Density Operators for Coherent Fields*

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We study the forms which the conditions for coherence impose upon the density operators for electromagnetic fields. All fields possessing first-order coherence, we show, may be regarded as ones in which only a single mode is excited; this mode need not, however, be a monochromatic one. The higher order coherence conditions may be regarded as specifying certain moments of the distribution of the number of photons in the single excited mode. Full coherence, in particular, is shown to require that the number of photons present follow a Poisson distribution. The class of fields which possess full coherence is shown to be larger than the class of eigenstates of the annihilation operators. The unique character of the eigenstates, on the other hand, is demonstrated by means of a number of simple theorems.

I. INTRODUCTION

THE statistical information which describes the state of the quantized electromagnetic field is implicitly contained in its density operator. We have shown in earlier papers[1–3] how the knowledge of the density operator enables us, for example, to evaluate all of the correlation functions for the field vectors. An important property of the correlation functions is that they furnish a concise description of the coherence properties of the field; when the correlation functions obey appropriate factorization conditions the fields are described as coherent. Such restrictions on the form of the correlation functions, it is clear, may also be regarded as constraints placed upon the density operator of the field. In the present paper, we shall study in some detail the constraints which the coherence conditions impose upon the density operator, and discuss some explicit features of the fields whose density operators satisfy them.

It is convenient at this point to recall the definitions of the correlation functions, and of the coherence properties which they describe. The first-order correlation function for a radiation field described by the density operator ρ is given by

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathrm{tr}\{\rho E_\mu^{(-)}(\mathbf{r}t)E_\nu^{(+)}(\mathbf{r}'t')\}\,, \quad (1.1)$$

in which the operators $E_\mu^{(+)}(\mathbf{r}t)$ and $E_\nu^{(-)}(\mathbf{r}t)$ are the positive and negative frequency parts, respectively, of the operators for the vector components of the electric field at position $\mathbf{r}$ and time t. The correlation function of nth order is defined as the average of a $2n$-fold product of components of the field strength. We may write it as

$$G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n}) =\mathrm{tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_n) \times E^{(+)}(x_{n+1})\cdots E^{(+)}(x_{2n})\}\,, \quad (1.2)$$

where each variable x_j is an abbreviation for the space and time coordinates $\mathbf{r}_j$ and t_j and a vector index μ_j as well. A field possesses mth order coherence if the first m of its correlation functions can be written in the factorized form

$$G^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})=\prod_{j=1}^{n}\mathcal{E}^*(x_j)\mathcal{E}(x_{j+n})\,, \quad (1.3)$$

where the function $\mathcal{E}(x)$ is a complex solution of the wave equation. In a previous paper[4] we have studied the restrictions which this succession of conditions imposes on the structure of the correlation functions. We shall show that an analysis closely related to the one we have used in discussing the correlation functions may be used to derive a number of properties of the density operators.

The density operators of fields which possess first-order coherence are discussed in Sec. II. There exists a sense, as we shall show, in which first-order coherence corresponds to the restriction that only one variety of photons be present in the field. This variety may correspond to the excitation of an arbitrary superposition of the various monochromatic modes of the field. If we use this superposition to define a new mode of the field, which is, in general, a nonmonochromatic one, we find that the density operator for a first-order coherent field reduces to a form which has only this single mode excited.

The reduction of the density operator to a single-mode form makes it possible to specify the operator by means of a single matrix B_{mn}, which is its occupation number

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[1] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[2] R. J. Glauber, *Quantum Optics and Electronics, Les Houches 1964*, edited by C. deWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach Science Publishers, Inc., New York, 1965), p. 63.
[3] R. J. Glauber, in *Proceedings of the Physics of Quantum Electronics Conference, San Juan, Puerto Rico, 1965*, edited by P. Kelley, B. Lax, and P. E. Tannenwald (McGraw-Hill Book Company, Inc., New York, 1966), p. 788.
[4] U. M. Titulaer and R. J. Glauber, Phys. Rev. **140**, B676 (1965). We will use the abbreviation CF in referring further to this paper. The main results of CF, and some of the present paper as well, are also described by U. M. Titulaer and R. J. Glauber, Ref. 3, p. 812.

representation. A diagonal element of this matrix, B_{nn}, is the probability for the presence of n photons in the one excited mode. We show in Sec. III that there exists a simple connection between these diagonal elements and the set of real constants g_n which were introduced in CF as a means of describing higher order coherence properties of the field. Each of the two sets of constants, $\{g_n\}$ and $\{B_{nn}\}$, can easily be evaluated if the other is given. We show, in particular, that for any fully coherent field, i.e., one which is characterized by $g_n=1$ for all n, the probabilities B_{nn} correspond to a Poisson distribution for the number of photons present in the field. The coherence conditions, on the other hand, do not impose any restrictions on the off-diagonal elements of the matrix B_{mn}. There exists accordingly a broad variety of states which fulfill the coherence conditions precisely.

A well-known example of a fully coherent field is one which is in an eigenstate of the annihilation operators a_k for all the modes of the field.[2,5] Such an eigenstate, denoted by $|\{\alpha_k\}\rangle$, is characterized by the relation

$$a_k|\{\alpha_k\}\rangle=\alpha_k|\{\alpha_k\}\rangle. \tag{1.4}$$

This state can be expressed in terms of the eigenstates of the number operator by means of the expression[5]

$$|\{\alpha_k\}\rangle=\prod_k\left[\sum_{n=0}^{\infty} e^{-\frac{1}{2}|\alpha_k|^2}\frac{(\alpha_k)^n}{(n!)^{1/2}}|n\rangle_k\right]. \tag{1.5}$$

Although these states possess a number of unique physical properties, they are not the only ones which possess full coherence. The wider class of states which satisfy the full set of coherence conditions is discussed in Sec. III.

In Sec. IV we describe some of the ways in which the eigenstates of the annihilation operators are distinguished from the other states which exhibit full coherence. Their first characterization, which follows immediately from their nature as eigenstates of the annihilation operators, is that they are the only ones in which the variance of any annihilation operator vanishes. The characterization proves to be useful in deriving a number of other unique properties of these states. We find, for example, that the eigenstates of the annihilation operator are the only ones for which the density operator for the field reduces to a product of density operators for the individual modes, regardless of which system of orthogonal modes we use. They are, furthermore, the only states with first-order coherence in which the function $\mathcal{E}(x)$ in the expression (1.3) can be taken to be the expectation value of $E^{(+)}(x)$.

II. EVALUATION OF THE DENSITY OPERATOR

In this section we begin the systematic study of the restrictions imposed upon the density operator by the requirement of first-order coherence. Before doing so, however, we shall find it instructive to examine some particular examples of fields which exhibit first-order coherence. These examples will later be useful as illustrations of the general theory. We consider them here since they suggest some of its important features.

[5] R. J. Glauber, Phys. Rev. **131**, 2766 (1963). See in particular Sec. III and Eq. (9.1).

One of the simplest examples of a state which has first-order coherence is an arbitrary pure single photon state, which we may denote by $|1\text{ phot}\rangle$. For this field the first-order correlation function $G^{(1)}(x_1,x_2)$ takes the form

$$G^{(1)}(x_1,x_2)=\langle 1\text{ phot}|E^{(-)}(x_1)E^{(+)}(x_2)|1\text{ phot}\rangle. \tag{2.1}$$

Since $E^{(+)}(x)$ is a photon annihilation operator, $E^{(+)}(x)|1\text{ phot}\rangle$ can only be a multiple of the vacuum state, $|\text{vac}\rangle$. We may therefore insert the projection operator on the vacuum state in Eq. (2.1) and write

$$G^{(1)}(x_1,x_2)=\langle 1\text{ phot}|E^{(-)}(x_1)|\text{vac}\rangle \times\langle\text{vac}|E^{(+)}(x_2)|1\text{ phot}\rangle.$$

This means that for a single-photon field $G^{(1)}(x_1,x_2)$ fulfills the requirement for first-order coherence,

$$G^{(1)}(x_1,x_2)=\mathcal{E}^*(x_1)\mathcal{E}(x_2), \tag{2.2}$$

when we choose the function $\mathcal{E}(x)$ to be

$$\mathcal{E}(x)=\langle\text{vac}|E^{(+)}(x)|1\text{ phot}\rangle. \tag{2.3}$$

We note that no restrictions whatever are placed upon the spectral properties of the state $|1\text{ phot}\rangle$; any pure one-photon wave packet will do, whatever its frequency distribution may be.

A second, and more familiar type of coherent field is one which is monochromatic and completely polarized. Examples of such fields are ones in which only a single mode, say the kth, is excited. To show that such a field fulfills the requirement for first-order coherence we make use of the mode expansion for the positive frequency part of the field,

$$\mathbf{E}^{(+)}(\mathbf{r},t)=i\sum_k(\tfrac{1}{2}\hbar\omega_k)^{1/2}\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t}a_k. \tag{2.4}$$

The summation is carried out over a complete set of modes, characterized by the orthonormal set of vector mode functions $\mathbf{u}_k(\mathbf{r})$ and the corresponding frequencies ω_k. The operator a_k is an annihilation operator for a photon in the kth mode.

If the kth mode is the only one excited, the density operator for the field must obey the relations $a_l\rho=\rho a_l^\dagger=0$ unless $l=k$. These relations imply that the first-order correlation function takes the form

$$G^{(1)}(x_1,x_2)=\text{tr}\{\rho E^{(-)}(x_1)E^{(+)}(x_2)\} =(\tfrac{1}{2}\hbar\omega_k)\mathbf{u}_k^*(\mathbf{r}_1)\mathbf{u}_k(\mathbf{r}_2)e^{i\omega_k(t_1-t_2)}\,\text{tr}\{\rho a_k^\dagger a_k\}. \tag{2.5}$$

Since the trace that appears here is independent of x_1 and x_2, we can obviously construct a field $\mathcal{E}(x)$ in terms of which the correlation function factorizes according to the coherence condition (2.2). A field confined to a single mode can still take a variety of quantum mechanical forms; its density operator can be written as

$$\rho=\sum_{m_k,n_k} c_{m_kn_k}\frac{(a_k^\dagger)^{m_k}}{(m_k!)^{1/2}}|\text{vac}\rangle\langle\text{vac}|\frac{(a_k)^{n_k}}{(n_k!)^{1/2}}. \quad (2.6)$$

Such density operators may describe pure states as well as mixtures.

To exhibit more general forms of coherent fields, let us note that the field operator $\mathbf{E}^{(+)}(\mathbf{r},t)$ may be expanded in terms of other solutions of the wave equation than the monochromatic mode functions $\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt}$. We may, for example, define a different set of solutions $\mathbf{v}_l(\mathbf{r}t)$ of the wave equation as the set of linear combinations

$$\mathbf{v}_l(\mathbf{r}t)=i\sum_k\gamma_{lk}(\tfrac{1}{2}\hbar\omega_k)^{1/2}\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt}, \quad (2.7)$$

where the coefficients γ_{lk} are the elements of a unitary matrix, i.e., they obey the unitarity relation

$$\sum_l\gamma_{lk}^*\gamma_{lm}=\delta_{km}.$$

If we let b_l be the annihilation operator for a photon in the mode $\mathbf{v}_l(\mathbf{r}t)$, then we require that the positive frequency field operator have the expansion

$$\mathbf{E}^{(+)}(\mathbf{r},t)=\sum_l\mathbf{v}_l(\mathbf{r}t)b_l. \quad (2.8)$$

Comparison of this expansion with Eq. (2.4) shows that we must then have

$$\sum_l b_l\gamma_{lk}=a_k.$$

By inverting this relation and its Hermitian adjoint, we see that b_l and $b_l^\dagger$ are given by the linear combinations

$$b_l=\sum_k\gamma_{lk}^*a_k, \quad (2.9a)$$

$$b_l^\dagger=\sum_k\gamma_{lk}a_k^\dagger. \quad (2.9b)$$

These operators are seen to satisfy the commutation relations

$$[b_l,b_m]=[b_l^\dagger,b_m^\dagger]=0,\quad [b_l,b_m^\dagger]=\delta_{lm}, \quad (2.10)$$

which are of the same form as those of the operators a and $a^\dagger$. The transformation (2.9) which is induced by Eq. (2.7) is therefore a canonical one.[6]

Let us now suppose that only one of the modes corresponding to the functions $\mathbf{v}_l$ is occupied, say the one for $l=l_0$. Then the density operator obeys the relations

$$b_l\rho=\rho b_l^\dagger=0$$

for $l\neq l_0$. By using these relations in conjunction with the expression (2.8) and its adjoint for the complex field operators, we find that the first-order correlation function takes the form

$$G^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathbf{v}_{l_0}^*(\mathbf{r}t)\mathbf{v}_{l_0}(\mathbf{r}'t')\,\text{tr}\{\rho b_{l_0}^\dagger b_{l_0}\}. \quad (2.11)$$

This form satisfies the requirements for first-order coherence just as the form in Eq. (2.5) did.

Since the fields we shall consider below are only excited in the single mode associated with a particular function $\mathbf{v}_{l_0}(\mathbf{r}t)$, we may simplify the notation a bit by dropping the indices l_0. It is evident that by defining the transformation coefficients suitably the function $\mathbf{v}$ for the single mode which is excited may be taken to be any function of the form

$$\mathbf{v}(\mathbf{r}t)=i\sum_k\gamma_k(\tfrac{1}{2}\hbar\omega_k)^{1/2}\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt}, \quad (2.12)$$

where the complex coefficients γ_k obey the normalization condition

$$\sum_k|\gamma_k|^2=1. \quad (2.13)$$

The corresponding annihilation and creation operators are then

$$b=\sum_k\gamma_k^*a_k, \quad (2.14a)$$

$$b^\dagger=\sum_k\gamma_ka_k^\dagger. \quad (2.14b)$$

An n photon state of the mode (2.12) may evidently be expressed in the form $(n!)^{-1/2}(b^\dagger)^n|\text{vac}\rangle$. It follows then that the most general density operator for which this mode alone is excited may be written in the form

$$\rho=\sum_{m,n}B_{mn}\frac{(b^\dagger)^n}{(n!)^{1/2}}|\text{vac}\rangle\langle\text{vac}|\frac{b^m}{(m!)^{1/2}}, \quad (2.15)$$

where the coefficients B_{mn}, like the $c_{m_kn_k}$ in Eq. (2.6), may be chosen in a great variety of ways.

By generalizing our definition of a mode to include nonmonochromatic solutions of the wave equation we have derived the density operators (2.15), which describe a broad class of fields possessing first-order coherence. We shall now show that the density operators (2.15) form the most general class possessing this property; the density operator for any first-order coherent field can be written in the form (2.15).

To demonstrate this theorem we make use of an identity, satisfied by the density operators of all first-order coherent fields, as shown in our previous paper. This is the relation CF (3.9a) which may be written as

$$E_\mu^{(+)}(\mathbf{r},t)\rho=\frac{G_{\mu_0\mu}^{(1)}(\mathbf{r}_0t_0,\mathbf{r}t)}{G_{\mu_0\mu_0}^{(1)}(\mathbf{r}_0t_0,\mathbf{r}_0t_0)}E_{\mu_0}^{(+)}(\mathbf{r}_0t_0)\rho,$$

where $\mathbf{r}_0$, t_0 is an arbitrary point for which the correlation function in the denominator does not vanish for an appropriate choice of μ_0. When both sides of this equation are multiplied by

$$-i(\tfrac{1}{2}\hbar\omega_k)^{-1/2}u_{k\mu}^*(\mathbf{r})e^{i\omega_kt},$$

summed over μ, and integrated over $\mathbf{r}$ the resulting equation may be written in the form

$$a_k\rho=\beta_kE^{(+)}(x_0)\rho, \quad (2.16)$$

[6] It may be noted that, although the transformation is canonical, the mode functions $\mathbf{v}_l(\mathbf{r},t)$ are not in general an orthonormal set, because of the frequency-dependent normalization factors in Eq. (2.7).

where the constant β_k is given by

$$\beta_k=\sum_\mu \int \frac{u_{k\mu}^*(\mathbf{r})e^{i\omega_k t}}{i(\frac{1}{2}\hbar\omega_k)^{1/2}} \frac{G_{\mu_0\mu}^{(1)}(\mathbf{r}_0t_0,\mathbf{r}t)}{G_{\mu_0\mu_0}^{(1)}(\mathbf{r}_0t_0,\mathbf{r}_0t_0)}d\mathbf{r}. \quad (2.17)$$

It is easily seen from the fact that $G^{(1)}$ satisfies the wave equation that the dependence on t in this equation cancels exactly, and the coefficient β_k is independent of time. We show further that $\sum_k|\beta_k|^2$ is finite as long as the mean total number of photons $\langle N\rangle$ in the field is finite[7]; this follows directly from the expression

$$\langle N\rangle=\mathrm{tr}\{\sum_k a_k^\dagger a_k\rho\}=\sum_k|\beta_k|^2G^{(1)}(x_0,x_0)<\infty.$$

With these preliminaries we are able to evaluate the density operator ρ in the occupation number representation,

$$\rho=\sum_{\{n_k\},\{m_l\}} |\{n_k\}\rangle\langle\{n_k\}|\rho|\{m_l\}\rangle\langle\{m_l\}|.$$

If we insert the definitions of the states $|\{n_k\}\rangle$ in this equation we have

$$\rho=\sum_{\{n_k\},\{m_l\}}\prod_k \frac{(a_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|\mathrm{vac}\rangle\langle\mathrm{vac}|\prod_k\frac{(a_k)^{n_k}}{(n_k!)^{1/2}}\rho \times\prod_l\frac{(a_l^\dagger)^{m_l}}{(m_l!)^{1/2}}|\mathrm{vac}\rangle\langle\mathrm{vac}|\prod_l\frac{(a_l)^{m_l}}{(m_l!)^{1/2}}.$$

We may now make use of relation (2.16) and its adjoint and of the commutation properties of creation and annihilation operators to rewrite this expression for ρ in the form

$$\rho=\sum_{\{n_k\},\{m_l\}}\prod_k\frac{(\beta_k a_k^\dagger)^{n_k}}{n_k!}|\mathrm{vac}\rangle\langle\mathrm{vac}|[E^{(+)}(x_0)]^{\Sigma_k n_k}\rho \times[E^{(-)}(x_0)]^{\Sigma_l m_l}|\mathrm{vac}\rangle\langle\mathrm{vac}|\prod_l\frac{(\beta_l^* a_l)^{m_l}}{m_l!}. \quad (2.18)$$

We next define $n=\sum_k n_k$, and $m=\sum_l m_l$ and use the multinomial theorem to carry out the partial summations in which n and m remain fixed. In this way we obtain

$$\rho=\sum_{n,m}\frac{(\sum_k\beta_k a_k^\dagger)^n}{n!}|\mathrm{vac}\rangle\langle\mathrm{vac}|\frac{(\sum_l\beta_l^* a_l)^m}{m!} \times\langle\mathrm{vac}|[E^{(+)}(x_0)]^n\rho[E^{(-)}(x_0)]^m|\mathrm{vac}\rangle. \quad (2.19)$$

This expression is seen to have exactly the form of Eq. (2.15) when the coefficients γ_k in the definitions (2.14) of b and $b^\dagger$ are identified as

$$\gamma_k=\beta_k[\sum_k|\beta_k|^2]^{-1/2}. \quad (2.20)$$

The coefficients β_k are given by Eq. (2.17) and the boundedness of $\sum_k|\beta_k|^2$ was shown for any field with an average occupation number which is bounded. When this identification is made, we see that the coefficients B_{nm} of Eq. (2.15) are given by

$$B_{nm}=(n!m!)^{-1/2}[\sum_k|\beta_k|^2]^{1/2(n+m)} \times\langle\mathrm{vac}|[E^{(+)}(x_0)]^n\rho[E^{(-)}(x_0)]^m|\mathrm{vac}\rangle. \quad (2.21)$$

It is clear from this expression for the coefficients B_{nm} that they form a Hermitian matrix. From the definition (2.14a) of b we find the relations $b|\mathrm{vac}\rangle=0$ and $[b,b^\dagger]=1$. They enable us to evaluate the trace of the density operator in terms of the B_{nm} as

$$\mathrm{tr}\rho=\sum_{n,m}B_{nm}(n!m!)^{-1/2}\langle\mathrm{vac}|b^m(b^\dagger)^n|\mathrm{vac}\rangle$$
$$=\sum_{n,m}B_{nm}(n!m!)^{-1/2}n!\delta_{nm}$$
$$=\sum_n B_{nn}=1.$$

The matrix B_{nm} therefore has unit trace. Its diagonal element B_{mm} is easily seen to be the probability that m photons occupy the mode. Our main result in this section can be summarized quite concisely. The density operator for the most general type of field possessing first order coherence is a simple generalization of the density operator (2.6) for a field with a single monochromatic mode excited. To achieve full generality it is only necessary to replace the creation operator $a_k^\dagger$ in the density operator (2.6) by the more general creation operator $b^\dagger$ which creates a photon in a particular superposition of modes. If we think of this superposition as specifying a particular type of photon wave packet, then we see that the field may be regarded as consisting entirely of photons of that type.[8]

We have proved both that a field specified by the density operator (2.15) has first-order coherence and that conversely every field with first-order coherence must have a density operator of the form (2.15). This means that we have obtained a third way of characterizing first-order coherent fields. The first two are the factorization condition (2.1) and the maximum fringe visibility condition. The latter condition and its equivalence with the factorization condition are discussed in CF, Sec. III.

III. FULLY COHERENT FIELDS

To discuss the higher order coherence properties of a field with first-order coherence we consider the sequence

[7] The assumption that the total number of photons is finite is a natural one in the present context. We have assumed that the modes of the field are a denumerable set, as is the case for the field in any finite enclosure. For such a field there exists a minimum frequency, and infrared divergences do not occur.

[8] It is perhaps worth noting that this situation strongly resembles the one encountered in the study of superfluids. The superfluid component has long-range correlations which may be expressed by means of a factorization of its correlation functions. This component consists, furthermore, entirely of particles in a small number of similar quantum states. The fact that these states may be nonstationary and quite arbitrary in form was stressed recently by P. C. Hohenberg and P. C. Martin, Ann. Phys. (N. Y.) 34, 291 (1965).

of numbers

$$g_n=\frac{G^{(n)}(x_0\cdots x_0)}{[G^{(1)}(x_0,x_0)]^n}. \tag{3.1}$$

In CF we showed that these numbers completely describe the higher order coherence properties of a field which has first-order coherence. Such a field is specified, as shown in Sec. II, by a density operator of the type (2.15). The correlation functions with all arguments set equal are therefore given by

$$G^{(n)}(x_0\cdots x_0)=\sum_m B_{mm}\langle\text{vac}|\frac{b^m}{(m!)^{1/2}}\times[E^{(-)}(x_0)]^n[E^{(+)}(x_0)]^n\frac{(b^\dagger)^m}{(m!)^{1/2}}|\text{vac}\rangle. \tag{3.2}$$

To evaluate this expression we make use of the commutator

$$[b^m,[E^{(-)}(x_0)]^n]=[m!/(m-n)!]b^{m-n}[b,E^{(-)}(x_0)]^n \tag{3.3}$$

for $m\geqslant n$. The terms of the sum (3.2) for $m<n$ are easily seen to vanish.

The relation (3.3) together with its adjoint enables us to write $G^{(n)}(x_0\cdots x_0)$ as

$$G^{(n)}(x_0\cdots x_0)=|[b,E^{(-)}(x_0)]|^{2n}\sum_{m=n}^{\infty}B_{mm}\frac{1}{m!}\left[\frac{m!}{(m-n)!}\right]^2\times\langle\text{vac}|b^{m-n}(b^\dagger)^{m-n}|\text{vac}\rangle$$

$$=|[b,E^{(-)}(x_0)]|^{2n}\sum_{m=n}^{\infty}B_{mm}\frac{m!}{(m-n)!}. \tag{3.4}$$

In particular for $n=1$, we have

$$G^{(1)}(x_0,x_0)=|[b,E^{(-)}(x_0)]|^2\sum_{m=1}^{\infty}B_{mm}m. \tag{3.5}$$

If we introduce the notation

$$\langle f(m)\rangle\equiv\sum_m f(m)B_{mm}, \tag{3.6}$$

we can express the coefficient g_n as

$$g_n=\frac{G^{(n)}(x_0\cdots x_0)}{[G^{(1)}(x_0,x_0)]^n}=\frac{\langle m!/(m-n)!\rangle}{\langle m\rangle^n}. \tag{3.7}$$

A fully coherent field, we have shown in CF, is characterized by $g_n=1$ for all n. This condition can be written as

$$\langle m!/(m-n)!\rangle=\langle m\rangle^n. \tag{3.8}$$

In other words, the factorial moments of the photon number distribution reduce to powers of the mean photon number. This characterization of the field leads uniquely to a Poisson distribution for the number of photons present in it. The distribution is most easily derived by making use of a generating function method. A number of details of this method which we shall not examine here are discussed elsewhere.[9]

If we define a generating function $F(\lambda)$ as

$$F(\lambda)=\sum_{m=0}^{\infty}\lambda^m B_{mm}, \tag{3.9}$$

then we see immediately that

$$B_{mm}=[(1/m!)(d/d\lambda)^m F(\lambda)]|_{\lambda=0}. \tag{3.10}$$

On the other hand, the factorial moments can also be expressed in terms of $F(\lambda)$ as

$$\left\langle\frac{m!}{(m-n)!}\right\rangle=\sum_{m=n}^{\infty}m(m-1)\cdots(m-n+1)B_{mm}=[(d/d\lambda)^n F(\lambda)]|_{\lambda=1}. \tag{3.11}$$

Since Eq. (3.8) tells us the derivatives of the generating function at $\lambda=1$, we may construct the function explicitly by means of a Taylor series expansion about $\lambda=1$. In this way, we find

$$F(\lambda)=\sum_{n=0}^{\infty}\frac{(\lambda-1)^n}{n!}\langle m\rangle^n=e^{(\lambda-1)\langle m\rangle}.$$

By carrying out the differentiations indicated in Eq. (3.10) we see that the B_{nn} are given by

$$B_{nn}=(\langle m\rangle^n/n!)e^{-\langle m\rangle}, \tag{3.12}$$

which is exactly a Poisson distribution.

The converse of this theorem is also true. If the density operator for a field takes the form of Eq. (2.15), and if the diagonal matrix elements B_{nn} form a Poisson distribution, then the field is fully coherent. To prove this we need only note that under the conditions stated the relations (3.8) will all hold, or the g_n, in other words, will all equal unity.

It is interesting to observe that once a field possesses first-order coherence, and the density operator consequently falls into the form of Eq. (2.15), the higher order coherence conditions only place constraints on the diagonal elements of the matrix B_{mn}. The great freedom which remains available in the choice of the off-diagonal elements means that a considerable variety of mixed states as well as pure ones is capable of satisfying the conditions for full coherence. (The definition of full coherence has intentionally been chosen to admit appropriate types of mixtures as well as pure states, since virtually no optical experiments deal with pure states.)

The eigenstates of the photon annihilation operators are obvious examples of pure states which are fully coherent. We may use the theorems we have proved to

[9] See Ref. 2, Lecture XVII.

derive a more general class of pure states which possess the same property. To do this we note that for any pure state, $|\,\rangle$, the density operator takes the form $\rho=|\,\rangle\langle\,|$. Its matrix elements in the occupation-number representation must therefore take the form

$$\langle\{n_k\}|\rho|\{m_k\}\rangle=r^*(\{n_k\})r(\{m_k\})\,, \qquad (3.13)$$

where $r(\{m_k\})$ is a suitably determined function of the occupation numbers. We see then that if a density operator of the form (2.15) is to represent a pure state, the matrix B_{mn} must factorize according to the scheme

$$B_{nm}=C_n^*C_m\,, \qquad (3.14)$$

in which the set of coefficients C_n obey the normalization condition

$$\sum_n|C_n|^2=1\,.$$

The density operator which results from substituting Eq. (3.14) into Eq. (2.15) may be regarded as a projection operator on the state

$$e^{i\phi}\sum_n[C_n/(n!)^{1/2}](b^\dagger)^n|\text{vac}\rangle\,, \qquad (3.15)$$

where ϕ is an arbitrary phase angle. This is the most general pure state which possesses first-order coherence. The conditions for full coherence impose the further requirement that the $B_{nn}=|C_n|^2$ be given by Eq. (3.12). By satisfying this requirement we see that the most general pure state which has full coherence may be written in the form

$$\sum_n(\langle m\rangle^{n/2}/n!)e^{-\langle m\rangle/2+i\theta_n}(b^\dagger)^n|\text{vac}\rangle\,, \qquad (3.16)$$

where the phases θ_n may be chosen arbitrarily. Comparison of this expression with that for an eigenstate of a photon annihilation operator, Eq. (1.4), shows that it is an eigenstate of the operator b if and only if the θ_m obey a linear relationship of the form

$$\theta_m=m\theta+\phi \quad (\text{mod}\,2\pi) \qquad (3.17)$$

for some θ and ϕ.

By making use of Eq. (3.18) of Ref. 5, we can easily show that an eigenstate of b, denoted by $|\beta\rangle$, is also an eigenstate of each of the annihilation operators a_k. We use the definition of b, Eq. (2.14a), and the commutativity of all $a_k^\dagger$ and $a_{k'}$ for $k\neq k'$ to write

$$\begin{aligned}|\beta\rangle&=\exp[\beta b^\dagger-\beta^*b]|\text{vac}\rangle\\&=\exp[\textstyle\sum_k(\beta\gamma_ka_k^\dagger-\beta^*\gamma_k^*a_k)]|\text{vac}\rangle\\&=\textstyle\prod_k\exp[\beta\gamma_ka_k^\dagger-\beta^*\gamma_k^*a_k]|\text{vac}\rangle\\&=|\{\beta\gamma_k\}\rangle.\end{aligned} \qquad (3.18)$$

Here $|\{\beta\gamma_k\}\rangle$ is the eigenstate of the operators a_k with eigenvalues $\beta\gamma_k$. The constants γ_k are defined in Eq. (2.20).

The difference between an eigenstate of the annihilation operators[10] and the more general type of fully coherent state, described by Eq. (3.16) does not become manifest in experiments, which can be described completely in terms of the correlation functions $G^{(n)}$, since these functions are the same for both. Examples of such experiments are measurements of the field intensity, photon coincidence rates and the distribution functions of the number of photons counted. The quantities these experiments measure are independent of the absolute phase of the field. A typical example of a quantity which does depend on the absolute phase is an average value of the product of unequal powers of the creation and annihilation operators $b^\dagger$ and b. If ρ is the projection operator on the state (3.16), such a quantity is given by

$$\text{tr}\{\rho b^{\dagger l}b^n\}=\langle m\rangle^{(l+n)/2}\sum_p\frac{\langle m\rangle^p}{p!}e^{-\langle m\rangle}e^{i(\theta_{p+n}-\theta_{p+l})}. \qquad (3.19)$$

It is interesting to compare this average value with the one we obtain if ρ corresponds to a mixture of eigenstates of b with the same amplitude, $|\beta|=\langle m\rangle^{1/2}$, but different phases,

$$\rho=\sum_j p_j|\langle m\rangle^{1/2}e^{i\theta_j}\rangle\langle\langle m\rangle^{1/2}e^{i\theta_j}|\,, \qquad (3.20)$$

where the p_j are non-negative numbers and have the normalization $\sum_j p_j=1$. For this density operator we find, in place of Eq. (3.19),

$$\text{tr}\{\rho(b^\dagger)^lb^n\}=\langle m\rangle^{(l+n)/2}\sum_j p_je^{i(n-l)\theta_j}. \qquad (3.21)$$

Comparison of this result with Eq. (3.19) shows us that for each l and n we can construct a set of coefficients p_j and θ_j such that the result (3.19) is reproduced by a ρ of the form (3.20). The average expressed in Eq. (3.19), in other words, is equivalent to one in which the phase of the field is allowed to vary while the modulus remains fixed. For different l and n, however, we must use different sets of p_j and θ_j.

The density operator (3.20) corresponds in the classical limit to an ensemble of fields with fixed modulus and a phase randomly distributed among certain discrete values. A state of the field of the form given by Eq. (3.16) can evidently be replaced, within the context of Eq. (3.19), by such a random-phase ensemble. While this observation is of help in picturing the physical character of the states (3.16), we must remember that the equivalent ensembles will be different for each pair of integers l and n in Eq. (3.19).

The subtle correlations which may be present in a quantum mechanical state cannot be reproduced faithfully by means of a single classical ensemble, as can be

[10] In Ref. 5 and elsewhere the eigenstates of the annihilation operator have been referred to as coherent states. We have used their lengthier and more specific designation in the present paper, however, in order to avoid confusion in discussing broader classes of states which satisfy the coherence conditions. Since the eigenstates of the annihilation operators are the only easily generated members of this set, and mathematically the most useful ones, their designation simply as coherent states will remain convenient in most other contexts.

seen by considering a state (3.16) for which

$$\theta_n = n\theta + (-1)^n\phi.$$

This state may show a large phase uncertainty in the expression for $\mathrm{tr}\{\rho b\}$, but none at all in that for $\mathrm{tr}\{\rho b^2\}$. States of such thoroughly unclassical nature[11] will, of course, not be produced by everyday light sources. However, in a complete quantum mechanical discussion we cannot exclude them from consideration.

IV. UNIQUE PROPERTIES OF THE ANNIHILATION OPERATOR EIGENSTATES

The eigenstates of the annihilation operators can be distinguished from the more general class of states which exhibit full coherence in a variety of ways. We shall discuss several of the respects in which these states are unique in the present section.

It is well known that the eigenstates of the annihilation operators are ones for which the product of the uncertainties of the coordinate and of the momentum of each field oscillator is a minimum. This minimum property does not characterize the eigenstates uniquely, however, since each oscillator has a continuum of states with the same value of the uncertainty product. A more useful way of characterizing the eigenstates of the annihilation operators is through the discussion of a rather different type of uncertainty. For this purpose we note that for any density operator ρ, and any set of complex numbers $\{\alpha_k\}$ we have the positive-definiteness inequalities[12]

$$\mathrm{tr}\{\rho(a_k^\dagger-\alpha_k^*)(a_k-\alpha_k)\}\geq 0, \tag{4.1}$$

which hold for all k. If the field is in a pure eigenstate of the annihilation operators corresponding to the eigenvalues $\{\alpha_k\}$, i.e., we have $\rho=|\{\alpha_k\}\rangle\langle\{\alpha_k\}|$, then it is clear that the averages given by Eq. (4.1) vanish for all k. Since the numbers $\{\alpha_k\}$ are in that case the mean values of the mode amplitudes, $\alpha_k=\mathrm{tr}\{\rho a_k\}$, the eigenstates of the annihilation operators may be said to minimize the uncertainties (i.e., the variances) of the complex field amplitudes. [There exists some ambiguity in defining the variance of a non-Hermitian operator. The expression (4.1) is chosen so that the zero-point oscillations do not contribute to the variance.]

To show that the minimum property we have just exhibited characterizes the eigenstates uniquely, let us assume that for some sets of amplitudes $\{\alpha_k\}$ we have

$$\mathrm{tr}\{\rho(a_k^\dagger-\alpha_k^*)(a_k-\alpha_k)\}=0 \tag{4.2}$$

for all k, and solve for the density operator ρ. As a first step we may make use of the theorem proved in Sec. II of CF which shows that the identities (4.2) imply

$$(a_k-\alpha_k)\rho=\rho(a_k^\dagger-\alpha_k^*)=0 \tag{4.3}$$

for all k. It is clear from these relations that α_k is the mean value of a_k and α_k^* is that of the conjugate operator. By using the brackets $\langle\ \rangle$ to denote statistical averages, we may therefore write Eq. (4.2) in the alternative form

$$\langle a_k^\dagger a_k\rangle-|\langle a_k\rangle|^2=0. \tag{4.4}$$

Let us consider the case of a field in which only the kth mode is excited. For brevity we shall temporarily drop the index k. The density operator for this field can be written in the occupation number representation as

$$\rho=\sum_{n,m}|n\rangle\langle n|\rho|m\rangle\langle m|. \tag{4.5}$$

We may now use the definitions of the states $|n\rangle$ and the identities (4.3) to bring the density operator to the form

$$\rho=\sum_{n,m}|n\rangle\langle\mathrm{vac}|\frac{a^n}{(n!)^{1/2}}\rho\frac{(a^\dagger)^m}{(m!)^{1/2}}|\mathrm{vac}\rangle\langle m|$$
$$=\langle\mathrm{vac}|\rho|\mathrm{vac}\rangle\textstyle\sum_n\frac{\alpha^n}{(n!)^{1/2}}|n\rangle\sum_m\langle m|\frac{(\alpha^*)^m}{(m!)^{1/2}}. \tag{4.6}$$

Since $\mathrm{tr}\rho=1$ we must have

$$\langle\mathrm{vac}|\rho|\mathrm{vac}\rangle=e^{-|\alpha|^2}. \tag{4.7}$$

We see then that ρ is a projection operator on the eigenstate $|\alpha\rangle$ given by Eq. (1.5). Hence the necessary and sufficient condition that Eq. (4.2) holds for a single-mode excitation is that the mode be in the eigenstate of the annihilation operator having eigenvalue α.

The generalization of this result to the case in which arbitrarily many modes are excited is immediate. By expanding the density operator in the occupation number representation for the full set of modes and again using the identities (4.3) we may see that ρ is the projection operator on the state $|\{\alpha_k\}\rangle$ which is an eigenstate of all the operators a_k.

The theorem we have just proved applies to non-monochromatic modes as well as monochromatic ones. Thus if the operators b_l are the set defined by Eq. (2.9a) and there exists a set of complex numbers β_l such that

$$\mathrm{tr}\{\rho(b_l^\dagger-\beta_l^*)(b_l-\beta_l)\}=0 \tag{4.8}$$

for all l, then the β_l must be the mean values of the b_l,

$$\beta_l=\mathrm{tr}\{\rho b_l\}, \tag{4.9}$$

and ρ must be the projection operator on the state

$$|\{\beta_l\}\rangle=\exp(-\tfrac{1}{2}\textstyle\sum_l|\beta_l|^2)\sum_n[(\sum_l\beta_l b_l^\dagger)^n/n!]|\mathrm{vac}\rangle$$
$$=\exp[\textstyle\sum_l(\beta_l b_l^\dagger-\beta_l^* b_l)]|\mathrm{vac}\rangle, \tag{4.10}$$

which is an eigenstate of the $\{b_l\}$. By using Eqs. (2.9a)

[11] We may note, incidentally, that this state is an example of one which cannot be represented by means of the P representation defined in Ref. 5. It is a particular case of the example stated in Ref. 3, Eq. (23).

[12] See, e.g., CF, Eq. (3.1c) or the Appendix of Ref. 1.

and (2.9b) to expand b_l in terms of the sets of operators $\{a_k\}$ and $\{a_k^\dagger\}$, respectively, we see that the state $|\{\beta_l\}\rangle$ may be written as

$$|\{\beta_l\}\rangle=\prod_k \exp[\sum_l(\beta_l\gamma_{lk}a_k^\dagger-\beta_l^*\gamma_{lk}^*a_k)]|\text{vac}\rangle. \quad (4.11)$$

We have shown in Eq. (3.18) that such a state is an eigenstate of the $\{a_k\}$. It may be written alternatively in the form $|\{\alpha_k\}\rangle$, where the eigenvalues α_k are given by

$$\alpha_k=\sum_l\beta_l\gamma_{lk}. \quad (4.12)$$

To obtain further insight into the meaning of the inequality (4.1), let us consider a single mode for which the annihilation operator is a, and introduce a coordinate and a momentum operator for it by writing

$$q=(\hbar/2\omega)^{1/2}(a^\dagger+a), \quad (4.13a)$$

$$p=i(\hbar\omega/2)^{1/2}(a^\dagger-a). \quad (4.13b)$$

If we then let the constant α be

$$\alpha=\langle a\rangle=(2\hbar\omega)^{-1/2}\{\omega\langle q\rangle+i\langle p\rangle\}, \quad (4.14)$$

where the brackets $\langle\ \rangle$ stand for statistical averages, we find that the inequality (4.1) may be written in the form

$$(\omega/2\hbar)\langle(q-\langle q\rangle)^2\rangle+(2\hbar\omega)^{-1}\langle(p-\langle p\rangle)^2\rangle-\tfrac{1}{2}\geq 0. \quad (4.15)$$

The eigenstates of the annihilation operator are the states which reduce this particular linear combination of the variances of q and p to its minimum value. The linear combination is a familiar one. For the case $\langle q\rangle=\langle p\rangle=0$, the inequality asserts simply that the energy of any state of the oscillator exceeds that of its ground state, $\frac{1}{2}\hbar\omega$. It is clear, furthermore, that for $\langle q\rangle$ and $\langle p\rangle$ different from zero, the state which minimizes the sum, i.e., the eigenstate of the annihilation operator, may be found simply by displacing the ground state in coordinate and momentum space. The latter property of the eigenstates of a has already been noted in Ref. 5. The inequality (4.15) may indeed be shown to be a consequence of the uncertainty principle. If we apply first the arithmetic mean-geometric mean inequality and then the uncertainty principle we find

$$\tfrac{1}{2}\omega^2\langle(q-\langle q\rangle)^2\rangle+\tfrac{1}{2}\langle(p-\langle p\rangle)^2\rangle \geq\omega\{\langle(q-\langle q\rangle)^2\rangle\langle(p-\langle p\rangle)^2\rangle\}^{1/2}\geq\tfrac{1}{2}\hbar\omega.$$

It is interesting to apply the inequality we have been discussing to the set of states which correspond to fully coherent fields. If we let b be the annihilation operator defined by Eq. (2.14a) and let β be an arbitrary complex number, then the inequality becomes

$$\operatorname{tr}\{\rho(b^\dagger-\beta^*)(b-\beta)\}\geq 0. \quad (4.16)$$

If, in particular, we let β be the average value of b, $\beta=\operatorname{tr}\{\rho b\}=\langle b\rangle$, then Eq. (4.16) reduces to the Schwarz inequality

$$\langle b^\dagger b\rangle\geq|\langle b\rangle|^2. \quad (4.17)$$

As we have noted earlier, all states which possess full coherence have Poisson distributions for their occupation numbers; the distributions which correspond, for example, to the states specified by Eq. (3.16) are all identical and have the mean occupation number $\langle b^\dagger b\rangle=\langle m\rangle$. The mean value of the operator b is therefore constrained in such states by the inequality

$$\langle m\rangle\geq|\langle b\rangle|^2, \quad (4.18)$$

and the upper bound of $|\langle b\rangle|$ is only attained when the state reduces to an eigenstate of b.

Our discussion at the end of Sec. III of the physical nature of the states (3.16) indicates that the difference between the two members of the inequality (4.18) arises largely from the phase uncertainty of the operator b. The phase uncertainty of the field evidently tends to be minimized, within the class of fully coherent states, by the eigenstate of the annihilation operator.[13]

One of the interesting features of the eigenstates of the annihilation operators is evident in Eq. (3.18). When a field with only the b mode occupied is in an eigenstate of b, then its state vector, when written in the representation associated with the a_k modes, factorizes into a product of eigenstates of the $\{a_k\}$. The density operator for the field, in other words, takes the product form $\rho=\prod_k\rho_k$, and measurements made on the individual modes yield statistically independent results. We shall show that this condition of statistical independence is actually sufficient to single out the eigenstates of the annihilation operator from among all states of the field which possess coherence of any order. To do this let us suppose that the density operator ρ describes a first-order coherent field. Then, as we have noted earlier, only a single mode of the field is excited, and we label it the b_{l_0}-mode. We shall assume that none of the orthogonal set of modes we label with the amplitudes $\{a_k\}$ is identical to the b_{l_0}-mode. These statements imply, as we shall show, that if ρ can be expressed in the factorized form $\prod_k\rho_k$, it is a projection operator on an eigenstate of the annihilation operator b_{l_0}. The factorization, in other words, is a sufficient condition for the field to be a pure eigenstate as well as a necessary one.

To begin the proof of this theorem we note that the b_{l_0}-mode may be regarded as a member of an orthogonal set of modes for which the b_l are given as linear combinations of the $\{a_k\}$ by Eqs. (2.9a) and (2.9b). These relations are

$$b_l=\sum\gamma_{lk}^*a_k, \quad (4.19a)$$

$$b_l^\dagger=\sum\gamma_{lk}a_k^\dagger, \quad (4.19b)$$

where the coefficients γ_{lk} form a unitary matrix. The

[13] Calculations verifying that the eigenstates of the annihilation operators possess relatively well-defined phases have been carried out by P. Carruthers and M. M. Nieto [Phys. Rev. Letters **14**, 387 (1965)]. The phase uncertainties are small in the sense that they approach the lower bound set by the commutation relations for $\langle m\rangle\gg 1$, and are not far from this limit for smaller values of $\langle m\rangle$.

b_l-modes with $l\neq l_0$ are not excited. We therefore have

$$b_l^\dagger\rho=\rho b_l=0\,,\quad \text{for}\quad l\neq l_0\,, \tag{4.20}$$

a statement which implies that the variance of b_l also vanishes;

$$\operatorname{tr}\{\rho(b_l^\dagger-\langle b_l^\dagger\rangle)(b_l-\langle b_l\rangle)\}=0\,,\ \text{for}\ l\neq l_0. \tag{4.21}$$

This variance can also be written as

$$\operatorname{tr}\{\rho\sum_k\gamma_{lk}(a_k^\dagger-\alpha_k^*)\sum_{k'}\gamma_{lk'}^*(a_{k'}-\alpha_{k'})\}=0\,, \tag{4.22}$$

where $\alpha_k=\langle a_k\rangle$ is the expectation value of a_k.

If ρ is assumed to factorize into the form $\prod_k\rho_k$ it is clear that the terms with $k\neq k'$ vanish in Eq. (4.22). The vanishing of the variance (4.21) therefore implies the condition

$$\sum_k|\gamma_{lk}|^2\operatorname{tr}\{\rho_k(a_k^\dagger-\alpha_k^*)(a_k-\alpha_k)\}=0,\ \text{for}\ l\neq l_0. \tag{4.23}$$

Since, according to Eq. (4.1), all the terms of this sum are non-negative, we must have

$$\operatorname{tr}\{\rho_k(a_k^\dagger-\alpha_k^*)(a_k-\alpha_k)\}=0\,, \tag{4.24}$$

for all k, providing there exists at least one value of $l\neq l_0$ such that $\gamma_{lk}\neq 0$. If, on the other hand, no such value of l exists, i.e., we have $\gamma_{lk}=0$ for $l\neq l_0$, then the relation which is inverse to Eq. (4.19a),

$$a_k=\sum_l b_l\gamma_{lk}\,, \tag{4.25}$$

reduces to the form

$$a_k=b_{l_0}\gamma_{l_0k}\,, \tag{4.26}$$

and the coefficient γ_{l_0k} is simply a phase factor. This statement contradicts our explicit assumption that the excited mode is not one of the set associated with the amplitudes a_k.

The relation (4.24) has been shown earlier in this section to imply that ρ_k is the projection operator on an eigenstate of the annihilation operator a_k. The over-all density operator ρ is therefore the projection operator on a simultaneous eigenstate of all the a_k. This result completes the proof of our theorem.[14]

It is perhaps worth noting that the general class of fully coherent fields only possesses a factorization property much weaker in form than the one we have demonstrated for the eigenstates of the annihilation operator. The density operators for fully coherent fields are given by Eq. (2.15) with the diagonal coefficients B_{nn} specified by Eq. (3.12). The diagonal elements of the density operator in the n-quantum-state representation are just the probabilities for having a specified number of quanta in each field mode. It is easily shown, by calculating the diagonal element that the joint probability distribution for the set of occupation numbers $\{n_k\}$ is

$$\begin{aligned}p(\{n_k\})&=\langle\{n_k\}|\rho|\{n_k\}\rangle\\&=\prod_k(\langle m_k\rangle^{n_k}/n_k!)e^{-\langle m_k\rangle}\,,\end{aligned} \tag{4.27}$$

where

$$\langle m_k\rangle=|\gamma_k|^2\langle m\rangle\,, \tag{4.28}$$

and γ_k is given by Eq. (2.20). The joint probability distribution for the occupation numbers of the modes reduces to a product of independent Poisson distributions. The factorization property of the density operator, however, need not extend to its off-diagonal matrix elements. The various modes will not, in general, contribute in a statistically independent way to physical processes which depend on such matrix elements.

We conclude this section with another corollary of the theorem proved at the beginning of this section. We note that the positive frequency part of the field at $x=(\mathbf{r},t,\mu)$ is an operator of the form (2.14a) with $\gamma_k^*=i(\frac{1}{2}\hbar\omega_k)^{1/2}u_{k\mu}(\mathbf{r})e^{-i\omega_kt}$. Vanishing of the variance of $E^{(+)}(x)$ therefore implies vanishing of the variance of a_k for all k such that $u_{k\mu}(\mathbf{r})\neq 0$. If we require the variance of $E^{(+)}(x)$ to vanish for all x, then the variance of a_k vanishes for all k, and the field is in a pure eigenstate of the annihilation operators.

On the other hand, the variance of $E^{(+)}(x)$ can be written as

$$\begin{aligned}&\operatorname{tr}\{\rho E^{(-)}(x)E^{(+)}(x)\}-\langle E^{(+)}(x)\rangle^*\langle E^{(+)}(x)\rangle\\&\qquad=G^{(1)}(x,x)-\langle E^{(+)}(x)\rangle^*\langle E^{(+)}(x)\rangle.\end{aligned} \tag{4.29}$$

Here we have used the definition (1.1) of $G^{(1)}$, and have written $\langle E^{(+)}(x)\rangle$ for the expectation value of $E^{(+)}(x)$. If we compare the condition for the vanishing of the variance,

$$G^{(1)}(x,x)=\langle E^{(+)}(x)\rangle^*\langle E^{(+)}(x)\rangle\,, \tag{4.30}$$

with the condition (2.2) for first-order coherence, we see that the field $\mathcal{E}(x)$ in terms of which $G^{(1)}$ factorizes, must be equal to $\langle E^{(+)}(x)\rangle$, apart from a phase factor. It is obvious, furthermore, that Eq. (4.30) is obeyed by any eigenstate of the annihilation operators. Thus we may formulate the following theorem.

A field which obeys the first-order coherence condition (2.2) is in a pure eigenstate of the annihilation operators if and only if the function $\mathcal{E}(x)$ in this condition may be taken to be the expectation value of $E^{(+)}(x)$.

This result may be formulated somewhat differently if we introduce a larger set of correlation functions defined by

$$\begin{aligned}&G^{(n,m)}(x_1\cdots x_n,x_{n+1}\cdots x_{n+m})\\&\qquad=\operatorname{tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_n)\\&\qquad\qquad\times E^{(+)}(x_{n+1})\cdots E^{(+)}(x_{n+m})\}\,.\end{aligned} \tag{4.31}$$

[14] For the particular case in which the excited mode is a superposition of two other modes a related result has been noted by Y. Aharonov, D. Falkoff, E. Lerner, and H. Pendleton, Ann. Phys. (N. Y.) (to be published); a partial report is contained in the conference proceedings mentioned in Ref. 3. These authors give a physical interpretation of the result in terms of a comparison between quantum mechanical and classical features of measurement processes. The demonstration we have given provides a rigorous basis for their theorem in that it overcomes the reservations they point out at the end of Sec. II of their paper. Furthermore, it extends their theorem to deal with arbitrary numbers of superposed modes.

It is seen immediately that if the field is in an eigenstate of the annihilation operators we have

$$G^{(n,m)}(x_1\cdots x_{n+m})=\prod_{j=1}^{n}\mathcal{E}^*(x_j)\prod_{j=n+1}^{n+m}\mathcal{E}(x_j)\qquad(4.32)$$

for all n and m. The function $\mathcal{E}(x)$ may be identified as

$$\mathcal{E}(x)=G^{(0,1)}(x)=\langle E^{(+)}(x)\rangle.\qquad(4.33)$$

Factorization of the $G^{(n,m)}$ as indicated in Eq. (4.32) implies, in turn, that the field is in a pure eigenstate of the annihilation operators. This result is in fact implied, according to the theorem proved earlier, by just the two conditions (4.32) for (n,m) equal to (0,1) and (1,1).

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Decay Properties of the ω Meson*

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The reaction $K^-p\rightarrow\Lambda\omega$, as observed in the Lawrence Radiation Laboratory's 72-in. hydrogen bubble chamber, has provided over 4600 examples of ω decay. The distribution of pion momenta in the decay $\omega\rightarrow\pi^+\pi^-\pi^0$, of which 4200 examples have been seen, is found to be consistent with C conservation. With the assumption that the ω spin is 1^-, the pion-momentum distributions predicted by two different decay matrix elements are compared with the experimental distributions, as a test for final-state interactions. It is also shown that a spin of 3^- for the ω is unlikely. Branching fractions of other decay modes with respect to the $\pi^+\pi^-\pi^0$ decay mode are: neutrals, 0.097±0.016; $\eta\gamma$, <0.017; $\eta\pi^0$, <0.017; $\pi^+\pi^-\gamma$, <0.05; e^+e^-, <0.0003; and $\mu^+\mu^-$, <0.0017. The $\omega\rightarrow\pi^+\pi^-$ branching fraction lies between $(0.17\pm0.03)^2=0.029$ (coherence between the ρ and the ω production amplitudes assumed) and 0.082±0.020 (incoherence assumed).

I. INTRODUCTION

We present here some measurements of ω decay as seen in the reaction $K^-p\rightarrow\Lambda\omega$, with incident K^- momentum between 1.2 and 1.8 BeV/c. The decay modes studied provide information on C conservation, $SU(3)$ symmetry and ϕ-ω mixing, the electromagnetic structure of particles, I conservation, and the spin of the ω.

TABLE I. Decay modes of the ω meson.

Decay mode of ω	Rate[a] Other experiments[b]	This experiment
$\pi^+\pi^-\pi^0$	12 MeV	···
All neutrals	0.106±0.010	0.097±0.016
$\pi^0\gamma$	≈0.10	···
$\eta\gamma\rightarrow$ neutrals	···	<0.011
$\pi^+\pi^-$, complete incoherence assumed	<0.008[c]	0.082±0.020
complete coherence assumed		$(0.17\pm0.03)^2=0.029$
$\pi^+\pi^-\gamma$	···	<0.05
$\eta\gamma$	···	<0.017
$\eta\pi^0$	···	<0.017
e^+e^-	≈0.0001	<0.0003
$\mu^+\mu^-$	<0.001	<0.0017

[a] All fractions represent the absolute rate divided by the $\omega\rightarrow\pi^+\pi^-\pi^0$ rate.
[b] See Ref. 4.
[c] See Ref. 11.

The conservation of C forbids the decays $\omega\rightarrow\eta\pi^0$ and $\omega\rightarrow\rho\gamma$, and also forbids an asymmetry between the π^+ and π^- in the decay $\omega\rightarrow\pi^+\pi^-\pi^0$. Measurement of these properties therefore tests C conservation. According to $SU(3)$ symmetry, the ω is a vector meson that is a linear combination of ω_8, a member of an octet of vector mesons, and ω_1, an $SU(3)$ singlet. The decay mode $\omega\rightarrow\eta\gamma$ bears upon this presumption, called ϕ-ω mixing.

The $\pi^+\pi^-$, $\pi^+\pi^-\gamma$, $\eta\gamma$, e^+e^-, and $\mu^+\mu^-$ decay modes of the ω have an intimate connection with the electromagnetic structure of mesons. The two-pi decay mode is crucial to an understanding of the possible electromagnetic mixing between ρ and ω.

The distribution over the internal-momentum variables in $\omega\rightarrow\pi^+\pi^-\pi^0$ allows us, because of our large amount of data, to look for final-state interactions and to explore the possibility that the ω spin is $J^P=3^-$ rather than $J^P=1^-$.

Table I summarizes the decay properties of the ω.

II. EXPERIMENTAL IDENTIFICATION OF THE REACTION

Approximately 4600 events of the reaction

$$K^-p\rightarrow\Lambda\omega$$

have been identified in a K^- exposure of the 72-inch hydrogen bubble chamber. The momentum settings were 1.22, 1.32, 1.42, 1.51, 1.60, and 1.70 BeV/c. The

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