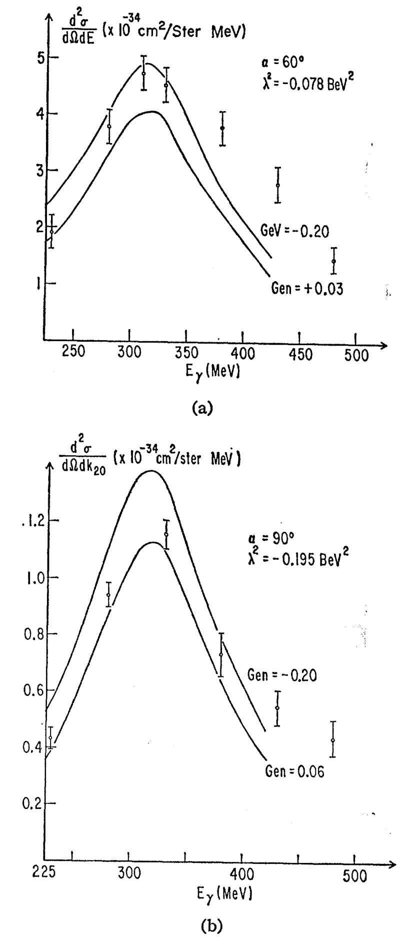

FIG. 13. Electroproduction of pions—differential cross sections. The abscissa represents the virtual photon "laboratory energy" $E_\gamma = (s-m^2)/2m$. The data are from Hand (Ref. 12).

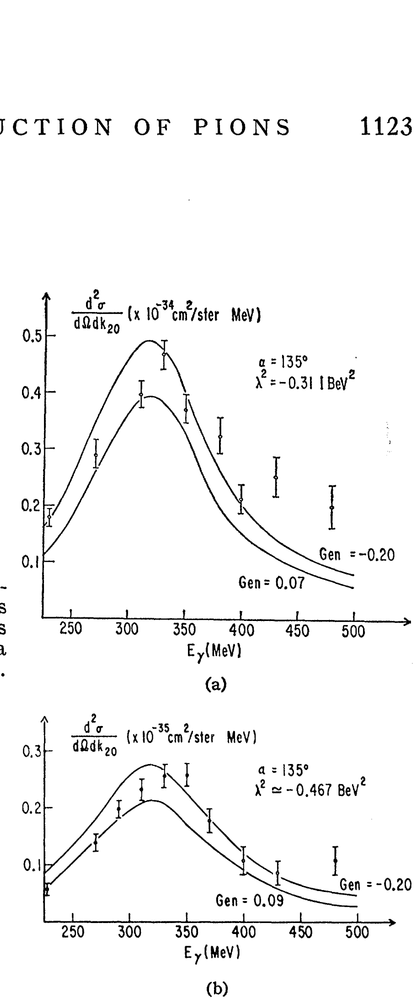

FIG. 14. Electroproduction of pions—differential cross sections. The data are from Hand (Ref. 12).

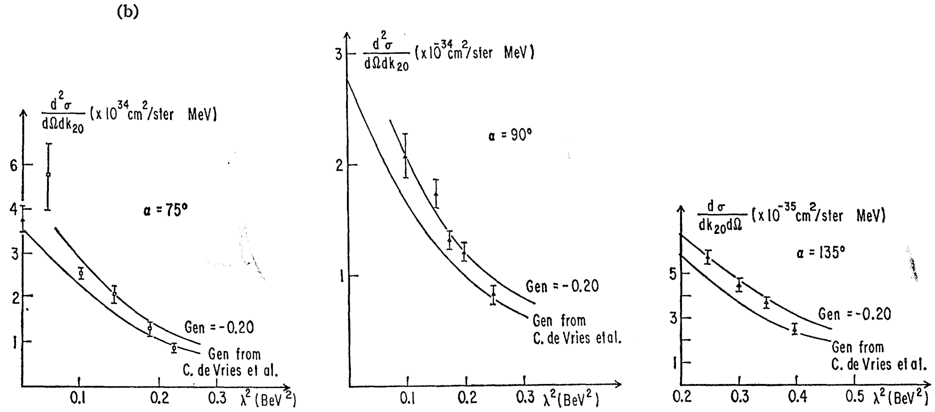

FIG. 15. Electroproduction of pions. Variation of the cross section with λ^2. The π-N center-of-mass energy is fixed at 1200 MeV. The data are from Panofsky and Allton (Ref. 39) (squares) and Olsen (Ref. 40) (circles).

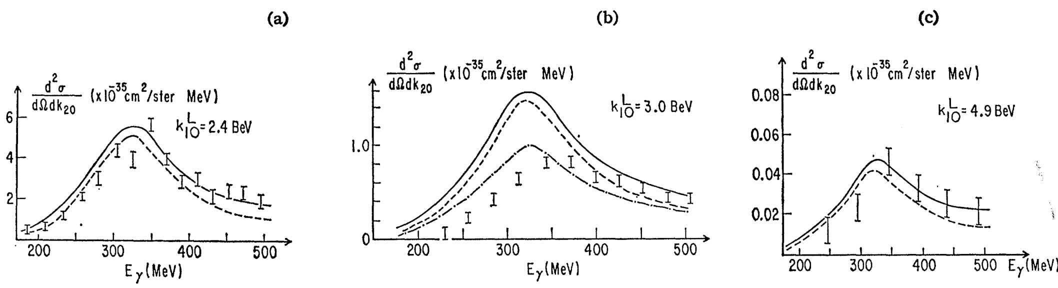

FIG. 16. Electroproduction of pions–differential cross sections. The dashed lines represent the σ_T contribution. The broken line in (b) is the cross section obtained by reducing the values of the form factors in Eq. (6.4) by 20%. The data are from Cone *et al.* (Ref. 14).

respectively, and α is the electron scattering angle in the laboratory system. At $\lambda^2=0$, $\sigma_L=0$. For $\lambda^2\neq 0$, the relative contribution of σ_L increases for small scattering angles α. We have plotted the separated contributions of σ_L and σ_T in Fig. 16 for $\alpha=31°$. For the data at 135°, 90°, and 60° its contribution is less than 10% in the entire region of energies considered.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor William R. Frazer for suggesting this problem and for his constant advice and guidance throughout the course of this work. He also wants to thank Professor D. Y. Wong, R. C. Vik, and S. Peyton for discussions.

APPENDIX I

The current j_μ can be split into three isotopic-spin independent currents: $j_\mu{}^s$ corresponding to the isoscalar part of the electromagnetic current and $j_\mu{}^{1/2}$ and $j_\mu{}^{3/2}$ corresponding to transitions $T=\frac{1}{2}\rightarrow T=\frac{1}{2}$ and $T=\frac{1}{2}\rightarrow T=\frac{3}{2}$ induced by the isovector part of the electromagnetic current. Instead of those, it is usual to introduce

$$j_\mu{}^+=\tfrac{1}{3}(j_\mu{}^{1/2}+2j_\mu{}^{3/2}),\qquad j_\mu{}^-=\tfrac{1}{3}(j_\mu{}^{1/2}-j_\mu{}^{3/2}). \tag{I1}$$

If α designates the isospin of the pion, we can write

$$\langle p_2,q;\alpha|j_\mu|p_1\rangle=j_\mu{}^s\tau_\alpha+j_\mu{}^+\delta_{\alpha 3}+j_\mu{}^-\tfrac{1}{2}[\tau_\alpha,\tau_3]. \tag{I2}$$

Following Dennery[4] we will write T in terms of 6 manifestly gauge-invariant amplitudes A_i (we are using $\gamma_5=i\gamma^0\gamma^1\gamma^2\gamma^3$):

$$T=i\gamma_5\sum_{i=1}^{6}M_iA_i \tag{I3}$$

where M_i have the following meaning

$$\begin{aligned}
M_1&=\tfrac{1}{2}\gamma_\mu\gamma_\nu F^{\mu\nu},\\
M_2&=(p_1+p_2)_\mu(q-k/2)_\nu F^{\mu\nu},\\
M_3&=\gamma_\mu q_\nu F^{\mu\nu},\\
M_4&=\gamma_\mu(p_1+p_2)_\nu F^{\mu\nu}-2mM_1,\\
M_5&=k_\mu q_\nu F^{\mu\nu},\\
M_6&=k_\mu\gamma_\nu F^{\mu\nu},\\
F^{\mu\nu}&=\epsilon^\mu k^\nu-\epsilon^\nu k^\mu.
\end{aligned} \tag{I4}$$

The amplitudes A_i are functions of the invariants λ^2, s, t, and u.

Analogously to the currents j^s, j^+, and j^- we define the isospin amplitudes A^s, A^+, and A^-. According to Dennery,[4] they satisfy the Mandelstam representation:

$$\begin{aligned}
A_i{}^\alpha={}&R_i{}^\alpha\left[\frac{1}{m^2-s}+\xi_i{}^\alpha\frac{1}{m^2-u}\right]+\frac{1}{\pi}\int_{(m+\mu)^2}^\infty ds'\rho_i{}^\alpha(s')\left[\frac{1}{s'-s}+\xi_i{}^\alpha\frac{1}{s'-u}\right]+\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty ds'\int_{4\mu^2}^\infty dt'\frac{a_i{}^\alpha(s',t')}{(s'-s)(t'-t)}\\
&+\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty ds'\int_{(m+\mu)^2}^\infty du'\frac{b_i{}^\alpha(s',u')}{(s'-s)(u'-u)}+\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty du'\int_{4\mu^2}^\infty dt'\frac{c_i{}^\alpha(u',t')}{(u'-u)(t'-t)},\quad\text{for }i=1,3,4,6
\end{aligned} \tag{I5}$$

and A_2 and A_5 satisfy the representation:

$$\begin{aligned}
A_i{}^\alpha={}&\frac{R_i{}^\alpha}{t-\mu^2}\left[\frac{1}{m^2-s}+\xi_i{}^\alpha\frac{1}{m^2-u}\right]+\frac{T_i{}^\alpha}{t-\mu^2}+\frac{1}{t-\mu^2}\frac{1}{\pi}\int_{(m+\mu)^2}^\infty ds'\rho_i{}^\alpha(s')\left[\frac{1}{s'-s}+\xi_i{}^\alpha\frac{1}{s'-u}\right]\\
&+\frac{1}{t-\mu^2}\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty ds'\int_{4\mu^2}^\infty dt'\frac{a_1{}^\alpha(s',t')}{(s'-s)(t'-t)}+\frac{1}{t-\mu^2}\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty ds'\int_{(m+\mu)^2}^\infty du'\frac{b_i{}^\alpha(s',u')}{(s'-s)(u'-u)}\\
&+\frac{1}{t-\mu^2}\frac{1}{\pi^2}\int_{(m+\mu)^2}^\infty du'\int_{4\mu^2}^\infty dt'\frac{c_i{}^\alpha(u',t')}{(u'-u)(t'-t)},
\end{aligned} \tag{I6}$$

where α refers to the superscript "+", "−" or "s" and

$$\begin{aligned}
\xi_i{}^\alpha&=+1\quad\text{for}\quad\alpha=\begin{cases}\text{"+" or "s"} & \text{and } i=1,2,4,\\ \text{"−"} & \text{and } i=3,5,6,\end{cases}\\
&=-1\quad\text{for}\quad\alpha=\begin{cases}\text{"+" or "s"} & \text{and } i=3,5,6,\\ \text{"−"} & \text{and } i=1,2,4.\end{cases}
\end{aligned} \tag{I7}$$

The residues of the poles are well known

$$\begin{aligned}
&R_2{}^\pm=2R_1{}^\pm=2R_5{}^\pm=-geF_{1v}, && R_2{}^s=2R_1{}^s=2R_5{}^s=-geF_{1s},\\
&R_3{}^\pm=R_4{}^\pm=geF_{2v}/2, && R_3{}^s=-R_4{}^s=geF_{2s}/2,\\
&T_5{}^-=\frac{2ge}{\lambda^2}(F_\pi-F_{1v}).
\end{aligned} \tag{I8}$$

F_{1v}, F_{1s} and F_{2v}, F_{2s} are the isovector (v) and isoscalar (s) nuclear form factors and F_π is the pion form factor. They are normalized as follows: $F_\pi(0)=F_{1v}(0)=F_{1s}(0)=1$, $F_{2v}(0)=\mu_p-\mu_n$ and $F_{2v}(0)=\mu_p+\mu_n$.

Following Fubini, Nambu, and Wataghin,[2] we added to the T matrix the term

$$ige(F_\pi-F_{1v})\frac{k_\mu\epsilon^\mu}{\lambda^2}\bar{u}(p_2)\gamma_5 u(p_1)\tfrac{1}{2}[\tau_\alpha,\tau_3] \tag{I9}$$

which is identically zero, in order to preserve the gauge invariance of the Born term.

The T matrix is related to the scattering amplitude in the center of mass of the pion-nucleon system by

$$T=4\pi\frac{W}{m}\chi^\dagger\left\{i\mathfrak{F}_1\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}+\mathfrak{F}_2\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\mathbf{k}\times\boldsymbol{\varepsilon}}{qk}+i\mathfrak{F}_3\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{qk}+i\mathfrak{F}_4\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{q}\cdot\boldsymbol{\varepsilon}}{q^2}+i\mathfrak{F}_5\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\mathbf{k}\cdot\boldsymbol{\varepsilon}}{k^2}+i\mathfrak{F}_6\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\mathbf{k}\cdot\boldsymbol{\varepsilon}}{qk}-i\mathfrak{F}_7\frac{\boldsymbol{\sigma}\cdot\mathbf{q}}{q}\epsilon_0-i\mathfrak{F}_8\frac{\boldsymbol{\sigma}\cdot\mathbf{k}\epsilon_0}{k}\right\}\chi, \tag{I10}$$

where W is the total energy, k and q are the momenta of the photon and pion, respectively, χ are the Pauli spinors, and the $\mathfrak{F}$'s are linear combinations of the invariant amplitudes.

Charge conservation implies the following relations among the $\mathfrak{F}$'s:

$$k_0\mathfrak{F}_8-k(\mathfrak{F}_5+\mathfrak{F}_1+z\mathfrak{F}_3)=0,\quad k_0\mathfrak{F}_7-k(\mathfrak{F}_6+z\mathfrak{F}_4)=0, \tag{I11}$$

where $z=\mathbf{q}\cdot\mathbf{k}/qk$.

In Appendix II we will give the relationship between the $\mathfrak{F}$'s and the A's. An inspection of those equations shows that the $\mathfrak{F}$'s satisfy the reflection relations

$$\mathfrak{F}_1(W)=\mathfrak{F}_2(-W),\qquad \mathfrak{F}_3(W)=\mathfrak{F}_4(-W),\qquad \mathfrak{F}_7(W)=\mathfrak{F}_8(-W). \tag{I12}$$

APPENDIX II

The invariant amplitudes A_i are related to the center of mass amplitudes $\mathfrak{F}_i$ by

$$\mathfrak{F}_1=\{[(W+m)^2-\lambda^2]^{1/2}[(W+m)^2-\mu^2]^{1/2}/16\pi s\}\times\{(W-m)(A_1-2mA_4)-\tfrac{1}{2}(t-\mu^2-\lambda^2)(A_3-A_4)+(s-m^2)A_4-\lambda^2A_6\}, \tag{II1}$$

$$\mathfrak{F}_2=\{[(W-m)^2-\lambda^2]^{1/2}[(W-m)^2-\mu^2]^{1/2}/16\pi s\}\times\{-(W+m)(A_1-2mA_4)-\tfrac{1}{2}(t-\mu^2-\lambda^2)(A_3-A_4)+(s-m^2)A_4-\lambda^2A_6\}, \tag{II2}$$

$$\mathfrak{F}_3=\{[(W-m)^2-\lambda^2]^{1/2}[(W-m)^2-\mu^2]^{1/2}/16\pi s\}(E_2+m)\{-(s-m^2-\lambda^2/2)A_2-\lambda^2A_5+(W+m)(A_3-A_4)\}, \tag{II3}$$

$$\mathfrak{F}_4=\{[(W+m)^2-\lambda^2]^{1/2}[(W+m)^2-\mu^2]^{1/2}/16\pi s\}(E_2-m)\{(s-m^2-\lambda^2/2)A_2+\lambda^2A_5+(W-m)(A_3-A_4)\}, \tag{II4}$$

$$\begin{aligned}\mathfrak{F}_7=-\{[(W+m)^2-\lambda^2]^{1/2}[(W-m)^2-\mu^2]^{1/2}/16\pi s\}\{&(E_1-m)(A_1-2mA_4)+\tfrac{1}{2}[(t-\mu^2-\lambda^2)+2q_0(W-m)](A_4-A_3)\\&-A_4(W-m)(E_1-m)+\tfrac{1}{2}[(t-\mu^2-\lambda^2)(\tfrac{1}{2}k_0-q_0)-(t-\mu^2)(E_1+E_2)+k_0(s-m^2)]A_2\\&+[\tfrac{1}{2}k_0(\lambda^2+\mu^2-t)-\lambda^2q_0]A_5-\lambda^2A_6\},\end{aligned} \tag{II5}$$

$$\begin{aligned}\mathfrak{F}_8=-\{[(W-m)^2-\lambda^2]^{1/2}[(W+m)^2-\mu^2]^{1/2}/16\pi s\}\{&-(E_1+m)(A_1-2mA_4)+\tfrac{1}{2}[(t-\mu^2-\lambda^2)+2q_0(W+m)](A_4-A_3)\\&-A_4(W+m)(E_1+m)+\tfrac{1}{2}[(t-\mu^2-\lambda^2)(q_0-\tfrac{1}{2}k_0)+(t-\mu^2)(E_1+E_2)-k_0(s-m^2)]A_2\\&+[-\tfrac{1}{2}k_0(\lambda^2+\mu^2-t)+\lambda^2q_0]A_5-\lambda^2A_6\}.\end{aligned} \tag{II6}$$

The expansion of $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$, and $\mathfrak{F}_4$ in transverse multipoles was given by CGLN[1] and their inversion by Ball.[3] The $\mathfrak{F}_7$ and $\mathfrak{F}_8$ amplitudes contain only contributions from the scalar multipoles. The expansion formulas are

$$\mathfrak{F}_1=\sum(lM_{l+}+E_{l+})P_{l+1}'(z)+[(l+1)M_{l-}+E_{l-}]P_{l-1}'(z), \tag{II7}$$

$$\mathfrak{F}_2=\sum[(l+1)M_{l+}+lM_{l-}]P_l'(z), \tag{II8}$$

$$\mathfrak{F}_3=\sum(E_{l+}-M_{l+})P_{l+1}''(z)+(E_{l-}+M_{l-})P_{l-1}''(z), \tag{II9}$$

$$\mathfrak{F}_4=\sum(M_{l+}-E_{l+}-M_{l-}-E_{l-})P_l''(z), \tag{II10}$$

$$\mathfrak{F}_7=\sum(S_{l-}-S_{l+})P_l'(z), \tag{II11}$$

$$\mathfrak{F}_8=\sum S_{l+}P_{l+1}'(z)-S_{l-}P_{l-1}'(z), \tag{II12}$$

where $z=\mathbf{q}\cdot\mathbf{k}/qk$ and the inversion formulas are

$$E_{l+}=\frac{1}{2(l+1)}\int_{-1}^{+1}dz\left\{\mathfrak{F}_1P_l(z)-\mathfrak{F}_2P_{l+1}(z)+\mathfrak{F}_3l\frac{P_{l-1}(z)-P_{l+1}(z)}{2l+1}+\mathfrak{F}_4(l+1)\frac{P_l(z)-P_{l+2}(z)}{2l+3}\right\},\quad l\geqslant 0,\qquad\text{(II13)}$$

$$E_{l-}=\frac{1}{2l}\int_{-1}^{+1}dz\left\{\mathfrak{F}_1P_l(z)-\mathfrak{F}_2P_{l-1}(z)-\mathfrak{F}_3(l+1)\frac{P_{l-1}(z)-P_{l+1}(z)}{2l+1}-\mathfrak{F}_4l\frac{P_{l-2}(z)-P_l(z)}{2l-1}\right\},\qquad l\geqslant 2,\qquad\text{(II14)}$$

$$M_{l+}=\frac{1}{2(l+1)}\int_{-1}^{+1}dz\left\{\mathfrak{F}_1P_l(z)-\mathfrak{F}_2P_{l+1}(z)-\mathfrak{F}_3\frac{P_{l-1}(z)-P_{l+1}(z)}{2l+1}\right\}.\qquad\text{(II15)}$$

$$M_{l-}=\frac{1}{2l}\int_{-1}^{+1}dz\left\{-\mathfrak{F}_1P_l(z)+\mathfrak{F}_2P_{l-1}(z)+\mathfrak{F}_3\frac{P_{l-1}(z)-P_{l+1}(z)}{2l+1}\right\},\quad l\geqslant 1,\qquad\text{(II16)}$$

$$S_{l+}=\frac{1}{2}\int_{-1}^{+1}dz\{\mathfrak{F}_7P_{l+1}(z)+\mathfrak{F}_8P_l(z)\},\quad l\geqslant 0,\qquad\text{(II17)}$$

$$S_{l-}=\frac{1}{2}\int_{-1}^{+1}dz\{\mathfrak{F}_7P_{l-1}(z)+\mathfrak{F}_8P_l(z)\},\quad l\geqslant 1.\qquad\text{(II18)}$$

With the aid of Eqs. (II7) to (II12) one can project out the Born term amplitudes in the 3-3 state. We obtain the following results

$$\mathfrak{M}_{1+}{}^B=\frac{ge}{32\pi W}\Bigg\{\Bigg[-\frac{2m(W-m)}{[(E_1-m))E_2-m)]^{1/2}}Q_1(x)-\frac{2m(W+m)}{[(E_1+m)(E_2+m)]^{1/2}}Q_2(x)$$
$$+\frac{2}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}(W+m)[Q_0(x)-Q_2(x)]\Bigg]G_{Mv}(\lambda^2)-\frac{1}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}\frac{(W-m)}{m}[Q_0(x)-Q_2(x)]F_{1v}(\lambda^2)$$
$$-\frac{2}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}[Q_0(y)-Q_2(y)]F_\pi(\lambda^2)\Bigg\},\qquad\text{(II.19)}$$

$$\mathcal{E}_{1+}{}^B=\frac{ge}{32\pi w}\Bigg\{\Bigg[-\frac{2m(W-m)}{[(E_1-m)(E_2-m)]^{1/2}}Q_1(x)-\frac{2m(W+m)}{[(E_1+m)(E_2+m)]^{1/2}}Q_2(x)$$
$$-\frac{2}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}(W+m)[Q_0(x)-Q_2(x)]-\frac{4}{5}\left(\frac{E_2-m}{E_1-m}\right)^{1/2}(W-m)[Q_1(x)-Q_3(x)]\Bigg]G_{Mv}(\lambda^2)$$
$$+\Bigg[\frac{1}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}\frac{(W-m)}{m}[Q_0(x)-Q_2(x)]+\frac{2}{5}\left(\frac{E_2-m}{E_1-m}\right)^{1/2}\frac{(W+m)}{m}[Q_1(x)-Q_3(x)]\Bigg]F_{1v}(\lambda^2)$$
$$+\Bigg[\frac{2}{3}\left(\frac{E_2+m}{E_1+m}\right)^{1/2}[Q_0(y)-Q_2(y)]-\frac{4}{5}\left(\frac{E_2-m}{E_1-m}\right)^{1/2}[Q_1(y)-Q_3(y)]\Bigg]F_\pi(\lambda^2)\Bigg\},\qquad\text{(II20)}$$

$$\mathcal{S}_{1+}{}^B=\frac{ge}{16\pi w}\Bigg\{-\Bigg[\frac{(W-m)(E_1+E_2-q_0)+2m(E_1-m)}{[(E_2+m)(E_1-m)]^{1/2}}Q_2(x)+\frac{(E_1+E_2-q_0)(W+m)-2m(E_1+m)}{[(E_2-m)(E_1+m)]^{1/2}}Q_1(x)\Bigg]G_{Mv}(\lambda^2)$$
$$+\frac{(E_1+E_2-q_0)}{2m}\Bigg[\frac{(W+m)}{[(E_2+m)(E_1-m)]^{1/2}}Q_2(x)+\frac{(W-m)}{[(E_2-m)(E_1+m)]^{1/2}}Q_1(x)\Bigg]F_{1v}(\lambda^2)$$
$$+(2q_0-k_0)\Bigg[\frac{Q_2(y)}{[(E_2+m)(E_1-m)]^{1/2}}+\frac{Q_1(y)}{[(E_2-m)(E_1+m)]^{1/2}}\Bigg]F_\pi(\lambda^2)\Bigg\},\qquad\text{(II21)}$$

where

$$G_{Mv}(\lambda^2)=F_{2v}(\lambda^2)+F_{1v}(\lambda^2)/2m,\quad x=(2E_1k_0-\lambda^2)/2qk,\quad y=(\lambda^2-2q_0k_0)/2qk, \tag{II22}$$

and

$$Q_l(z)=\frac{1}{2}\int_{-1}^{1}dt\frac{P_l(t)}{t+z} \tag{II23}$$

are the Legendre functions of the second kind.

APPENDIX III

The calculation of the cross section is simplified if one uses in the T matrix the relation $\epsilon_\mu k^\mu=0$ from the beginning. If we set $k_\mu\epsilon^\mu=0$, we can rewrite

$$T=iT_\mu\epsilon^\mu,\qquad T_\mu=\gamma_5[\gamma_\mu kV_1+p_{1\mu}V_2+p_{2\mu}V_3+\gamma_\mu V_4+kp_{1\mu}V_5+kp_{2\mu}V_6]. \tag{III1}$$

In the limit $k\cdot\epsilon=0$ the relation between the V's and A's is

$$\begin{aligned} &V_1=A_1-2mA_4, && V_2=(u-m^2-\tfrac{1}{2}\lambda^2)A_2-\lambda^2A_5, && V_3=(s-m^2-\tfrac{1}{2}\lambda^2)A_2+\lambda^2A_5,\\ &V_4=\tfrac{1}{2}(s-u)A_4-\tfrac{1}{2}(t-\mu^2-\lambda^2)A_3-\lambda^2A_6, && V_5=-(A_3+A_4), && V_6=A_3-A_4. \end{aligned} \tag{III2}$$

The summation over the electron spin gives

$$\tfrac{1}{2}\sum\epsilon_\mu\epsilon_\nu^\dagger=(e^2/2m_e^2\lambda^4)[k_{1\mu}k_{2\nu}+k_{1\nu}k_{2\mu}+\tfrac{1}{2}\lambda^2g_{\mu\nu}]. \tag{III3}$$

The summation over the nucleon spin will be a symmetrical tensor with the general form

$$\tfrac{1}{2}\sum T^\mu T^{\nu\dagger}=(1/2m^2)[Ag_{\mu\nu}+Bp_{1\mu}p_{1\nu}+Cp_{2\mu}p_{2\nu}+D\tfrac{1}{2}(p_{1\mu}p_{2\nu}+p_{1\nu}p_{2\mu})], \tag{III4}$$

where A, B, C, D are quadratic forms in the V's. Defining

$$V_{ig}=\mathrm{Re}V_iV_g^*, \tag{III5}$$

we have

$$A=\tfrac{1}{2}[(s-m^2)(u-m^2)-\mu^2\lambda^2]V_{11}+\tfrac{1}{2}(t-4m^2)V_{44}-m(s-u)V_{14}, \tag{III6}$$

$$B=-\tfrac{1}{2}tV_{22}+\tfrac{1}{2}[\lambda^2(\mu^2-4m^2)-(s-m^2)(u-m^2)]V_{55}+(u-m^2-\lambda^2)(V_{12}-V_{45}) -2mV_{24}+2m\lambda^2V_{15}+m(t-\mu^2-\lambda^2)V_{25}, \tag{III7}$$

$$C=-\tfrac{1}{2}tV_{33}+\tfrac{1}{2}[\lambda^2(\mu^2-4m^2)-(s-m^2)(u-m^2)]V_{66}+(s-m^2-\lambda^2)(V_{13}+V_{46}) +2mV_{34}+2m\lambda^2V_{16}+m(t-\mu^2-\lambda^2)V_{36}, \tag{III8}$$

$$D=2[-\lambda^2V_{11}+V_{44}-\tfrac{1}{2}tV_{23}+\tfrac{1}{2}(s-m^2-\lambda^2)(V_{12}+V_{45}-mV_{26}-mV_{35})+\tfrac{1}{2}(u-m^2-\lambda^2)(V_{13}-V_{46}-mV_{26}-mV_{35}) +m(V_{24}-V_{34}+\lambda^2V_{15}+\lambda^2V_{16})+\tfrac{1}{2}[\lambda^2(\mu^2-4m^2)-(s-m^2)(u-m^2)]V_{56}]. \tag{III9}$$

Starting from (II3) and (II4) Gourdin[44] has derived the formula for the cross section as a function of A, B, C, and D. In our notation his result is

$$\frac{d^2\sigma}{d\Omega_2{}^Ldk_{20}{}^L}=-\frac{e^2}{8\pi^3}\frac{k_2{}^L}{mk_1{}^L}\frac{Wk}{\lambda^2}\left\{\sigma_T(\lambda^2,s)\left[1-\frac{\lambda^2m^2}{2sk^2}\cot^2\!\left(\frac{\alpha}{2}\right)\right]-\frac{\lambda^2m^2}{2sk^2}\cot^2\!\left(\frac{\alpha}{2}\right)\sigma_L(\lambda^2,s)\right\}, \tag{III10}$$

where α is the scattering angle of the final electron in the lab system and $\sigma_T(\lambda^2,s)$ and $\sigma_L(\lambda^2,s)$ are the total cross section for production of pions by a vector meson with mass λ^2 polarized in the transverse or longitudinal directions. For $\lambda^2=0$, σ_T reduces to the photoproduction cross section.

In terms of A, B, C, and D we have[44]

$$\sigma_T(\lambda^2,s)=\frac{1}{16\pi s}\frac{q}{k}\int_{-1}^{+1}\{-A+\tfrac{1}{2}(1-z^2)q^2C\}dz, \tag{III11}$$

$$\sigma_L(\lambda^2,s)=\frac{1}{16\pi s}\frac{q}{k}\int_{-1}^{+1}\left\{-A+\frac{sk^2}{\lambda^2}\left[B+\frac{1}{W^2}\left(E_2+\frac{qk_0}{k}z\right)^2C+\frac{1}{W}\left(E_2+\frac{qk_0}{k}z\right)D\right]\right\}dz. \tag{III12}$$

---

[44] M. Gourdin, Nuovo Cimento **26**, 1094 (1961).