expect that

$$\lambda(v)=(1-v^2/c^2)^{1/2}(1+\delta)\,,$$

where, for not too high values of v, δ is of the order of magnitude

$$\delta\sim\alpha((nP)-Mc)/\hbar\,,$$

where M is the mass of the particle and P its energy momentum vector. This becomes in the lab frame when $n=(1,0)$

$$\delta\sim\alpha T/\hbar c\,,$$

where T is the kinetic energy. As an example let us put

$$T=400 \text{ MeV};$$

then

$$T/\hbar c=2\times10^{13}\text{ cm}^{-1}.$$

If the universal length is

$$\alpha=\hbar/mc=2\times10^{-14}\text{ cm}\,,$$

then one would have about 40% deviation from the usual formula. If on the other hand

$$\alpha=6\times10^{-17}\text{ cm}\,,$$

the characteristic length for weak interaction,[1] one would have

$$\delta\sim1.2\times10^{-3}\,,$$

which is just about the accuracy of the present day value of the muon lifetime.[4]

[4] See, e.g., G. Källén, *Elementary Particle Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1964), p. 4.

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Gravitational Collapse and Relativistic Magnetohydrodynamics*

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Einstein's field equations for a perfect fluid coupled to a frozen-in magnetic field are studied in the high-density limit of gravitational collapse. The assumption of infinite electrical conductivity is used to integrate Maxwell's equations and the fluid entropy conservation equation; and the integrals obtained show that there are certain general, physically reasonable conditions under which the electromagnetic energy density can become much larger than the fluid energy density as the collapse proceeds, even when the electromagnetic field was initially very weak. The widest possible range of cases is discussed under the assumption that the equation of state is asymptotically linear. Ways in which the hypotheses used might go wrong are mentioned.

I. INTRODUCTION

THEORETICAL studies of the gravitational-collapse problem have shown that it may be necessary to allow for the effects of magnetic fields in any real astrophysical case. In particular, the work of Ginzburg and his colleagues[1–3] has pointed to the fact that the behavior and appearance of the quasistellar radio sources may be profoundly affected by the presence of frozen-in magnetic fields.

In the present paper we shall use the general theory of relativity to analyze a collapsing system composed of a perfect fluid with asymptotically linear equation of state coupled to a frozen-in magnetic field, and we shall employ geometric considerations to see whether the magnetic field or the fluid becomes the dominant dynamic element of the system as the collapse proceeds.

This will be accomplished by following a given fluid element as it is crushed in the collapse and examining the local ratio of the fluid energy density ϵ to the electromagnetic-field energy density $F^2=F_{ab}F^{ab}$. We abbreviate $\epsilon/F^2=\beta$. We shall show that if the electrical conductivity is infinite, as is usually the case for stellar material, then in a frame comoving with the fluid, $-T_4{}^4=\text{total energy density}=\epsilon+\frac{1}{4}F^2$. Then if $\beta\to\infty$ as the collapse continues, one may neglect the effect of the magnetic field on the motion of the system; whereas if $\beta\to0$, then the effect of the fluid may be neglected.

The principle aim of this paper will be to show that there are a rather large number of physically reasonable cases where the magnetic field will finally become the dominant dynamic component of the system, no matter how weak it was initially. Thus for these cases any dynamical analysis of the collapse problem will be incomplete unless one allows for the effect of such a magnetic field. This is especially relevant for astro-

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[1] V. L. Ginzburg, L. M. Ozernoi, and S. I. Syrovatskii, Dokl. Akad. Nauk SSSR **154**, 557 (1964) [English transl.: Soviet Phys.—Doklady **9**, 3 (1964)]. Also in I. Robinson, A. Schild, and E. L. Schucking, *Quasi-Stellar Sources and Gravitational Collapse* (University of Chicago Press, Chicago, 1965), p. 277.

[2] V. L. Ginzburg, Dokl. Akad. Nauk SSSR **156**, 43 (1964) [English transl.: Soviet Phys.—Doklady **9**, 329 (1964)]. Also in Robinson, Schild, and Schucking, Ref. 1, p. 283.

[3] V. L. Ginzburg and L. M. Ozernoi, Zh. Eksperim. i Teor. Fiz. **47**, 1030 (1964) [English transl.: Soviet Phys.—JETP **20**, 689 (1964)].

physics since frozen-in magnetic fields are known to pervade great numbers of stellar and galactic objects.

We shall begin by presenting in Sec. II an outline of the theory of relativistic magnetohydrodynamics (MHD) with the assumption of infinite electrical conductivity σ. Most of our calculations will be done relative to a co-moving frame of reference, which always exists for a one-component fluid. By "co-moving frame" we do not mean locally Galilean, but rather a frame in which the contravariant spatial components of the fluid 4-velocity vanish identically over all of space-time. This implies no restrictions on the form of the metric tensor g_{ij}, but means simply that the network of spatial coordinates moves fixed in the fluid. It will then be shown that in the comoving frame the covariant electric field components F_{4j} vanish and that the magnetic field components F_{23}, F_{31}, and F_{12} do not depend upon the time.

It will also be shown in Sec. II that for infinite σ the proper entropy of the fluid is conserved locally, there being no Joule heating. This will also necessitate the assumption of zero thermal conductivity, and we shall assume that we are dealing with time intervals so short that the dynamical effects of heat conduction and thermal radiation may be neglected.[4]

In Sec. III we introduce the use of a well-known type of spatial metric which has invariant significance when restricted to comoving frames of reference. This spatial metric will then be used to simplify the law of the conservation of entropy and an expression for F^2.

In Sec. IV we apply the foregoing analysis to a rigorous discussion of the behavior of the ratio $\beta=\epsilon/F^2$. It will be shown that the cases $\beta\rightarrow 0,\ \infty$, respectively, do not depend on the precollapse value of β, but only on the limiting form of the equation of state and on the geometric type of collapse. The dependence will be derived without assuming any special symmetries, either spatial or temporal; e.g., we will not assume that the cross terms $g_{4\rho}$ vanish.

Section V will be devoted to a discussion of ways in which the electrical resistivity and radiation losses might become important in the high-density limit, and various conclusions and recommendations will be made.

II. RELATIVISTIC MAGNETOHYDRODYNAMICS

We now develop some results of relativistic MHD relevant to the collapse problem. We shall assume the validity of the nonvacuum Maxwell equations together with the relativistic form of Ohm's law. A local form of the law of conservation of magnetic flux through a closed contour moving with the fluid will be derived, and Einstein's field equations will be used to examine the conservation of fluid entropy.

[4] The assumption of infinite electrical conductivity has been investigated by Ginzburg (see Ref. 2) and found to be generally valid for the collapse problem. However, although it is usually thought that one may neglect the thermal conductivity, a detailed analysis of this point appears to be lacking.

Maxwell's equations for the electromagnetic field tensor $F^{ij}=-F^{ji}$ are, with a comma representing partial differentiation and a semicolon covariant differentiation,

$$F_{ij,k}+F_{ki,j}+F_{jk,i}=0\,, \tag{1}$$

and

$$F^{ij}{}_{;j}=J^i\,, \tag{2}$$

where J^i is the charge-current 4-vector.

Now let u^k be the 4-velocity of the fluid and σ its electrical conductivity. Then with $u^k u_k=-1$, Ohm's law may be written[5]

$$\sigma^{-1}(J^k+u^kJ^mu_m)=u_mF^{mk}.$$

If we now assume that σ is infinite, we conclude

$$u_mF^{mk}=0. \tag{3}$$

Let us express this result in a comoving frame of reference, in which $u^\rho=0$, $\rho=1,\,2,\,3$, everywhere (see Sec. I). From now on, we shall indicate nonscalar expressions referred to a comoving frame by a bar, so that in a co-moving frame we have $\bar{u}^k=\delta_4{}^k(-\bar{g}_{44})^{-1/2}$.

We thus obtain from Eq. (3) $\bar{u}^4\bar{F}_{4k}=0$, or

$$\bar{F}_{4\rho}=0,\ \rho=1,\,2,\,3. \tag{4}$$

We call these the covariant electrical-field components in the co-moving frame.

Next one may use Eqs. (1) and (4) to get

$$\bar{F}_{\mu\nu,4}+\bar{F}_{4\mu,\nu}+\bar{F}_{\nu4,\mu}=\bar{F}_{\mu\nu,4}=0.$$

Hence the magnetic field components $\bar{F}_{\mu\nu}$ are independent of the time $\bar{x}^4$, or

$$\bar{F}_{\mu\nu}=\bar{F}_{\mu\nu}(\bar{x}^\rho),\ \mu,\ \nu,\ \rho=1,\,2,\,3. \tag{5}$$

This result is invariant with respect to transformations which preserve the co-moving character of the frame. Note that now any "divergence-free" $\bar{F}_{\mu\nu}$ may be assigned as an initial condition, so that the electromagnetic field tensor may be completely eliminated as an unknown from the equations of motion.

Let us now consider the law of conservation of fluid entropy in this context. We write the field equations in gravitational units (velocity of light and Newtonian gravitational constant both unity) as

$$G_j{}^k=-8\pi T_j{}^k$$
$$=-8\pi[(p+\epsilon)u^ku_j+\delta_j{}^kp+F_{aj}F^{ak}-\tfrac{1}{4}\delta_j{}^kF^2], \tag{6}$$

where $G_j{}^k$ is the Einstein tensor, ϵ is the proper energy density per unit volume of the fluid, and p is the fluid pressure. Equation (6) implies the local conservation law $T_j{}^k{}_{;k}=0$, which gives, on contracting with u^j and using $u^ju_j=-1$, $u^ju_{j;k}=0$, and Eqs. (1) and (2),

$$u^jT_j{}^k{}_{;k}=-\epsilon_{,j}u^j-(p+\epsilon)u^k{}_{;k}+F_{aj}J^au^j=0.$$

[5] R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1962), p. 104.

By Eq. (3) the last term on the right vanishes, so that we finally obtain

$$\epsilon_{,j}u^j+(p+\epsilon)u^k{}_{;k}=0. \qquad (7)$$

For changes of state involving no heat conduction we may write the proper fluid entropy density $s=s(\epsilon,\bar{x}^\rho)$ as a function of the energy density and of the spatial coordinates $\bar{x}^\rho$ in a co-moving frame as[6]

$$s(\epsilon,\bar{x}^\rho)=s_0\exp\int_{\epsilon_0}^{\epsilon}[p(\epsilon,\bar{x}^\rho)+\epsilon]^{-1}d\epsilon, \qquad (8)$$

where $p=p(\epsilon,\bar{x}^\rho)$ is the local adiabatic equation of state and s_0 and ϵ_0 may depend on $\bar{x}^\rho$. This form has also been used to represent a baryon number density.[7] We write the divergence expression $(su^j)_{;j}$ in the co-moving frame as

$$(s\bar{u}^j)_{;j}=\bar{s}_{,j}\bar{u}^j+s\bar{u}^j{}_{;j}=\bar{s}_{,4}\bar{u}^4+s\bar{u}^j{}_{;j}.$$

From Eq. (8) we see that $\bar{s}_{,4}=\partial s/\partial\bar{x}^4=(\partial s/\partial\epsilon)\epsilon_{,4}$, and thus

$$(s\bar{u}^j)_{;j}=(p+\epsilon)^{-1}s[\bar{\epsilon}_{,4}\bar{u}^4+(p+\epsilon)\bar{u}^j{}_{;j}],$$
$$=(p+\epsilon)^{-1}s[\bar{\epsilon}_{,j}\bar{u}^j+(p+\epsilon)\bar{u}^j{}_{;j}].$$

This is a tensor expression and hence is true in any system of coordinates. Using Eq. (7), we obtain then

$$(su^j)_{;j}=(p+\epsilon)^{-1}s[\epsilon_{,j}u^j+(p+\epsilon)u^j{}_{;j}]=0. \qquad (9)$$

Thus the entropy density given by Eq. (8) satisfies a local conservation equation. This result may be expressed in a more useful form by going back to a co-moving system:

$$(su^j)_{;j}=(-g)^{-1/2}[(-g)^{1/2}su^j]_{,j}=0$$
$$=[(-\bar{g})^{1/2}s\bar{u}^j]_{,j}=\partial[(-\bar{g})^{1/2}s\bar{u}^4]/\partial\bar{x}^4.$$

One may then easily obtain, using $\bar{u}^4=(-\bar{g}_{44})^{-1/2}$ and then integrating and squaring,

$$s^2(\epsilon,\bar{x}^\rho)\bar{g}/\bar{g}_{44}=f(\bar{x}^\rho), \qquad (10)$$

where $f(\bar{x}^\rho)$ is an arbitrary function independent of $\bar{x}^4$.

This completes our survey of relativistic MHD with infinite electrical conductivity. We note that Eqs. (4) and (5) constitute a local form of the law of conservation of magnetic flux through a closed contour moving with the material, expressed in nonrelativistic form as $\Phi=\int\int_A\mathbf{B}\cdot d\mathbf{A}=\text{constant}$.[8] It is very easy to derive a similar scalar integral form which expresses the flux conservation law in general relativity, but we shall not bother to do that here.

It would be interesting to find exact solutions to Eq. (6) which correspond to the well-known Alfvén waves. There might also exist other kinds of MHD gravitational waves which could lead to new ways of testing general relativity.

III. THE SPATIAL METRIC AND THE ENERGY DENSITIES

We now introduce a spatial metric which will be very useful for the analysis of the collapse geometry. Again restricting ourselves to a frame co-moving with the fluid we define[9]

$$\gamma_{\alpha\beta}=\bar{g}_{\alpha\beta}-\bar{g}_{4\alpha}\bar{g}_{4\beta}(\bar{g}_{44})^{-1},\quad \alpha,\beta=1,2,3, \qquad (11)$$

and write for a spatial "length"

$$dl^2=\gamma_{\alpha\beta}d\bar{x}^\alpha d\bar{x}^\beta. \qquad (12)$$

This dl may be shown to represent physically the following type of distance[9]: Suppose that we are given two neighboring points fixed spatially in the co-moving frame, p_1 at $\bar{x}^\alpha$ and p_2 at $\bar{x}^\alpha+d\bar{x}^\alpha$, and suppose that at time $\bar{x}^4$ a photon is sent from p_1 to p_2 and then instantaneously reflected and sent back again to p_1, where it is received at time $\bar{x}^4+d\bar{x}^4$. At p_1 the lapse of proper time is $d\tau_1=(-\bar{g}_{44})^{1/2}d\bar{x}^4$. The distance between p_1 and p_2 is then defined as, with the "velocity of light" $c=1$, $dl=\frac{1}{2}cd\tau_1=\frac{1}{2}(-\bar{g}_{44})^{1/2}d\bar{x}^4$. One may then use the geodesic equation for the photon $ds^2=0$ to arrive at Eqs. (11) and (12).

It is easy to show that the form (12) is invariant with respect to transformations which preserve the co-moving character of the frame, given by $\bar{u}^\rho=0$. In fact, if we pass from $\bar{x}^k$ to $\bar{x}^{k'}$, both co-moving, we must have

$$\bar{u}^{\rho'}=(\partial\bar{x}^{\rho'}/\partial\bar{x}^n)\bar{u}^n=(\partial\bar{x}^{\rho'}/\partial\bar{x}^4)\bar{u}^4=0,\text{ or }\partial\bar{x}^{\rho'}/\partial\bar{x}^4=0.$$

Conversely, $\partial\bar{x}^\rho/\partial\bar{x}^{4'}=0$. Using these relations one may easily show the invariance of Eqs. (11) and (12). Thus it is natural to use this dl in co-moving systems.

Further, from Eq. (11) it follows that, with $\gamma=|\gamma_{\alpha\beta}|$,

$$\bar{g}^{\alpha\beta}\gamma_{\beta\sigma}=\delta_\sigma{}^\alpha,\quad |\bar{g}^{\alpha\beta}|\gamma=1. \qquad (13)$$

Since $\bar{g}_{44}=|\bar{g}^{\alpha\beta}|/|\bar{g}^{ki}|=\bar{g}|\bar{g}^{\alpha\beta}|$, we get $\bar{g}_{44}\gamma=\bar{g}$; thus we may write Eq. (10) as

$$s^2(\epsilon,\bar{x}^\rho)\gamma=f(\bar{x}^\rho). \qquad (14)$$

We define a collapse as a situation in which, at some fixed spatial point $\bar{x}_0{}^\rho$ in a co-moving frame, we have $\lim\epsilon(\bar{x}_0{}^\rho,\bar{x}^4)=\infty$ for some sequence of times $\bar{x}^4$. There may be situations in which ϵ remains finite and F^2 diverges, but it is not easy to see how they might occur.

It will now be demonstrated that ϵ diverges for fixed $\bar{x}_0{}^\rho$ if and only if γ goes to zero simultaneously: We must have[10] $0\le p\le\epsilon$, and hence Eq. (8) implies, if $\epsilon\ge\epsilon_0$,

$$s_0\exp\int_{\epsilon_0}^{\epsilon}\frac{d\epsilon}{2\epsilon}\le s\le s_0\exp\int_{\epsilon_0}^{\epsilon}\frac{d\epsilon}{\epsilon},$$

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[6] W. J. Cocke, Ann. Inst. Henri Poincare A2, 283 (1965).

[7] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, *Gravitation Theory and Gravitational Collapse* (University of Chicago Press, Chicago, 1965), p. 100.

[8] See for example H. Alfvén and C.-G. Fälthammar, *Cosmical Electrodynamics* (Oxford University Press, New York, 1963), pp. 101–102.

[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959), pp. 257–258.

[10] B. K. Harrison, K. S. Thorne, M. Wakano, and J. A. Wheeler, Ref. 7, pp. 105–107.

or $s_0(\epsilon/\epsilon_0)^{1/2}\leq s\leq s_0\epsilon/\epsilon_0$. Therefore, for fixed $\bar{x}_0{}^\rho$, $\epsilon\rightarrow\infty$ if and only if $s\rightarrow\infty$. Equation (14) then implies that $\epsilon\rightarrow\infty$ if and only if $\gamma\rightarrow 0$. This fact characterizes local fluid crushing in geometric terms.

We now examine the electromagnetic energy density in the co-moving frame: $-(\bar{T}_{em})_4{}^4=-\bar{F}_{4a}\bar{F}^{4a}+\frac{1}{4}F^2=\frac{1}{4}F^2$ by Eqs. (4) and (6). To simplify the discussion we will suppose that at $\bar{x}_0{}^\rho$ we have oriented our coordinate system so that $\bar{F}_{12}=\bar{F}_{31}=0$. The covariant magnetic field "vector" then points in the $\bar{x}^1$ direction. Since $\bar{F}_{\mu\nu}$ does not depend on $\bar{x}^4$, we may easily arrange this orientation at any fixed $\bar{x}_0{}^\rho$ for all $\bar{x}^4$ by a simple time-independent transformation.

Then $F^2=\bar{F}_{ab}\bar{F}^{ab}=2(\bar{F}_{23})^2[\bar{g}^{22}\bar{g}^{33}-(\bar{g}^{23})^2]$. But from Eqs. (13) one may get $\bar{g}^{22}\bar{g}^{33}-(\bar{g}^{23})^2=|\bar{g}^{\alpha\beta}|\gamma_{11}=\gamma_{11}/\gamma$. Then we have the important result, for the magnetic field pointing locally in the $\bar{x}^1$ direction,

$$F^2\gamma/\gamma_{11}=h(\bar{x}^\rho)\,, \tag{15}$$

where $h(\bar{x}^\rho)=2(\bar{F}_{23})^2$ is independent of $\bar{x}^4$.

IV. RELATIVE MAGNITUDE OF FLUID AND MAGNETIC FIELD

We now analyze the relative magnitude of the fluid energy density versus the magnetic field energy density. Equations (14) and (15) will allow us to do this by stipulating the various ways in which γ and γ_{11} go to zero.

It will first be necessary to consider asymptotic forms of the equation of state. We shall assume that as $\epsilon\rightarrow\infty$, the pressure behaves like $p\sim\alpha\epsilon$, where α is a constant and $0\leq\alpha\leq 1$. More complicated dependencies could be discussed, but there seems to be no point in doing so at this stage. Then if ϵ_0 is well into the region where $p\sim\alpha\epsilon$, Eq. (8) gives

$$s\sim s_0\exp\int_{\epsilon_0}^{\epsilon}\frac{d\epsilon}{\alpha\epsilon+\epsilon}=s_0\left(\frac{\epsilon}{\epsilon_0}\right)^{1/(\alpha+1)}, \tag{16}$$

and Eq. (14) becomes $\epsilon^{2/(\alpha+1)}\gamma\sim f(\bar{x}^\rho)$.

We now suppose that $\epsilon\rightarrow\infty$ and $\gamma\rightarrow 0$ at the fixed $\bar{x}_0{}^\rho$. Then there are three possibilities for γ_{11}: as in case I where γ_{11} remains nonzero (or even diverges); or as in case II where $\gamma_{11}\rightarrow 0$ but still $\gamma_{11}/\gamma\rightarrow\infty$; or as in case III where $\gamma_{11}\rightarrow 0$ and γ_{11}/γ remains finite (or vanishes). There are no other possibilities, and we now discuss them one by one.

Case I

In this case, where $\gamma\rightarrow 0$, but γ_{11} remains nonzero, or even diverges, the invariant length of the fluid element in the $\bar{x}^1$ direction does not go to zero, so that the crushing occurs in directions perpendicular to the field. Dividing Eq. (16) by Eq. (15) we get at the fixed $\bar{x}_0{}^\rho$ with $\delta=2(\alpha+1)^{-1}$,

$$\epsilon^\delta\gamma_{11}/F^2=(\epsilon/F^2)\epsilon^{(1-\alpha)/(\alpha+1)}\gamma_{11}\sim\text{const.}$$

Then if $\alpha<1$, we see that $\epsilon\rightarrow\infty$ implies $\beta=\epsilon/F^2\rightarrow 0$. Thus in this case F^2 becomes much larger than ϵ, no matter how small F^2 was initially, provided, of course, that it did not vanish altogether.

If $\alpha=1$, then $\beta\sim\gamma_{11}{}^{-1}$ and the answer depends in more detail on the behavior of γ_{11}. $\alpha=1$ is considered to be the "stiffest possible" equation of state.[10]

Case II

In this case, where γ_{11} and γ both go to zero, but still $\gamma_{11}/\gamma\rightarrow\infty$, the following must hold: Either there exists some number $n\geq 1$ such that $(\gamma_{11})^m/\gamma\rightarrow\infty$ for $m<n$ and $(\gamma_{11})^m/\gamma\rightarrow 0$ for $m>n$; or $(\gamma_{11})^m/\gamma\rightarrow\infty$ for all m. Here the fluid element is crushed in the magnetic field direction as well as in directions perpendicular to the field. The case $n=3$ includes the possibility that the element is crushed isotropically.

Now let $m>1$, and use Eqs. (15) and (16) at constant $\bar{x}_0{}^\rho$ to obtain, with $\delta=2(\alpha+1)^{-1}$,

$$\epsilon^\delta\gamma(\gamma_{11})^{m/(m-1)}(F^2\gamma)^{m/(1-m)}\sim\text{const},$$

or

$$\beta^{m/(m-1)}\epsilon^{\delta-[m/(m-1)]}\sim(\gamma_{11}{}^m/\gamma)^{1/(1-m)}. \tag{17}$$

We now break case II up into four separate categories. (A) $\gamma_{11}{}^m/\gamma\rightarrow\infty$ for all m. Then if $\alpha<1$, we can always pick m in Eq. (17) so large that $\delta>m/(m-1)$. Then

$$\beta^{m/(m-1)}\epsilon^{\delta-[m/(m-1)]}\sim(\gamma_{11}{}^m/\gamma)^{1/(1-m)}\rightarrow 0$$

implies that since $\epsilon\rightarrow\infty$, we must have $\beta\rightarrow 0$. Thus the magnetic field density F^2 again dominates. However, if $\alpha=1$, then we can reason as in case I to get $\beta\sim\gamma_{11}{}^{-1}\rightarrow\infty$, and hence the fluid energy density will dominate. (B) n exists, but $n>1$ and $\delta>n/(n-1)$. Then since $d[n/(n-1)]/dn=-(n-1)^{-2}<0$, there exists p such that $\delta>p/(p-1)$ and $1<p<n$. Then Eq. (17) implies $\beta^{p/(p-1)}\epsilon^{\delta-[p/(p-1)]}\sim(\gamma_{11}{}^p/\gamma)^{1/(1-p)}$. Since $1<p<n$, the right-hand side goes to zero by the principal hypothesis of case II. Then $\epsilon\rightarrow\infty$ implies $\beta\rightarrow 0$. Thus the magnetic field is again the predominant dynamic element. (C) n exists; and either $n>1$ with $n/(n-1)>\delta$, or $n=1$. Then there exists q such that $q>n$, $q>1$, and $\delta<q/(q-1)$. If $q=m$ is then used in Eq. (17), the right-hand side will diverge since $q>n$; and since the exponent of ϵ is now negative, we conclude that β diverges also. Thus ϵ becomes much greater than F^2, and the effect of the magnetic field may be neglected. (D) n exists, $n>1$, and $\delta=n/(n-1)$. We now see that by Eq. (17) $\beta^{n/(n-1)}\sim(\gamma_{11}{}^n/\gamma)^{1/(1-n)}$, and the behavior of β depends on the limit of $\gamma_{11}{}^n/\gamma$.

Case III

In this case, where γ_{11} and $\gamma\rightarrow 0$, but γ_{11}/γ remains finite, or even vanishes, Eq. (15) tells us the answer immediately: F^2 remains finite, or vanishes, and thus β definitely diverges. Here we seem to have in hand the possibility that the fluid is "running down the magnetic field lines," so that the field does not participate in the collapse.

Let us now briefly discuss some special situations which fall under case II. First, suppose that the collapse is in some sense isotropic, so that $n=3$. Then if $\alpha=\frac{1}{3}$, we have $\delta=n/(n-1)=\frac{3}{2}$, which comes under category (D). But if $0\leq\alpha<\frac{1}{3}$, then $\delta>n/(n-1)$, and category (B) applies, and the magnetic field dominates the scene; and if $\frac{1}{3}<\alpha\leq 1$, then $n=3$ gives $n/(n-1)=\frac{3}{2}>\delta$, and the field may be neglected. In general, the "stiffer" is the equation of state, the more likely is it that the fluid will dominate. The case $\alpha=\frac{1}{3}$ is of course the equation of state of an extremely relativistic system of charged particles plus radiation field.

Secondly, suppose $\alpha=\frac{1}{3}$, and let n vary: For $n<3$, we have $\delta=\frac{3}{2}<n/(n-1)$, and hence the field may be neglected; whereas if $n>3$, the fluid may be neglected.

For the limit $\alpha=1$ we have seen that in case II the fluid always dominates and that in case I the fluid never dominates absolutely, since then β might go to zero, but never diverges.

In this section we have covered all the various ways in which a fluid element can be crushed. However, there appears to be no way to tell which of the various cases or categories will occur for given initial conditions without actually integrating the equations of motion.

V. CONCLUSIONS

The analysis of the preceding section leads us to the conclusion that a small frozen-in magnetic field may under certain rather general conditions become the predominant dynamical element in an MHD system undergoing gravitational collapse. There are, however, certain limitations to this result, the main one being that in the limit $\epsilon\rightarrow\infty$ the electrical resistivity of the fluid may become important, since the particle collision rates might out-run the particle number densities.[11] This would happen more readily for a stiff equation of state; i.e., a material in which the interparticle forces are sharply repulsive at close range.

I. M. Khalatnikov[12] has also criticized the notion of infinite conductivity for highly compressed matter. However, his arguments seem to hold only for a fluid with a single-charge component; whereas, an electrically neutral astrophysical plasma must have both positive- and negative-charge components.

Another limitation, however, would originate in the violation of fluid-entropy conservation (nonadiabatic motion) from bremsstrahlung and synchrotron radiation. Also, as F^2 increases rapidly, the induced electric fields might cause considerable radiation loss because of particle betatron acceleration. These effects would presumably become larger for increasing F^2, and it would be very interesting to know more about them in detail.

Another difficulty might be that the presence of a strong magnetic field would produce anisotropic pressures and conductivities, so that, strictly speaking, pressure and conductivity tensors would be more appropriate than simple scalars. However, this would not be apt to change the general conclusions of this paper.

Thus it appears that magnetic fields may be important for studies of the collapse problem. However, a magnetic field could very likely not halt a gravitational collapse, since the field would contribute its own gravitational forces originating in its mass-energy and stress tensor.

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[11] In general, electrical conductivities are proportional to number density times single-particle collision time. See for example Alfvén and Fälthammar, Ref. 8, pp. 146–150.

[12] I. M. Khalatnikov, Zh. Eksperim. i Teor. Fiz. **48**, 261 (1965) [English transl.: Soviet Phys.—JETP **21**, 172 (1965)].