Electron Scattering from an Aligned Deformed Nucleus*

H. B. GREENSTEIN
Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California
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The problem of scattering high-energy electrons from a deformed nucleus which has been aligned in a magnetic field is considered from the point of view of the Schiff approximation for high-energy potential scattering. Integrals for the scattering amplitude are evaluated in the asymptotic limit of large β ($\beta \equiv QR$) assuming a model for the charge-density distribution which is equivalent to a uniformly charged ellipsoid of revolution. It is found that terms bilinear in the quadrupole strength must be retained. The formalism is extended to take account of nuclear spin-state transitions. Low-lying rotational levels are summed over, using the Bohr-Mottelson model in the strong-coupling limit. Application is made to the case of the holmium nucleus.

I. INTRODUCTION

THE problem of scattering high-energy electrons from a deformed nucleus aligned in a magnetic field is now within experimental bounds and offers a new method for probing this type of nuclear structure. Asymmetries in the cross section which are a reflection of nuclear asymmetries are expected to be enhanced according to the degree of nuclear alignment.

The theoretical description of such scattering has until now been based on either of two methods, the Born approximation or the distorted-wave approximation.[1] A complete phase-shift analysis is feasible only for spherically symmetric potentials. Another approximation for high-energy potential scattering was developed by Schiff[2] who suggested that it may have special value for nonspherical potentials. The work here is based upon this suggestion. There are two modifications. First, the integrals which appear are evaluated asymptotically. This approximation was first made by Tiemann[3] for the case of spherical potentials. Secondly the formalism is extended so as to account for spin-state transitions of the nucleus.

As a point of departure to a discussion of the Schiff approximation, and because it provides a natural comparison, the first Born approximation is discussed in detail in Sec. II. The nucleus is described by means of a density matrix. Nuclear spin-state transitions are described using the Bohr-Mottelson collective model in the strong-coupling limit.

In Sec. III the Schiff-Tiemann method is applied to a classical deformed-charge distribution. It is then shown how the effects of nuclear spin transitions can be handled.

Section IV is devoted to a summary and discussion of the general results. It also includes a special application to the case of the holmium nucleus.

A number of assumptions have been made throughout. Those which shall not receive further mention are: (1) only the Coulomb part of the interaction is important for the excitation of collective modes[4] (except in the backward direction); (2) the effect on the motion of the electrons of the magnetic field used to align the nucleus is negligible; (3) the electron mass is negligible at the energies considered here and will be set equal to zero; (4) electron polarizations are of no interest since only the scalar part of the interaction is treated; (5) the recoil of the nucleus can be ignored.

Some of the material of the following sections is discussed in a more expansive fashion in the thesis[5] upon which this paper is based.

II. FIRST BORN APPROXIMATION

A. Classical Scattering

Suppose we are scattering high-energy electrons from a nucleus and wish to consider the scattering amplitude in first Born approximation. For the moment we ignore nuclear spin and represent the scattering center by a classical charge distribution $\rho(\mathbf{x})$, in which case the amplitude for scattering may be written as

$$f_D(\mathbf{Q}) = (\bar{u}_f \gamma_0 u_i) f(\mathbf{Q}), \tag{2.1}$$

$$f(\mathbf{Q}) = 2me\rho(\mathbf{Q})/Q^2. \quad (e>0) \tag{2.2}$$

(Units: $\alpha = e^2 = 1/137$.) Here $\rho(\mathbf{Q})$ is the Fourier transform of the charge density $\rho(\mathbf{x})$,

$$\rho(\mathbf{Q}) = \int d^3x\, e^{i\mathbf{Q}\cdot\mathbf{x}} \rho(\mathbf{x}) \tag{2.3}$$

and $\mathbf{Q}$ is the momentum transfer

$$\mathbf{Q} = \mathbf{k}_i - \mathbf{k}_f.$$

For the Dirac scattering amplitude $f_D(\mathbf{Q})$ we have factored out the part involving the Dirac spinors u_i and u_f (in order to facilitate comparison with the Schiff-Tiemann approximation, to be described later).

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[1] A good summary of approximation schemes pertinent to the problem of Coulomb scattering may be found in D. R. Yennie, F. L. Boos, and D. G. Ravenhall, Phys. Rev. **137**, B882 (1965). A brief discussion of some of these is given in Sec. IV.

[2] L. I. Schiff, Phys. Rev. **103**, 443 (1956).

[3] J. J. Tiemann, Phys. Rev. **109**, 183 (1958).

[4] L. I. Schiff, Phys. Rev. **96**, 765 (1954).

[5] H. B. Greenstein, Ph.D. thesis, Stanford University, 1965 (unpublished).

The amplitude $f(\mathbf{Q})$ would be appropriate for electrons satisfying a Schrödinger equation. For the cross section we have

$$d\sigma/d\Omega=|\bar{u}_f\gamma_0 u_i|^2|f(\mathbf{Q})|^2 \tag{2.4}$$

(normalization: $\bar{u}u=1$). If we average over initial and sum over final electron spins, we obtain in the limit of relativistic electrons ($v\simeq c$),

$$\langle\sum_{s_f}|\bar{u}_f\gamma_0 u_i|^2\rangle_{s_i}=(E^2/m^2)\cos^2(\tfrac{1}{2}\Theta), \tag{2.5}$$

where Θ is the angle of scattering. From this it follows that

$$d\sigma/d\Omega=(E^2/m^2)\cos^2(\tfrac{1}{2}\Theta)|f(\mathbf{Q})|^2. \tag{2.6}$$

A description in terms of multipole distributions may be obtained through the expansions[6]

$$\begin{aligned}\rho(\mathbf{Q})&=\textstyle\sum_l \rho_l(Q)Y_{l,0}{}^*(\Omega_Q),\\ \rho(\mathbf{x})&=\textstyle\sum_l \rho_l(r)P_l(\cos\theta).\end{aligned} \tag{2.7}$$

Equation (2.3) then implies that

$$\rho_l(Q)=4\pi i^l\int r^2dr\; j_l(Qr)\rho_l(r)\left[\frac{4\pi}{2l+1}\right]^{1/2}. \tag{2.8}$$

If we regard as the input the radial density functions $\rho_l(r)$, then use of the above equations allows us to find the scattering amplitude (2.2). Assuming only monopole and quadrupole distributions are present, the cross section then takes the form ($\mu_Q\equiv\cos\theta_Q$)

$$\begin{aligned}\frac{d\sigma}{d\Omega}=\frac{e^2}{\pi}\frac{E^2\cos^2(\Theta/2)}{Q^4}\Big\{&|\rho_0(Q)|^2\\ &+2(5)^{1/2}\rho_0(Q)\rho_2(Q)P_2(\mu_Q)+5|\rho_2(Q)|^2[P_2(\mu_Q)]^2\Big\}.\end{aligned} \tag{2.9}$$

The functions $\rho_l(Q)$ are essentially form factors for the nucleus at rest. In the limit of $Q\to 0$ they are related to the static moments, which for $l=0$ and $l=2$ are the total charge Ze and the quadrupole moment Q_0. The relations are

$$\begin{aligned}\lim_{Q\to 0}\rho_0(Q)&=(4\pi)^{1/2}Ze,\\ \lim_{Q\to 0}\rho_2(Q)&=-(Q_0/6)(\tfrac{4}{5}\pi)^{1/2}Q^2.\end{aligned} \tag{2.10}$$

It is of interest to consider the following two models:

$$\begin{aligned}(i):\quad &\rho_0(r)=\rho_0,\quad r\le R\quad(\rho_0=3Ze/4\pi R^3)\\ &\quad=0,\quad r>R\\ &\rho_2(r)=q_2\delta(r-R)/r^2.\end{aligned} \tag{2.11}$$

[6] These expansions assume that the charge distribution possesses a symmetry axis. If in addition there is invariance under the transformation $\theta\to\pi-\theta$, then only even values of l will enter.

Here $\rho_0(Q)$ and $\rho_2(Q)$ are easily computed via Eq. (2.8). In terms of the quadrupole moment Q_0,

$$Q_0=(8\pi/5)q_2R^2, \tag{2.12}$$

we find that

$$\begin{aligned}\rho_0(Q)&=(4\pi)^{1/2}3Zej_1(\beta)/\beta,\\ \rho_2(Q)&=-(5\pi)^{1/2}(Q_0/R^2)j_2(\beta),\end{aligned} \tag{2.13}$$

where

$$\beta\equiv QR. \tag{2.14}$$

(*ii*): a uniformly charged ellipsoid of revolution. Assume $\mathbf{Q}$ makes an angle θ_Q with the major axis. By direct calculation of $\rho(\mathbf{Q})$, using Eq. (2.3), we find that

$$\rho(\mathbf{Q})=3Zej_1(x)/x, \tag{2.15}$$

where

$$x^2=\beta^2[1+\epsilon^2P_2(\mu_Q)], \tag{2.16}$$

$$\beta=Q\bar{R}, \tag{2.17}$$

$$\epsilon^2=\tfrac{2}{3}((a^2-b^2)/(\bar{R})^2), \tag{2.18}$$

$$(\bar{R})^2=\tfrac{1}{3}(a^2+2b^2). \tag{2.19}$$

Assuming $\epsilon^2\ll 1$, we can expand this expression to first order in ϵ^2, whereupon we find that $\rho(\mathbf{Q})$ assumes the form (2.7) restricted to $l=0$ and $l=2$. The functions $\rho_0(Q)$ and $\rho_2(Q)$ are again as given in Eqs. (2.13) except that the radius R must now be replaced by the average radius $\bar{R}$. The eccentricity ϵ^2 is related to the quadrupole moment Q_0 by

$$Q_0=\tfrac{2}{5}Ze\epsilon^2(\bar{R})^2. \tag{2.20}$$

We conclude that to first order in ϵ^2 the quadrupole behavior of an ellipsoid of revolution is equivalent to that of a surface delta function, provided the surface is taken at $r=\bar{R}$.

B. Scattering from a Nucleus with Spin

1. Amplitude

If we wish now to account for the nuclear spin, we must replace the classical quantity $\rho(\mathbf{Q})$ by the matrix element of a nuclear charge-density operator taken between the initial and final nuclear spin states. Denoting this operator by $\hat{J}_0(\mathbf{Q})$, we have then that

$$\begin{aligned}\rho(\mathbf{Q})&\to\langle j'\mu'|\hat{J}_0(\mathbf{Q})|j\mu\rangle,\\ f(\mathbf{Q})&\to(2me/Q^2)\langle j'\mu'|\hat{J}_0(\mathbf{Q})|j\mu\rangle.\end{aligned} \tag{2.21}$$

We have designated only two nuclear quantum numbers, the total angular momentum j and its component μ along a fixed z axis. As before, we wish to describe the nuclear charge density in terms of multipole distributions. To do this we make a multipole expansion of the operator $\hat{J}_0(\mathbf{Q})$:

$$\hat{J}_0(\mathbf{Q})=\sum_{l,m}T^l{}_m(Q)Y_{l,m}{}^*(\Omega_Q). \tag{2.22}$$

The operator $T^l{}_m(Q)$ is an irreducible tensor operator of

rank l (component m) for the nucleus. The amplitude (2.21) may now be written

$$f(\mathbf{Q})=(2me/Q^2)\sum_{l,m}\langle j'\mu'|T^l{}_m(Q)|j\mu\rangle Y_{l,m}{}^*(\Omega_Q). \quad (2.23)$$

To evaluate the nuclear matrix elements we apply the Wigner-Eckart theorem,[7]

$$\langle j'\mu'|T^l{}_m|j\mu\rangle=(-)^{j'-\mu'}\begin{pmatrix} j' & l & j \\ -\mu' & m & \mu\end{pmatrix}\langle j'\|T^l\|j\rangle \quad (2.24)$$

and consider the reduced matrix elements $\langle j'\|T^l\|j\rangle$; it is necessary to consider separately $j'=j$ and $j'\neq j$. In the former case we observe that the amplitude (2.23) for the transition $|jj\rangle\rightarrow|jj\rangle$ has the form

$$f^j(\mathbf{Q})=(2me/Q^2)\sum_l \rho_l{}^j(Q)Y_{l,0}{}^*(\Omega_Q), \quad (2.25)$$

where we have used the definition

$$\rho_l{}^j(Q)\equiv\langle jj|T^l{}_0(Q)|jj\rangle. \quad (2.26)$$

Comparison with Eqs. (2.2) and (2.7) shows that the amplitude $f^j(\mathbf{Q})$ differs from the classical amplitude only in that the classical density $\rho_l(Q)$ has been replaced by the matrix element (2.26). Hence all the equations of part A can be considered appropriate here provided we make the substitution

$$\rho_l(r)\rightarrow\rho_l{}^j(r),$$

where $\rho_l{}^j(r)$ is interpreted to mean the radial charge density for the nucleus in the state $|jj\rangle$. These functions are now to be regarded as the input, and the densities $\rho_l{}^j(Q)$ are to be computed according to Eq. (2.8). Then from Eq. (2.26) we have that

$$\langle j\|T^l(Q)\|j\rangle=\begin{pmatrix} j & l & j \\ -j & 0 & j\end{pmatrix}^{-1}\rho_l{}^j(Q). \quad (2.27)$$

Evaluation of $\langle j'\|T^l\|j\rangle$ for $j'\neq j$ is more conveniently deferred to the next section [see Eq. (2.34)].

2. Cross Section

We shall now derive an expression for the cross section for scattering from a nucleus with spin starting from the amplitude (2.21). We shall proceed as follows:

(1). For the charge-density operator $\hat{J}_0(\mathbf{Q})$ we shall restrict the multipole expansion (2.8) to $l=0$ and $l=2$; i.e., we consider monopole and quadrupole charge densities only.

(2). The orientation of the initial nucleus will be described in terms of statistical weights p_μ giving the occupational probability for the state $|j\mu\rangle$.

(3). For final states $|j'\mu'\rangle$, all orientations μ' will be summed over.

(4). Transitions to states of $j'=j+1$ and $j'=j+2$ (which are the only two states available by quadrupole excitation) will be treated using the Bohr-Mottelson model to describe the permanently deformed nucleus.

For purposes of actually performing the calculation we shall use the density-matrix formalism to describe the initial nucleus. The density matrix[8] may be defined according to

$$\rho^i{}_{\mu\nu}=\langle\langle j\mu|i\rangle\langle i|j\nu\rangle\rangle_i,$$

where $\langle\ \rangle_i$ indicates an ensemble average over initial states $|i\rangle$. If the symmetry axis of the nuclear charge distribution (we always assume one to exist) is chosen as the quantization axis, the density matrix will then be diagonal and will have as diagonal matrix elements the statistical weights p_μ. Calculation of the cross section involves the quantity

$$\langle\textstyle\sum_f|f(\mathbf{Q})|^2\rangle_i=(2me/Q^2)^2\times\mathrm{Tr}\{\rho^i\hat{J}_0{}^\dagger(\mathbf{Q})\hat{J}_0(\mathbf{Q})\}, \quad (2.28)$$

where the trace is understood to be carried out in a matrix representation corresponding to the initial spin j, i.e., a $(2j+1)$ by $(2j+1)$ representation. In this expression we have used Eq. (2.21) for the amplitude $f(\mathbf{Q})$. The role of the final states may be seen explicitly by inserting a complete set of states between the operators $\hat{J}_0{}^\dagger(\mathbf{Q})$ and $\hat{J}_0(\mathbf{Q})$.

The trace (2.28) may be computed in a straightforward way if the operators appearing there are expressed in a spherical basis. For the operators $\hat{J}_0(\mathbf{Q})$ and $\hat{J}_0{}^\dagger(\mathbf{Q})$ this is accomplished directly through the expansion (2.22) in which the nuclear multipole operators $T^l{}_m(Q)$ are irreducible tensor operators of rank l (component m). The density matrix ρ^i may be written in an operator representation according to[8]

$$\rho^i=\sum_{k=0}^{2j}\sum_{q=-k}^{k}a_{k,q}J^k{}_q, \quad (2.29)$$

where the $J^k{}_q$ are a linearly independent set of $(2j+1)^2$ irreducible tensor operators of rank k and are built up[9]

[7] The notation and conventions for all angular momentum coupling coefficients are those of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[8] For a general discussion of the density matrix see U. Fano, Rev. Mod. Phys. **29**, 74 (1957). Some properties relevant to the spherical basis representation are given in Appendix A of Ref. 5.

[9] This may be accomplished in the following way. For $k=0$, we have the identity operator:

$$J^0{}_0\equiv I.$$

For $k=1$, the description in terms of a spherical basis is already familiar:

$$J^1{}_{\pm1}=\mp J_\pm/\sqrt{2},\quad J_\pm=J_x\pm iJ_y,$$
$$J^1{}_0=J_z.$$

We use the Clebsch-Gordan series [the inverse of (2.31)] to define

$$J^2{}_q\equiv[J^1\times J^1]^2{}_q=\textstyle\sum_{m,m'}\langle 1m1m'|112q\rangle J^1{}_m J^1{}_{m'}.$$

Similarly we define

$$J^3{}_q\equiv[J^1\times J^2]^3{}_q,$$
$$J^4{}_q\equiv[J^2\times J^2]^4{}_q.$$

It will be seen later that $J^k{}_q$ for $k>4$ will not be needed.

from the angular momentum operators J_x, J_y, J_z. For a diagonal density matrix the parameters $a_{k,q}$ are related to the statistical weight p_μ according to the equations

$$a_{k,q}=\frac{(2k+1)\delta_{q,0}}{\langle j\|J^k\|j\rangle}\sum_\mu p_\mu(-)^{j-\mu}\begin{pmatrix} j & k & j\\ -\mu & 0 & \mu\end{pmatrix},$$
$$p_\mu=\sum_{k=0}^{2j} a_{k,0}(-)^{j-\mu}\begin{pmatrix} j & k & j\\ -\mu & 0 & \mu\end{pmatrix}\langle j\|J^k\|j\rangle. \tag{2.30}$$

Use of these expansions allows the quantity $\rho^i J_0^\dagger \hat{J}_0$ to be expanded into a sum of products of irreducible tensor operators. Each such product is reducible but may be expanded into its irreducible components by means of the Clebsch-Gordan series:

$$T^{l_1}{}_{m_1}T^{l_2}{}_{m_2}=\sum_{l,m}\langle l_1m_1l_2m_2|l_1l_2lm\rangle[T^{l_1}\times T^{l_2}]^l{}_m. \tag{2.31}$$

To take the trace we merely apply the rule that all irreducible tensor operators are traceless unless they are of rank zero:

$$\mathrm{Tr}\{S_m{}^l\}=\delta_{l,0}\delta_{m,0}(2j+1)^{1/2}\langle j\|S^0\|j\rangle. \tag{2.32}$$

The result may be expressed in terms of individual reduced matrix elements by use of the rule

$$\langle j'\|[S^{k_1}\times S^{k_2}]^k\|j\rangle=\sum_{j''}(2k+1)^{1/2}\begin{Bmatrix} k_1 & k_2 & k\\ j & j' & j''\end{Bmatrix}$$
$$\times(-)^{k+j+j'}\langle j'\|S^{k_1}\|j''\rangle\langle j''\|S^{k_2}\|j\rangle. \tag{2.33}$$

Transitions to final states having $j'\neq j$ will be reflected in the presence of factors $\langle j'\|T^l(Q)\|j\rangle$. These may be evaluated by reference to the theory[10] which treats the collective modes of oscillation of a nucleus having both a permanently deformed core and a structure due to single-particle motion. Such nuclei are characterized by energy-level schemes resembling rotational bands. Within each band the levels j, $j+1$, $j+2,\cdots$, etc. are separated typically by 50–100 keV. Since in most experimental situations it would be difficult to resolve these levels, it is desirable to consider their effect on the cross section. In the Bohr-Mottelson theory the rotational motion is described by means of a $\mathfrak{D}$ function and is treated in the adiabatic approximation. This assumption is sufficient to yield the following relationship which holds for states having angular momentum J and belonging to the ground-state band:

$$\langle J\|T^2(Q)\|j\rangle=f(J,j)\langle j\|T^2(Q)\|j\rangle, \tag{2.34}$$

where

$$f(J,j)=(-)^{J-j}\left(\frac{2J+1}{2j+1}\right)^{1/2}$$
$$\times\begin{pmatrix} J & 2 & j\\ -j & 0 & j\end{pmatrix}\Big/\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix} \tag{2.35}$$

[10] J. D. Walecka (unpublished); A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 27, 16 (1953).

TABLE I. $|f(J,j)|^2$ for $J=j$, $j+1$, $j+2$.

J	$\lvert f(J,j)\rvert^2$
j	1
$j+1$	$\dfrac{6(2j+3)}{(2j+4)(2j-1)}$
$j+2$	$\dfrac{12(2j+2)}{(2j+4)(2j)(2j-1)}$

and j is the ground-state spin. The function $|f(J,j)|^2$ may be tabulated for $J=j$, $j+1$, $j+2$ (see Table I). It satisfies the following two sum rules[11]:

$$\text{(i)}\quad \sum_J|f(J,j)|^2=\left[(2j+1)\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2\right]^{-1}$$
$$=\frac{(2j+3)(2j+2)}{(2j)(2j-1)},$$
$$\text{(ii)}\quad \sum_J\begin{Bmatrix} 2 & 2 & k\\ j & j & J\end{Bmatrix}(-)^{J+j}|f(J,j)|^2 \tag{2.36}$$
$$=\frac{1}{(2j+1)}\frac{\begin{pmatrix} 2 & 2 & k\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} j & k & j\\ -j & 0 & j\end{pmatrix}}{\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2}.$$

We note that

$$\sum_{\text{all } J}=\sum_{J=j,j+1,j+2}$$

since $|f(J,j)|^2$ is zero for any other value of J. In the expression for the cross section the sums over the rotational states J can all be expressed in terms of the function $\Sigma_n(j,k)$:

$$\Sigma_n(j,k)\equiv\sum_{J=j}^{J=j+n}\begin{Bmatrix} 2 & 2 & k\\ j & j & J\end{Bmatrix}(-)^{J+j}|f(J,j)|^2. \tag{2.37}$$

For elastic scattering we put $n=0$. Then

$$\Sigma_0(j,k)=(-)^{2j}\begin{Bmatrix} 2 & 2 & k\\ j & j & j\end{Bmatrix}.$$

To include the effects of transitions to the two states which are available by quadrupole excitation we set $n=2$. Σ_2 can be evaluated by means of the second of two sum rules given above in (2.36).

[11] Proof of these sum rules rests on Eqs. (3.7.7) and (6.2.6) of Edmonds (Ref. 7).

The final result for the scattering cross section may be written

$$\frac{d\sigma}{d\Omega}=\frac{4E^2\cos^2(\frac{1}{2}\Theta)}{Q^4}\Big\{F_0+\Big[A_2\Big/5\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}\Big]\times F_Q P_2(\mu_Q)+\sum_{k=0} A_k\Big[\begin{pmatrix}2 & 2 & k\\ 0 & 0 & 0\end{pmatrix}\Big/\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2\Big]\times\Sigma_n(j,k)F_{QQ}P_k(\mu_Q)\Big\}, \quad (2.38)$$

where

$$F_0=(e^2/4\pi)\,|\rho_0{}^j(Q)|^2,$$
$$F_Q=2(5)^{1/2}(e^2/4\pi)[\rho_0{}^j(Q)\rho_2{}^j(Q)], \quad (2.39)$$
$$F_{QQ}=5(e^2/4\pi)\,|\rho_2{}^j(Q)|^2,$$

and

$$A_k\equiv a_{k,0}\langle j\|J^k\|j\rangle=(2k+1)\textstyle\sum_\mu p_\mu(-)^{j-\mu}\begin{pmatrix} j & k & j\\ -\mu & 0 & \mu\end{pmatrix}. \quad (2.40)$$

The effect of electron spin has been included by means of Eq. (2.1); spin summations were carried out as in Eq. (2.5).

In the following section these calculations will be repeated, but with the Schiff-Tiemann approximation for the amplitude replacing the Born approximation (2.21). The density-matrix formalism, the trace technique, and the method of summation over low-lying rotational levels will be applied in the same manner as they were here.

III. SCHIFF-TIEMANN APPROXIMATION

Permanent deformations are characteristically associated with high-Z nuclei for which the first Born approximation is not generally reliable. A more sophisticated approximation scheme, valid for large Z, has been developed by Schiff[2] in which the infinite Born series is summed after each term is approximated by the method of stationary phase. An asymptotic expansion for the Schiff amplitude, useful for $QR\gg1$, was developed by Tiemann.[3] We shall apply these approximation methods to the scattering of high-energy electrons from an aligned, deformed nucleus. As in the treatment of the first Born approximation, we ignore for the time being the effects of nuclear spin and first compute the scattering from a classical charge distribution. Then we shall extend the formalism to cover scattering from a nucleus with spin.

A. Classical Scattering

Expressed as an integral over the charge density, the Schiff approximation to the scattering amplitude is

$$f_D(\mathbf{k}_f,\mathbf{k}_i)=(\bar{u}_f\gamma_0 u_i)f(\mathbf{k}_f,\mathbf{k}_i), \quad (3.1)$$

where $f(\mathbf{k}_f,\mathbf{k}_i)$, computed on the basis of the Schrödinger equation, is

$$f(\mathbf{k}_f,\mathbf{k}_i)=(2me/Q^2)\int e^{i\mathbf{Q}\cdot\mathbf{x}}e^{iL(\mathbf{x})}\times[1-2\mathbf{Q}\cdot\nabla L(\mathbf{x})/Q^2]\rho(\mathbf{x})d^3x, \quad (3.2)$$

and

$$L(\mathbf{x})=-\int_0^\infty[V(\mathbf{x}-\hat{\mathbf{k}}_i s)+V(\mathbf{x}+\hat{\mathbf{k}}_f s)]\,ds. \quad (3.3)$$

(We assume throughout that the electrons are fully relativistic, i.e., that $v=c=1$.) This approximation is expected to be valid when $kR\gg1$ ($k=|\mathbf{k}_i|=|\mathbf{k}_f|$) and for scattering angles Θ which are large compared with $(kR)^{-1/2}$. However, there are no restrictions on the range of Ze^2, which in the Born approximation was required to be much less than one.

Because of the presence of the factors containing $L(\mathbf{x})$ [the Born approximation obtains in the limit of $L(\mathbf{x})\to0$] it is no longer possible to give simple relations between the radial charge densities and the amplitudes as was done for the Born approximation in Eq. (2.8). For this reason we shall discuss the Schiff approximation in terms of a model, i.e., a particular choice for $\rho_0(r)$ and $\rho_2(r)$. As before, we regard these two functions as the input. The model we choose is that of Eqs. (2.11). In Sec. II we discussed this model and showed that it was equivalent to the uniformly charged ellipsoid of revolution. Hence we expect that it provides a realistic description of a deformed charge distribution except for the neglect of surface-thickness effects. This point will receive further attention in Sec. IV.

The expansion (2.7), with $l=0$ and 2 only, will be assumed for the potential-energy function $V(\mathbf{x})$ and for the phase function $L(\mathbf{x})$, so that

$$L(\mathbf{x})=L_0(\mathbf{x})+L_2(\mathbf{x}), \quad (3.4)$$

where $L_l(\mathbf{x})$ is obtained by insertion of $V_l(\mathbf{x})$ into (3.3), with

$$V_l(\mathbf{x})=-e\int\rho_l(\mathbf{x}')d^3x'/|\mathbf{x}-\mathbf{x}'|.$$

The calculation of $L_0(\mathbf{x})$ and $L_2(\mathbf{x})$ for the model (2.11) is described in Appendix A. Obviously $L_0\propto Ze^2$, and $L_2\propto(q_2e)$. Since q_2e, the dimensionless quadrupole strength, will be small (for holmium, which has a large deformation, a value[12] of $Q_0=2\cdot10^{-24}e$ cm^2 implies that $eq_2=0.92\alpha$, $\alpha=1/137$), it is feasible to evaluate the amplitude (3.2) in a power-series expansion in (q_2e). At the same time Tiemann's asymptotic expansion in powers of $1/\beta(\beta\equiv QR)$ will be applied. For convenience in classifying the types of behavior to be found, we shall

[12] K. Alder, A. Bohr, T. Huus, B. Mottelson, and A. Winther, Rev. Mod. Phys. 28, 432 (1956); see Table V.3, p. 531.

write the amplitude (3.2) in the following way:

$$f(\mathbf{k}_f,\mathbf{k}_i)=f_0(\mathbf{k}_f,\mathbf{k}_i)+f_2(\mathbf{k}_f,\mathbf{k}_i), \tag{3.5}$$

where

$$f_l(\mathbf{k}_f,\mathbf{k}_i)=(2me/Q^2)\int\rho_l(\mathbf{x})E(\mathbf{k}_f,\mathbf{k}_i,\mathbf{x})d^3x \tag{3.6}$$

and

$$E(\mathbf{k}_f,\mathbf{k}_i,\mathbf{x})=e^{i\mathbf{Q}\cdot\mathbf{x}}e^{iL(\mathbf{x})}[1-2\mathbf{Q}\cdot\nabla L(\mathbf{x})/Q^2]. \tag{3.7}$$

It is found (see Appendix B for details) that f_0 and f_2 have the following asymptotic behavior (Ze^2 is here assumed to be of order 1):

f_0: $1/\beta^2$, $1/\beta^3,\cdots;(q_2e)/\beta^2$
$(q_2e)/\beta^3,\cdots;(q_2e)^2/\beta^2$, $(q_2e)^2/\beta^3,\cdots$; etc.

f_2: $(q_2e)/\beta$, $(q_2e)/\beta^2,\cdots;(q_2e)^2/\beta^2$,
$(q_2e)^2/\beta^3,\cdots$; etc.

We note that whereas the leading monopole term goes as $1/\beta^2$, the leading term linear in quadrupole strength goes as $1/\beta$. Since β is assumed large (for $A=165$, $Q=300$ MeV, we would have $\beta=9$), these terms can be of the same order of magnitude. Hence, calculations based on the model (2.11) predict a quadrupole enhancement whereby quadrupole scattering can be as important as monopole. In Sec. IV this point is given further consideration; it is argued that the introduction of a nonzero surface thickness would probably not alter this behavior.

For the calculation of the cross section, which is proportional to $|f_0+f_2|^2$, we shall require that all neglected terms are smaller than the retained ones at least by a factor of $1/\beta$ or (q_2e). In addition we shall include the first $1/\beta$ correction in the Tiemann asymptotic expansion. Therefore we keep the following terms:

$$1/\beta^4,\quad 1/\beta^5;(q_2e)/\beta^3,\quad (q_2e)/\beta^4;(q_2e)^2/\beta^2,\quad (q_2e)^2/\beta^3.$$

The terms in the amplitude which contribute to the above terms in the cross section are:

$$f_0:\ 1/\beta^2,\quad 1/\beta^3,\quad (q_2e)/\beta,$$
$$f_2:\ (q_2e)/\beta,\quad (q_2e)/\beta^2,\quad (q_2e)^2/\beta.$$

Hence we have calculated f_0 and f_2 to these orders. The results and some of the details are given in Appendix B. The cross section may be computed according to [cf. (3.1)]

$$d\sigma/d\Omega=|\bar{u}_f\gamma_0u_i|^2|f(\mathbf{k}_f,\mathbf{k}_i)|^2,$$

the result being

$$d\sigma/d\Omega=4E^2\cos^2(\tfrac{1}{2}\Theta)/Q^4\times\{F_0+P_2(\mu_Q)F_Q+P_2(\mu_T)F_T+[P_2(\mu_Q)]^2F_{QQ}+P_2(\mu_Q)P_2(\mu_T)F_{QTE}\}. \tag{3.8}$$

Here μ_Q and μ_T are defined by

$$\mu_Q=\hat{\mathbf{Q}}\cdot\hat{n},\qquad \mu_T=\hat{T}\cdot\hat{n}, \tag{3.9}$$

where $\hat{n}$ is a unit vector along the symmetry axis of the charge distribution, and

$$\mathbf{Q}=\mathbf{k}_i-\mathbf{k}_f,\qquad \mathbf{T}=\mathbf{k}_i+\mathbf{k}_f. \tag{3.10}$$

The quantitites F_0, F_Q, etc., are written out in full in Sec. IV [cf. Eqs. (4.3)].

B. Scattering from a Nucleus with Spin

1. Amplitude

We wish to extend the treatment of the complete Born series which was made by Schiff for a classical density $\rho(\mathbf{x})$ to the case that the nucleus is described by a charge-density operator $\hat{J}_0(\mathbf{x})$. To do this we assume for the time being that the electrons satisfy the Schrödinger equation and consider formally the entire Born series, the nth term of which will contain the operator $\hat{J}_0$ as a factor n times. Then we make the multipole expansion (2.22) for $\hat{J}_0(\mathbf{Q})$, assuming only $l=0$ and $l=2$, and retaining terms no higher than second order in the quadrupole (rank two) part, in accordance with our findings in part A which indicated the necessity of second-order terms. Hence, the terms which we keep contain, aside from any number of rank-zero operators $T^0{}_0$, either zero , one, or two rank-two operators $T^2{}_m$. This observation allows us to determine the angular momentum properties of the operator $\hat{S}$, the matrix element of which gives the amplitude for the transition $|j\mu\rangle\rightarrow|j'\mu'\rangle$ according to

$$\text{amp}=\langle j'\mu'|\hat{S}|j\mu\rangle. \tag{3.11}$$

Indicating various multiple sums and the integrals over all internal coordinates by a symbolic sum over λ, we assert the most general form of $\hat{S}$ to be

$$\hat{S}=\hat{S}^{(0)}+\hat{S}^{(1)}+\hat{S}^{(2)}, \tag{3.12}$$

where

$$\hat{S}^{(0)}=\sum_\lambda[A^{(\lambda)}]^0{}_0Y_{0,0}{}^*,$$

$$\hat{S}^{(1)}=\sum_{\lambda,m}\{[B_Q{}^{(\lambda)}]^2{}_mY_{2,m}{}^*(\Omega_Q)+[B_T{}^{(\lambda)}]^2{}_mY_{2,m}{}^*(\Omega_T)\},$$

$$\hat{S}^{(2)}=\sum_{\lambda,m,m'}\{[C_Q{}^{(\lambda)}]^2{}_m[D_Q{}^{(\lambda)}]^2{}_{m'}Y_{2,m}{}^*(\Omega_Q)Y_{2,m'}{}^*(\Omega_Q)+[C_T{}^{(\lambda)}]^2{}_m[D_T{}^{(\lambda)}]^2{}_{m'}Y_{2,m}{}^*(\Omega_T)Y_{2,m'}{}^*(\Omega_T)$$
$$+[C_{QT}{}^{(\lambda)}]^2{}_m[D_{QT}{}^{(\lambda)}]^2{}_{m'}\times\tfrac{1}{2}[Y_{2,m}{}^*(\Omega_Q)Y_{2,m'}{}^*(\Omega_T)+Y_{2,m}{}^*(\Omega_T)Y_{2,m'}{}^*(\Omega_Q)]\}. \tag{3.13}$$

[In this notation the order in quadrupole strength is indicated as a superscript for $\hat{S}^{(0)}$, $\hat{S}^{(1)}$, and $\hat{S}^{(2)}$.] This assertion is justified as follows: For $\hat{S}^{(0)}$ only rank-zero operators are involved, so the result can always be represented as another rank-zero operator. For $\hat{S}^{(1)}$ there is always one operator of rank two, all the others being of rank zero, so these may be represented as some rank-two operator. Since the amplitude $\hat{S}^{(1)}$ must be a scalar, the nuclear operator $\sum_\lambda [B^{(\lambda)}]^2_m$ must be contracted with a rank-two operator from the rest of the system. The only possibilities are $Y_{2,m}^*(\Omega_Q)$ and $Y_{2,m}^*(\Omega_T)$ [**Q** and **T** are defined in Eqs. (3.10)]; hence the most general form for $\hat{S}^{(1)}$ includes both these possibilities. For $\hat{S}^{(2)}$ the reasoning is similar.[13]

Application of the Wigner-Eckart theorem [cf. Eq. (2.24)] to Eqs. (3.11)–(3.13) shows that the amplitude for a transition $|j\mu\rangle \rightarrow |j'\mu'\rangle$ will be known provided we have determined the reduced matrix elements of the operators

$$\sum_\lambda [A^{(\lambda)}]^0_0, \quad \sum_\lambda [B_Q{}^{(\lambda)}]^2_m, \text{ etc.}$$

To do this we imagine a scattering process in which all the nuclear operators $T^l{}_m$ are assumed to connect only to the one state $|jj\rangle$. Although such a scattering has no physical realization except in the classical limit of very large j, it is useful to consider since we can write two different expressions for this amplitude which, when equated, allow us to evaluate the reduced matrix elements in question.

Let us consider first that part of the amplitude which contains monopole (rank-zero) factors only. Terms which contribute here will contain nuclear matrix elements in products of the form

$$\langle jj|T^0_0(Q_1)|jj\rangle\langle jj|T^0_0(Q_2)|jj\rangle\cdots\langle jj|T^0_0(Q_n)|jj\rangle.$$

However, this expression is also equal to[14]

$$\langle jj|T^0_0(Q_1)T^0_0(Q_2)\cdots T^0_0(Q_n)|jj\rangle.$$

The operator product appearing here, when generalized to include all terms of the Born series, was represented by

$$\sum_\lambda [A^{(\lambda)}]^0_0 Y_{0,0}^*$$

so as one expression for the monopole part of the amplitude we have

$$\langle jj|\sum_\lambda [A^{(\lambda)}]^0_0|jj\rangle Y_{0,0}^*.$$

Similar expressions obtain for the terms linear and bilinear in quadrupole strength [see Eq. (3.15), below].

Now it is obvious from the preceding equations that the amplitude for the process we are considering is formally identical to what would be computed classically except that the matrix element $\rho_l{}^j(Q)=\langle jj|T^l_0(Q)|jj\rangle$ now plays the role of the classical $\rho_l(Q)$. Hence we see that a second expression for this amplitude is the classical amplitude (3.2) with the functions $\rho_l(r)$ replaced by $\rho_l{}^j(r)$, the radial densities for the nucleus in the state $|jj\rangle$. Using the notation of Eq. (3.5) and indicating (as in Appendix B) with a superscript the order in quadrupole strength, we write

$$\begin{aligned} f_0&=f_0{}^{(0)}+f_0{}^{(1)}, \\ f_2&=f_2{}^{(1)}+f_2{}^{(2)}. \end{aligned} \tag{3.14}$$

Then to each order in quadrupole strength we equate the two expressions for this amplitude:

$$\begin{aligned} f_0{}^{(0)}|_{[\rho_l(r)=\rho_l{}^j(r)]}&=\langle jj|\sum_\lambda [A^{(\lambda)}]^0_0|jj\rangle Y_{0,0}^*, \\ (f_0{}^{(1)}+f_2{}^{(1)})|_{[\rho_l(r)=\rho_l{}^j(r)]}&=\langle jj|\sum_\lambda [B_Q{}^{(\lambda)}]^2_0|jj\rangle Y_{2,0}^*(\Omega_Q)+\langle jj|\sum_\lambda [B_T{}^{(\lambda)}]^2_0|jj\rangle Y_{2,0}^*(\Omega_T), \\ f_2{}^{(2)}|_{[\rho_l(r)=\rho_l{}^j(r)]}&=\sum_\lambda\langle jj|[C_Q{}^{(\lambda)}]^2_0|jj\rangle\langle jj|[D_Q{}^{(\lambda)}]^2_0|jj\rangle Y_{2,0}^*(\Omega_Q)Y_{2,0}^*(\Omega_Q) \\ &\quad+\sum_\lambda\langle jj|[C_T{}^{(\lambda)}]^2_0|jj\rangle\langle jj|[D_T{}^{(\lambda)}]^2_0|jj\rangle Y_{2,0}^*(\Omega_T)Y_{2,0}^*(\Omega_T) \\ &\quad+\sum_\lambda\langle jj|[C_{QT}{}^{(\lambda)}]^2_0|jj\rangle\langle jj|[D_{QT}{}^{(\lambda)}]^2_0|jj\rangle Y_{2,0}^*(\Omega_Q)Y_{2,0}^*(\Omega_T). \end{aligned} \tag{3.15}$$

Finally, application of the Wigner-Eckart theorem to all of the above matrix elements allows us to evaluate the reduced matrix elements

$$\sum_\lambda\langle j\|[A^{(\lambda)}]^0\|j\rangle, \quad \sum_\lambda\langle j\|[B_Q{}^{(\lambda)}]^2\|j\rangle, \text{ etc.}$$

For purposes of calculation we regard the left-hand sides of these equations as having been computed by Eqs. (3.6) and (3.7) for a particular choice of $\rho_0{}^j(r)$ and $\rho_2{}^j(r)$.

Transition matrix elements of the type $\langle J\|[B_Q{}^{(\lambda)}]^2\|j\rangle$ may be evaluated for rotational states J belonging to the ground-state band according to Eqs. (2.34) and (2.35), since these equations are valid for any rank-two operator. For virtual transitions the analogous relation is

$$\langle j\|C^2\|J\rangle\langle J\|D^2\|j\rangle=g(J,j)\langle j\|C^2\|j\rangle\langle j\|D^2\|j\rangle, \tag{3.16}$$

[13] Here, however, in the term containing both Ω_Q and Ω_T, we have kept only the symmetric combination. The reason for this is that our method of evaluating the reduced matrix elements of these operators, to be described shortly, is applicable only to terms having a classical analog, and the antisymmetric combination does not have this property. It will be seen, however, that when low-lying rotational states are summed over, these terms make no contribution anyhow. For further discussion of this point, see Sec. IV.

[14] This assumes that the states $|j\mu\rangle$ form a complete set. If other quantum numbers γ are required to label a state, the closure relation must be written

$$I=\sum_{\gamma j\mu}|\gamma j\mu\rangle\langle\gamma j\mu|.$$

If this is the case, we make the assumption that all transitions, both final state and virtual, in which the quantum numbers γ change can be neglected. Then the label γ can be suppressed, and the above assertion is valid. For states which are described in terms of the collective model, this assumption means that we neglect transitions outside of a rotational band.

where

$$g(J,j)=(-)^{J-j}|f(J,j)|^2 \tag{3.17}$$

and C^2 and D^2 are any irreducible tensor operators of rank two.

As noted earlier, the form of $\hat{S}$ given in Eqs. (3.12) and (3.13) was derived on the assumption that the electrons obey the Schrödinger equation. In the case of classical potential scattering, Schiff[2] has shown that the extension to a Dirac electron involves only multiplication by the factor $(\bar{u}_f\gamma_0 u_i)$ [cf. Eq. (3.1)]. However, it is obvious from the above discussion that any matrix element of $\hat{S}$ is linearly related (by a 3-j symbol) to a classical amplitude, so we conclude that the rule giving the Dirac amplitude is the same, i.e., multiply the Schrödinger amplitude by $(\bar{u}_f\gamma_0 u_i)$.

Specific application to the model [cf. (2.11)]

$$\begin{aligned}\rho_0{}^j(r)&=\rho_0, \quad r\leq R\\ &=0, \quad r>R,\\ \rho_2{}^j(r)&=q_2\delta(r-R)/r^2\end{aligned} \tag{3.18}$$

requires only the use of Eqs. (3.15) together with the calculation of $f_0{}^{(0)}$, $f_0{}^{(1)}$, etc. which is presented in Appendix B. We note that although, as before, q_2 is related to the quadrupole moment Q_0 through (2.12), Q_0 now means the "measured quadrupole moment" for the nucleus in the state $|jj\rangle$. It is related to the "intrinsic quadrupole moment" $Q_0{}^{(\text{int})}$ by

$$Q_0=\eta_j Q_0{}^{(\text{int})}, \tag{3.19}$$

where

$$\eta_j\equiv(2j+1)\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2=\frac{(2j)(2j-1)}{(2j+3)(2j+2)}. \tag{3.20}$$

2. Cross Section

The calculation of the Schiff-Tiemann cross section proceeds in a way analogous to that for the Born approximation which was based upon Eq. (2.28). Now however the Born amplitude (2.21) has been replaced by the amplitude (3.11), so therefore we must compute

$$\sum_f\langle|f(\mathbf{k}_f,\mathbf{k}_i)|^2\rangle_i=\mathrm{Tr}\{\rho^i\hat{S}^\dagger\hat{S}\}, \tag{3.21}$$

with $\hat{S}$ being defined by Eqs. (3.12) and (3.13). In order to define $\hat{S}^\dagger$ we use the following definition of the adjoint of a tensor operator:

$$[T]^{l\dagger}{}_m=(-)^m[T^\dagger]^l{}_{-m}, \tag{3.22}$$

where

$$\langle jj|[T^\dagger]^l{}_0|jj\rangle=\langle jj|T^l{}_0|jj\rangle^*.$$

The calculation of the trace (3.21) may be performed using the same techniques discussed in Sec. II. Again the density-matrix representation (2.29) is assumed. Low-lying rotational levels are summed over within the framework of the Bohr-Mottelson model. As before, these sums may all be expressed in terms of the quantity $\Sigma_n(j,k)$ [cf. Eq. (2.37)]. Although it is possible in the final result to distinguish between virtual- and final-state transitions, we shall not do this. Rather we assume that whichever states are available virtually are also available finally, and vice versa. The inclusion of electron spins, which we saw in Sec. III B1 to involve the factor $|\bar{u}_f\gamma_0 u_i|^2$, is again handled according to Eq. (2.5). The full expression for $d\sigma/d\Omega$ is given in Eq. (4.1).

IV. RESULTS AND DISCUSSION

A. Summary of Results

We may summarize the results for both the Born and the Schiff-Tiemann approximations in the following way:

$$\begin{aligned}\frac{d\sigma}{d\Omega}=\frac{4E^2\cos^2(\tfrac{1}{2}\Theta)}{Q^4}\Bigg\{&F_0+\left[A_2\Big/5\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}\right][P_2(\mu_Q)F_Q+P_2(\mu_T)F_T]\\ &+\sum_{k=0}A_k\left[\begin{pmatrix}2 & 2 & k\\ 0 & 0 & 0\end{pmatrix}\Big/\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2\right]\Sigma_n(j,k)P_k(\mu_Q)F_{QQ}\\ &+\sum_{k=0,2,4}\sum_{m=0}^{2}A_k\left[\begin{pmatrix}2 & 2 & k\\ m & -m & 0\end{pmatrix}\Big/\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2\right]\Sigma_n(j,k)N_mP_{2,m}(\mu_Q)P_{2,m}(\mu_T)\cos m\Phi F_{QTE}\\ &+\sum_{k=1,3}\sum_{m=0}^{2}A_k\left[\begin{pmatrix}2 & 2 & k\\ m & -m & 0\end{pmatrix}\Big/\begin{pmatrix} j & 2 & j\\ -j & 0 & j\end{pmatrix}^2\right]\Sigma_n(j,k)N_mP_{2,m}(\mu_Q)P_{2,m}(\mu_T)\sin m\Phi F_{QTO}\Bigg\}.\end{aligned} \tag{4.1}$$

For the Born approximation

$$\begin{aligned}&F_0=9\gamma^2[j_1(\beta)/\beta]^2, && F_T=0,\\ &F_Q=-6\gamma(4\pi eq_2)j_1(\beta)j_2(\beta)/\beta, && F_{QTE}=F_{QTO}=0,\\ &F_{QQ}=(4\pi eq_2)^2[j_2(\beta)]^2.\end{aligned} \tag{4.2}$$

For the Schiff-Tiemann approximation

$$F_0=(9\gamma^2/\beta^4)\{\cos^2\alpha-(2\cos\alpha/\beta)(\sin\alpha+10\gamma\cos\alpha\sin^3(\tfrac{1}{2}\Theta))\},$$

$$F_Q=-[6\gamma(4\pi eq_2)/\beta^3]\{\sin\alpha\cos\alpha+(4\cos^2\alpha-1)/\beta+(2\gamma/\beta)\sin\alpha\cos\alpha\sin(\tfrac{1}{2}\Theta)$$
$$\times[\tfrac{1}{5}(1-4\sin^2(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta))-9\sin^2(\tfrac{1}{2}\Theta)]\},$$

$$F_T=-[6\gamma(4\pi eq_2)/\beta^3](4\gamma/5\beta)\sin\alpha\cos\alpha\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)],$$

$$F_{QQ}=(4\pi eq_2/\beta)^2\{\sin^2\alpha+6\sin\alpha\cos\alpha/\beta+(4\gamma/5\beta)\sin(\tfrac{1}{2}\Theta)$$
$$\times[(1-2\cos^2\alpha)(1-4\sin^2(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta))-20\sin^2\alpha\sin^2(\tfrac{1}{2}\Theta)]\}, \tag{4.3}$$

$$F_{QTE}=(4\pi eq_2/\beta)^2(8\gamma/5\beta)(1-2\cos^2\alpha)\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)],$$

$$F_{QTO}=-(4\pi eq_2/\beta)^2(4\gamma/5\beta)\cos\alpha\sin\alpha\left\{2\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)]+\frac{1-\sin(\tfrac{1}{2}\Theta)}{1+\sin(\tfrac{1}{2}\Theta)}\right\}.$$

The symbol A_k is defined by

$$A_k\equiv a_{k,0}\langle j\|J^k\|j\rangle$$
$$=(2k+1)\sum_\mu p_\mu(-)^{j-\mu}\begin{pmatrix} j & k & j \\ -\mu & 0 & \mu\end{pmatrix}. \tag{4.4}$$

The constants N_m are

$$\begin{aligned} N_m&=1, && m=0\\ &=-1/3, && m=1\\ &=1/12, && m=2.\end{aligned} \tag{4.5}$$

$\Sigma_n(j,k)$ is for $n=0$ (inclusion of ground state only)

$$\Sigma_0(j,k)=(-)^{2j}\begin{Bmatrix} 2 & 2 & k \\ j & j & j\end{Bmatrix} \tag{4.6}$$

and for $n=2$ (inclusion of ground state plus first two excited states)

$$\Sigma_2(j,k)=\begin{pmatrix} 2 & 2 & k \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} j & k & j \\ -j & 0 & j\end{pmatrix}\Big/$$
$$(2j+1)\begin{pmatrix} j & 2 & j \\ -j & 0 & j\end{pmatrix}^2. \tag{4.7}$$

Other quantities are

Θ = angle of scattering;

Q = momentum transfer $=2E\sin(\frac{1}{2}\Theta)$;

$\psi=2\gamma\sin(\frac{1}{2}\Theta)[1+\frac{1}{3}\sin^2(\frac{1}{2}\Theta)]$;

$\beta=QR$;

$\alpha=\beta+\psi$; (4.8)

$\gamma=Ze^2$ $(e^2=1/137)$;

$\mathbf{Q}=\mathbf{k}_i-\mathbf{k}_f$; $\mathbf{T}=\mathbf{k}_i+\mathbf{k}_f$;

$\hat{n}$ = unit vector along symmetry axis of charge distribution;

$\mu_Q=\hat{n}\cdot\hat{Q}$, $\mu_T=\hat{n}\cdot\hat{T}$.

The angle Φ, which in the frame having $\hat{n}$ as a polar axis is the azimuth difference $\varphi_Q-\varphi_T$, may be defined in terms of invariants by

$$\cos\Phi=-\mu_Q\mu_T/[(1-\mu_Q^2)(1-\mu_T^2)]^{1/2},$$
$$\sin\Phi=\hat{T}\times\hat{Q}\cdot\hat{n}/[(1-\mu_Q^2)(1-\mu_T^2)]^{1/2}. \tag{4.9}$$

It may be verified that

$$\cos^2\Phi+\sin^2\Phi=1,$$

provided it is recognized that

$$\hat{Q}\cdot\hat{T}=0.$$

We note finally that $\begin{pmatrix} 2 & 2 & k \\ 0 & 0 & 0\end{pmatrix}$ is zero unless k is even, and that

$$\Sigma_2(j,k)\propto\begin{pmatrix} 2 & 2 & k \\ 0 & 0 & 0\end{pmatrix}$$

so that in the case that we include transitions to states $J=j+1$ and $J=j+2$, only A_0, A_2, and A_4 will enter the above sums. The parameter A_0 is always present and is required by the unitarity condition

$$\mathrm{Tr}\rho=1$$

to be

$$A_0=1/(2j+1)^{1/2}. \tag{4.10}$$

B. Special Cases

1. The Randomly Oriented Nucleus

It is of interest to examine our result for the special case of a randomly oriented nucleus. When all substates are equally populated, we have

$$p_\mu=1/(2j+1),$$
and
$$A_k=\delta_{k,0}/(2j+1)^{1/2}. \tag{4.11}$$

It is then possible to perform the sums over m appearing in Eq. (4.1) with the aid of the spherical-harmonic addition theorem. It is convenient to use a coordinate

system in which $\hat{n}$ is along the polar axis. In this case

$$\sum_{m=-2}^{2}\frac{4\pi}{5}\begin{pmatrix}2 & 2 & k\\ m & -m & 0\end{pmatrix}Y_{2,m}^{*}(\Omega_Q)Y_{2,-m}^{*}(\Omega_T)$$
$$=\sum_{m=0}^{2}\begin{pmatrix}2 & 2 & k\\ m & -m & 0\end{pmatrix}N_m P_{2,m}(\mu_Q)P_{2,m}(\mu_T)$$
$$\times\begin{cases}\cos m\Phi, & k \text{ even}\\ -i\sin m\Phi, & k \text{ odd}.\end{cases} \quad (4.12)$$

For $n=2$ we find that

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{random}}=\frac{4E^2\cos^2(\frac{1}{2}\Theta)}{Q^4}$$
$$\times\left\{F_0+\frac{1}{5\eta_j^2}[F_{QQ}-\tfrac{1}{2}F_{QTE}]\right\}, \quad (4.13)$$

where η_j is given by Eq. (3.20). Since both F_{QQ} and F_{QTE} are proportional to the square of the observed quadrupole moment [cf. Eqs. (4.2), (4.3), (2.12)], we can express the result in terms of the intrinsic moment by factoring out η_j^2 [cf. Eq. (3.19)]:

$$F_{QQ}=\eta_j^2 F_{QQ}^{(\text{int})},$$
$$F_{QTE}=\eta_j^2 F_{QTE}^{(\text{int})}. \quad (4.14)$$

Then Eq. (4.13) becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{random}}=\frac{4E^2\cos^2(\frac{1}{2}\Theta)}{Q^4}$$
$$\times\{F_0+\tfrac{1}{5}[F_{QQ}^{(\text{int})}-\tfrac{1}{2}F_{QTE}^{(\text{int})}]\}. \quad (4.15)$$

Hence we find that for randomly oriented nuclei the cross section, summed over the three rotational states of the ground-state band and expressed in terms of the intrinsic quadrupole moment, is independent of the nuclear spin. The same result was obtained by Schiff for the Born approximation by observing a sum rule which is equivalent (they are the same except for different normalizations) to the first of two sum rules we gave in Eq. (2.36). Although our result for the inclusion of the three rotational levels ($n=2$) required the use of the second sum rule, for the case of $k=0$ the two rules are easily shown to be equivalent. Therefore we may regard the second sum rule as the extension of the sum rule noticed by Schiff for unoriented ($k=0$) nuclei to the case in which nuclear alignments ($k\neq 0$) are present. These results are a direct consequence of having assumed the Bohr-Mottelson model to describe the collective motion of the nucleus. They would not be expected to follow using a model which, for example, treated individual particle motions.

2. *The Schiff-Tiemann Approximation in the Limit of $\gamma\ll 1$*

The cross section (4.1) as computed for the Schiff-Tiemann approximation using Eqs. (4.3) reduces to the result for the first Born approximation [Eqs. (4.2)] when γ becomes very much less than one. Since the Schiff-Tiemann cross section has been computed keeping only the first two terms in the expansions in $1/\beta$, the comparison with the Born cross section must be made on the same basis. It is then a simple matter to show that for γ negligible compared to unity Eqs. (4.3) reduce to (4.2).

3. *The Classical Limit*

The classical result (3.8) can be recovered from the expression (4.1) if we consider a nucleus in the state $|jj\rangle$ (i. e., we set $p_\mu=\delta_{\mu,j}$) and take the limit for j very large. Optimal quantum-mechanical alignment in the state $|jj\rangle$ becomes perfect alignment in the classical limit ($j\to\infty$). Reference to the tabulation of $|f(J,j)|^2$ (cf. Table I) will show that in the limit of $j\to\infty$ transitions to states of $J\neq j$ become negligible compared to those for $j\to j$. Nevertheless it is more convenient to consider formally the summation over all three levels by setting $\sum_n=\sum_2$; then this effect is taken care of automatically. One simple limit is required:

$$\lim_{j\to\infty}\begin{pmatrix}j & k & j\\ -j & 0 & j\end{pmatrix}=\frac{1}{(2j)^{1/2}}, \quad (\text{any } k).$$

The limit of $\sum_2(j,k)$ for $j\to\infty$ is then obtained from Eq. (4.7). Use of Eq. (4.12), together with the relations

$$\sum_k(2k+1)\begin{pmatrix}2 & 2 & k\\ 0 & 0 & 0\end{pmatrix}^2 P_k(\mu_Q)=[P_2(\mu_Q)]^2,$$
$$\sum_k(2k+1)\begin{pmatrix}2 & 2 & k\\ m & -m & 0\end{pmatrix}\begin{pmatrix}2 & 2 & k\\ 0 & 0 & 0\end{pmatrix}=\delta_{m,0}$$

makes it a simple matter to verify that the classical limit is indeed obtained.

C. Discussion

1. *A Comparison of Three Approximation Schemes*

We can compare the types of terms of the complete Born series which are treated in the (i) first Born approximation; (ii) distorted-wave approximation; (iii) Schiff-Tiemann approximation. The complete Born series for the amplitude contains the following types of terms:

Order	Terms
1	γ, (q_2e)
2	γ^2, $\gamma(q_2e)$, $(q_2e)^2$
3	γ^3, $\gamma^2(q_2e)$, $\gamma(q_2e)^2$, $(q_2e)^3$ etc.

The first Born approximation for the cross section contains terms γ^2, $\gamma(q_2e)$, and $(q_2e)^2$. In the distorted-wave approximation the monopole scattering is computed exactly while the quadrupole amplitude is treated to first order. The zero-order wave functions used for these calculations are the exact solutions to the Dirac equation for the spherical part of the potential. This approach does not take account of quadrupole distortions in the wave fronts which are treated in an approximate manner by Schiff through the phase function $L_2(\mathbf{x})$. However, both of these methods pick up terms $\gamma^N(q_2e)$ $(N\geq 1)$ in the cross section. When γ is of the order of $\frac{1}{2}$ (as it is for holmium), the corrections to the Born approximation are seen to be of the same order as the original term. For the terms bilinear in (q_2e) the Schiff approximation is able to include virtual as well as real quadrupole transitions, whereas the distorted-wave approximation describes only the latter (insofar as present application has not gone beyond first order in the amplitude). Both treatments bring in terms $\gamma^N(q_2e)^2(N\geq 0)$ in the cross section, so that again, relative to the first Born approximation $(N=0)$, the corrections are of the same order as the original Born contribution.

Application of the Born approximation to this problem was first made by Fubini and Ferroni and by Bernardini *et al.*[15] Similar considerations have been made by Weigert and Rose and by Burleson.[16] Schiff[4] has discussed the case of unaligned nuclei. The distorted-wave approximation was made by Downs *et al.*[17] for a randomly oriented nucleus and by Wright and Onley[18] for an aligned nucleus, in both instances assuming a nonzero surface thickness.

2. *The Necessity of Second-Order Terms*

An important result of the work here is the fact that it was necessary to treat second-order quadrupole terms in both the amplitude and the cross section. Assuming a uniformly charged ellipsoid of revolution (which was shown above to be equivalent in polar coordinates to a uniformly charged sphere plus a surface-delta-function quadrupole density) and making asymptotic expansions in QR for the amplitude, we found that some of the second-order quadrupole terms could be comparable to first-order and monopole terms. This was discussed in detail in Sec. III. A question of interest is whether or not this behavior is general. Would it persist if, for example, we eliminated the sharp cutoff inherent in a uniformly charged nucleus and instead considered a nuclues with a fuzzy edge? We offer the following arguments to the effect that it would. Recall that we found the asymptotic behavior of the integrals for f_0 to be as $1/\beta^2$, while that for f_2 was as $1/\beta$ (see Appendix B). Since in the extreme asymptotic limit the Born approximation is expected to provide a reasonable description of scattering, the amplitude will go as the Fourier transform of the charge density. Elimination of the sharp cutoff in $\rho_0(\mathbf{x})$ will result in the attenuation of high frequencies, so that in this case the amplitude must fall off even faster than $1/\beta^2$. Now the integrals for f_2 all had the factor $\rho_2(r)$ appearing linearly in the integrand, and for

$$\rho_2(r)=q_2\delta(r-R)/r^2$$

the remaining angular integrations gave rise to a $1/\beta$ asymptotic behavior. If we smear out this radial quadrupole distribution but still restrict it to rather narrow limits in the vicinity of $r=R$, we can approximate the radial integral by the mean-value theorem so that the remaining angular integrations will be performed over a surface very near $r=R$. Therefore we would again expect a $1/\beta$ asymptotic dependence, which in turn implies a quadrupole enhancement and the consequent need for second-order terms. This argument is not rigorous since there may be phase interferences in the radial integral which prohibit application of the mean-value theorem.

3. *A Remark Concerning Virtual Quadrupole Excitations*

It is necessary to make some qualifying remarks about the second-order term in the Schiff amplitude. It will be recalled that in the derivation of its operator properties terms were present which were both symmetric and antisymmetric with respect to interchange of Ω_Q and Ω_T, but that only the symmetric combination was chosen (cf. Ref. 13). Suppose we had kept the antisymmetric part also; in that case the integrand for $\hat{S}^{(2)}$ would have had the additional term [cf. Eq. (3.13)]

$$[E_{QT}{}^{(\lambda)}]^2{}_m[F_{QT}{}^{(\lambda)}]^2{}_{m'}\times\tfrac{1}{2}[Y_{2,m}{}^*(\Omega_Q)Y_{2,m'}{}^*(\Omega_T) - Y_{2,m}{}^*(\Omega_T)Y_{2,m'}{}^*(\Omega_Q)].$$

Our method of evaluating the reduced matrix elements of these operators is to consider the scattering process whereby the nuclear operators $T^l{}_m$ connect only to the one state $|jj\rangle$. Obviously the contribution of the above term vanishes identically, so it is not possible to evaluate its reduced matrix elements by this technique. Only terms having a classical analog can be treated by our prescription. Of course the antisymmetric term may exist. If it were kept, it would make an additional contribution to F_{QTO} [see Eq. (4.3)] and would enter the cross section [see Eq. 4.1], but only for odd[19] values of k. However, we see that if the three rotational levels are summed over $(\Sigma_n=\Sigma_2)$ only even values of k enter the cross section since Σ_2 vanishes for k odd.

[15] S. Fubini and S. Ferroni, Nuovo Cimento **1**, 263 (1955); M. Bernardini, P. Brovetto, and S. Ferroni, *ibid.* **5**, 1292 (1957).

[16] L. J. Weigert and M. E. Rose, Nucl. Phys. **51**, 529 (1964). P. Burleson (unpublished); M. R. Yearian (private communication).

[17] B. W. Downs, D. G. Ravenhall, and D. R. Yennie, Phys. Rev. **106**, 1285 (1957).

[18] L. E. Wright and D. S. Onley, Nucl. Phys. **64**, 231 (1965).

[19] The symmetric term, which was retained, is wholly contained in F_{QTE} and enters only for even values of k.

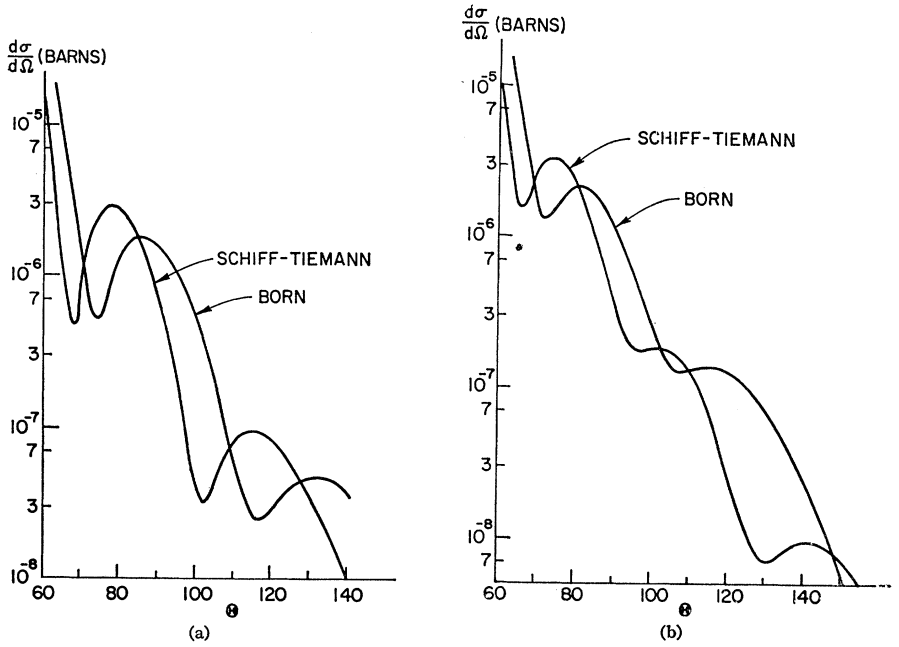

FIG. 1. $d\sigma/d\Omega$ versus Θ (in deg) for a holmium nucleus completely polarized in the $\mu=7/2$ state with the magnetic field (a) normal to the scattering plane and (b) parallel to the incident beam. The (observed) quadrupole moment was assigned a value $+2e\times10^{-24}$ cm^2.

Hence we conclude that for the odd-k contributions to $d\sigma/d\Omega$ we have neglected possible virtual quadrupole excitations, but that when all three rotational levels are summed over, odd values of k are not present, so that in this case the point becomes moot.

4. *The Presence of Even and Odd Values of k*

It was noted by Weigert and Rose[16] that for elastic scattering only even[20] values of k enter the Born approximation to the cross section. Our result (4.1) verifies this conclusion. We see however that this rule is not valid for the Schiff-Tiemann approximation. On the other hand, if the three rotational levels are summed over, both approximations involve only even values of k.

5. *Time-Reversal Invariance*

It was pointed out by Schiff[2] that his approximation displayed time-reversal invariance. We can readily verify that our expression (4.1) for $d\sigma/d\Omega$ has this property. We note that under time reversal $\mathbf{k}_f\rightarrow-\mathbf{k}_i$, $\mathbf{k}_i\rightarrow-\mathbf{k}_f$, $\hat{n}$ (the direction of the magnetic field) $\rightarrow-\hat{n}$, and $p_\mu\rightarrow p_{-\mu}$. It then follows that $\mathbf{Q}$ and Θ remain invariant, but $\mathbf{T}$ is taken into $(-\mathbf{T})$. The transformation of the other parameters may be deduced directly from Eqs. (4.4) and (4.9). It is then a simple matter to show that $d\sigma/d\Omega$ is left invariant under this transformation.

6. *Numerical Calculations for Holmium*

We have made numerical evaluations for $d\sigma/d\Omega$ for the scattering of 200-MeV electrons from an aligned holmium ($Z=67$, $A=165$, ground-state spin$=\frac{7}{2}$) target assuming a radius of 6.58 F, as computed from the $A^{1/3}$ law,

$$R=r_0A^{1/3}$$

with $r_0=1.2$ F. Plots of $d\sigma/d\Omega$ versus scattering angle Θ are shown in Fig. 1. Here it was assumed that the nucleus was completely polarized in the state $\mu=j$ (i.e., $p_\mu=\delta_{\mu,j}$). In Fig. 2 are displayed some graphs of R versus Θ, where

$$R=[\sigma_{\text{aligned}}-\sigma_{\text{random}}]/\sigma_{\text{random}}.$$

In this case we assumed the alignments could be described by a Maxwell-Boltzmann distribution (see Table II) characterized by $f_1=0.688$ and $f_2=0.639$, where f_k is defined by

$$f_k\equiv\langle\mu^k\rangle/j^k=\mathrm{Tr}\{\rho(J_z)^k\}/j^k.$$

It is expected to provide a realistic experimental description under optimal conditions.[21] The measured quadrupole moment was assigned a value of $+2$ (in units of $e\times10^{-24}$ cm^2, $e>0$). The intrinsic moment is greater by a factor of η_j^{-1} [see Eq. (3.19)]; for holmium ($j=\frac{7}{2}$), $\eta_j^{-1}=15/7$. In all cases we have included the sums over the three low-lying rotational levels.

Because our forms for $\rho(\mathbf{x})$ imply a sharp edge in the nuclear density, some extra high-frequency components

TABLE II. Maxwell-Boltzmann distribution assumed for the graphs of Fig. 2. $f_1=0.688$, $f_2=0.639$.

μ	$-7/2$	$-5/2$	$-3/2$	$-1/2$	$1/2$	$3/2$	$5/2$	$7/2$
p_μ	0.006	0.011	0.020	0.038	0.071	0.134	0.251	0.469

[20] Weigert and Rose (Ref. 16) speak of A_k for k-even as describing alignments, and for k-odd, polarizations. This nomenclature is not universal, however.

[21] R. Safrata (private communication).

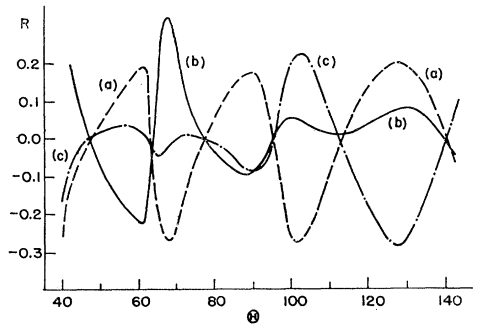

FIG. 2. R versus Θ (in deg) for a holmium nucleus aligned according to the Maxwell-Boltzmann distribution shown in Table II, assuming an (observed) quadrupole moment of $+2e\times10^{-24}$ cm^2, with the magnetic field (a) normal to the scattering plane, (b) in the scattering plane but normal to the incident beam, and (c) along the incident beam.

are thereby introduced. These are reflected in the curves which we obtain. They would presumably be eliminated if the effects of a surface thickness were included.

It should be remarked that the cross-section curves are not expected to be reliable in the vicinity of local minima. This is because at these points the terms retained in the asymptotic expansions tend to cancel, their sum thereby becoming comparable to the higher order terms which were dropped. A further restriction is a condition of applicability of the Schiff approximation, namely that

$$\Theta\gg(kR)^{-1/2},\quad k=|\mathbf{k}_i|=|\mathbf{k}_f|\,.$$

For the parameters given above this becomes

$$\Theta\gg25^\circ.$$

As mentioned earlier, neglect of magnetic-moment scattering is justified everywhere except in the backward direction.

D. Conclusion

We have shown that the Schiff-Tiemann approximation can be applied to describe scattering from a deformed nucleus. In order to treat nuclear alignments and spin-state transitions it was necessary to extend the formalism to the case of a nucleus having spin. This was done through second order in the quadrupole part of the interaction, since we found that second-order terms could be important. All of the integrals were evaluated analytically (as well as asymptotically), and it was to this end the nucleus was described by a uniformly charged ellipsoid of revolution. Finally, we have seen that alignment effects are significant for a greatly deformed nucleus such as holmium.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to Professor C. K. Iddings for first suggesting this problem and for subsequently guiding the progress of the research to its completion. His numerous suggestions and patient questioning have provided many insights of both a physical and mathematical nature.

Thanks are also due Professor L. I. Schiff for his enduring support and several suggestions. Conversations with Professor J. D. Walecka and Dr. T. Griffy proved very valuable, as did those with Professor M. R. Yearian and Dr. R. Safrata concerning the relationship of this work to experimental expectations.

APPENDIX A: CALCULATION OF THE INTEGRALS FOR $L(x)$

The phase function $L(\mathbf{x})$ is defined to be

$$L(\mathbf{x})\equiv-\int_0^\infty[V(\mathbf{x}-\hat{\mathbf{k}}_i s)+V(\mathbf{x}+\hat{\mathbf{k}}_f s)]\,ds\,,\quad\text{(A1)}$$

where $\hat{\mathbf{k}}_i$ and $\hat{\mathbf{k}}_f$ are unit vectors in the direction of the momenta of the initial and final electrons, and $V(\mathbf{x})$ is given in terms of the charge distribution $\rho(\mathbf{x})$ by

$$V(\mathbf{x})=-e\int\frac{\rho(\mathbf{x}')d^3x'}{|\mathbf{x}-\mathbf{x}'|}\,.$$

We employ harmonic expansions for both $\rho(\mathbf{x})$ [cf. Eq. (2.7)] and the inverse distance. Assuming the model (2.11) we find that

$$V(\mathbf{x})=V_0(\mathbf{x})+V_2(\mathbf{x})\,,\quad\text{(A2)}$$

where

$$V_0(\mathbf{x})=-\frac{3Ze^2}{2R}\left(1-\frac{r^2}{3R^2}\right),\qquad r\leq R$$
$$=-\frac{Ze^2}{r},\qquad r>R \quad\text{(A3)}$$

$$V_2(\mathbf{x})=-\frac{4\pi q_2 e}{5}\frac{r^2}{R^3}P_2(\cos\theta),\qquad r\leq R$$
$$=-\frac{4\pi q_2 e}{5}\frac{R^2}{r^3}P_2(\cos\theta),\qquad r>R. \quad\text{(A4)}$$

The angle θ is measured with respect to the symmetry axis. For $L(\mathbf{x})$ we can write

$$L(\mathbf{x})=L_0(\mathbf{x})+L_2(\mathbf{x})\,,\quad\text{(A5)}$$

were $L_k(\mathbf{x})$ is the contribution from $V_k(\mathbf{x})$.

Calculation of $L_0(x)$

We compute

$$L_0(\mathbf{x})=-\int_0^\infty[V_0(\mathbf{y}_i)+V_0(\mathbf{y}_f)]\,ds\,,\quad\text{(A6)}$$

where

$$\mathbf{y}_i(s)=\mathbf{x}-\hat{\mathbf{k}}_i s\,,$$
$$\mathbf{y}_f(s)=\mathbf{x}+\hat{\mathbf{k}}_f s\,. \quad\text{(A7)}$$

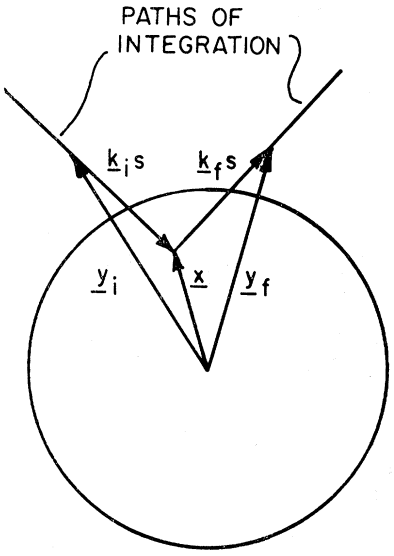

FIG. 3. Paths of integration for the integrals defining $L(\mathbf{x})$.

The path of integration is illustrated in Fig. 3. Discontinuities at the surface of the sphere correspond to $s=s_i$ and $s=s_f$, where

$$y_i(s_i)=R,$$
$$y_f(s_f)=R,$$

and $y=|\mathbf{y}|$. Explicitly,

$$s_i=-\zeta+(R^2-r^2+\zeta^2)^{1/2},$$
$$s_f=-\eta+(R^2-r^2+\eta^2)^{1/2}, \tag{A8}$$

where

$$\zeta\equiv-\mathbf{x}\cdot\hat{\mathbf{k}}_i, \quad \eta\equiv\mathbf{x}\cdot\hat{\mathbf{k}}_f. \tag{A9}$$

In the limit that $s\to\infty$ these integrals diverge logarithmically. This problem is handled by replacing the upper limit by $s=S$, where $S\gg R$. It will then be seen that the number S appears as an over-all phase in the scattering amplitude, so it is without physical significance.[22] The expression for $L_0(\mathbf{x})$ may be written in terms of invariants according to

$$L_0(\mathbf{x})=\frac{\gamma}{4R^3}[(5R^2-r^2)(s_i+s_f)+\tfrac{1}{3}(s_i^3+s_f^3)]+\gamma\ln\frac{2S}{R+s_i+\zeta}+\gamma\ln\frac{2S}{R+s_f+\eta}. \tag{A10}$$

Calculation of $L_2(x)$

$L_2(\mathbf{x})$ is defined by

$$L_2(\mathbf{x})=-\int_0^\infty[V_2(\mathbf{y}_i)+V_2(\mathbf{y}_f)]\,ds. \tag{A11}$$

[22] This prescription for the handling of the divergence was given by Tiemann (Ref. 3), who also calculated L_0 for a uniformly charged sphere. Our results below do not entirely agree with those given in that paper, but we do arrive at the same expression for the cross section, Eq. (18) of Ref. 3.

The procedure is straightforward. We find that

$$L_2(\mathbf{x})=(2\pi q_2e/5)\{s_iR^{-3}[\tfrac{1}{3}s_i^3(3b_i^2-1)-s_i(3ab_i+\zeta)+(3a^2-r^2)]+R^2[3a^2I_2-6ab_iI_3+3b_i^2I_4-I_1]\}+(2\pi q_2e/5)\{b_i\to b_f, s_i\to s_f, \zeta\to\eta\}, \tag{A12}$$

where

$$a=\mathbf{x}\cdot\hat{S},$$
$$b_i=\hat{\mathbf{k}}_i\cdot\hat{S}, \quad b_f=\hat{\mathbf{k}}_f\cdot\hat{S}, \tag{A13}$$

$\hat{S}$=unit vector along symmetry axis,

$$I_1=1/R^2(1+u), \quad I_2=(2+u)/[3R^4(1+u)^2],$$
$$I_3=\frac{1}{3R^3}\left[1-\frac{\zeta}{R}\frac{2+u}{(1+u)^2}\right],$$
$$I_4=\frac{1}{3R^2}\left[\frac{(1+s_i/R)^2}{(1+u)^2}+\frac{(s_i/R)^2}{1+u}\right], \tag{A14}$$
$$u=(1/R)[R^2-r^2+\zeta^2]^{1/2}.$$

The remaining parameters were previously defined in Eqs. (A8) and (A9).

APPENDIX B: INTEGRALS FOR THE SCATTERING AMPLITUDE

We give here a brief discussion[23] of the integrals (3.6). Introduction of the expansion (A5) allows us to write the amplitude $f(\mathbf{k}_f,\mathbf{k}_i)$ as in (3.5) and (3.14). The assertions concerning the asymptotic properties of f_l [see discussion following Eq. (3.7)] will be evident

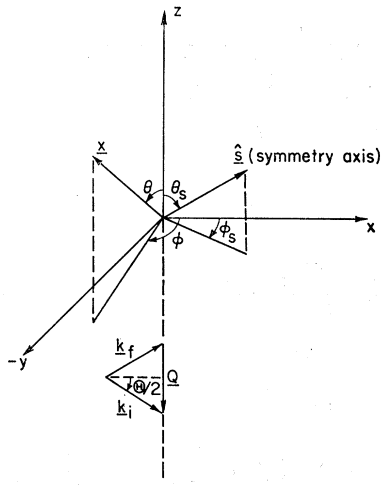

FIG. 4. Coordinate system used to perform the integrations for the scattering amplitude. The scattering is in the x-z plane, with $\mathbf{k}_i-\mathbf{k}_f=\mathbf{Q}$. The scattering angle is Θ.

[23] A more detailed treatment of these integrals may be found in the appendices of Ref. 5.

after a few of these integrals have been evaluated. All calculations will be carried to the orders indicated in Sec. IIIA. In all cases it is found that for purposes of obtaining asymptotic forms it is best to work in the coordinate system illustrated in Fig. 4. We still assume, of course, the model of Eqs. (2.11).

Integrals for f_0

In the coordinate system of Fig. 4 we may express $f_0^{(0)}$ according to the equations

$$f_0^{(0)}=(2me\rho_0/Q^2)\int_{-R}^{R}dze^{-iQz}F(z), \tag{B1}$$

$$F(z)=\iint_{(r\leq R)}dxdy\left[1+\frac{2}{Q}\frac{\partial L_0}{\partial z}\right]e^{iL_0(\mathbf{x})}. \tag{B2}$$

The integral on z may be developed into an asymptotic series by successive integrations by parts[24]:

$$\int_{-R}^{R}dze^{-iQz}F(z)=\sum_{n=0}\frac{-1}{(iQ)^{n+1}}[e^{-iQz}F^{(n)}(z)]^{z=R}_{z=-R}. \tag{B3}$$

It can be shown[23] that $F(z)$ possesses the analyticity at $z=\pm R$ required in order that this series make sense. It is also immediately obvious from Eq. (B2) that the leading term ($n=0$) vanishes, so the first nontrivial term ($n=1$) goes as $1/\beta^2$. The integrals for $f_0^{(1)}$ possess the same properties; here however $F(z)$ is defined by

$$F(z)=\iint_{(r\leq R)}dxdye^{iL_0(\mathbf{x})}\times\left[iL_2(\mathbf{x})\left(1+\frac{2}{Q}\frac{\partial L_0}{\partial z}\right)+\frac{2}{Q}\frac{\partial L_2}{\partial z}\right]. \tag{B4}$$

We find that [all symbols are defined in Eq. (4.8)]

$$f_0^{(0)}=\frac{6m\gamma}{Q^2\beta^2}e^{iL_0(0,0,R)}e^{i\psi}\left\{-\cos\alpha+\frac{1}{\beta}[\sin\alpha+10\gamma\sin^3(\tfrac{1}{2}\Theta)\cos\alpha-i\gamma\sin\alpha(7-10\sin^3(\tfrac{1}{2}\Theta))]\right\}, \tag{B5}$$

$$\begin{aligned}f_0^{(1)}=&\frac{4m\gamma(4\pi eq_2)}{5Q^2\beta^2}e^{iL_0(0,0,R)}e^{i\psi}\\&\times\{\sin\alpha[\sin(\tfrac{1}{2}\Theta)(1-4\sin^2(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta))P_2(\mu_Q)+2\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)]P_2(\mu_T)]\\&-i\cos\alpha[(1+\sin(\tfrac{1}{2}\Theta)[1-4\sin^2(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)]+[1+\sin(\tfrac{1}{2}\Theta)]^{-1})P_2(\mu_Q)\\&+(2\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)]+[1-\sin(\tfrac{1}{2}\Theta)][1+\sin(\tfrac{1}{2}\Theta)]^{-1})P_2(\mu_T)]\}.\end{aligned} \tag{B6}$$

Integrals for f_2

Because these integrals all have a factor $\delta(r-R)$, we work here in spherical polar coordinates and absorb the radial integral immediately. Then we find that $f_2^{(1)}$ can be defined by

$$f_2^{(1)}=(2meq_2/Q^2)\sum_{m=0}^{2}P_{2,m}(\cos\theta_s)(-)^mN_mI_m, \tag{B7}$$

where

$$I_m(\beta)=\int_0^{\pi}\sin\theta d\theta\, e^{-i\beta\cos\theta}\mathfrak{F}_m(\theta)P_{2,m}(\cos\theta), \tag{B8}$$

$$\mathfrak{F}_m(\theta)=\int_0^{2\pi}F(\theta,\varphi)\cos m(\varphi-\varphi_s)\,d\varphi, \tag{B9}$$

and

$$F(\theta,\varphi)=\left\{e^{iL_0(\mathbf{x})}\left[1+\frac{2}{\beta}R\frac{\partial L_0}{\partial z}\right]\right\}_{r=R}. \tag{B10}$$

[N_m is defined by Eq. (4.5).] The integrals for $I_m(\beta)$ can be developed asymptotically by successive integrations by parts, as in Eq. (B3), whenever $\mathfrak{F}_m(\theta)P_{2,m}(\cos\theta)$ is analytic at $\theta=0$ and $\theta=\pi$. However this is found not always to be the case. In these instances an asymptotic series may be developed by an application[25] of Bromwich's theorem,[26] which states that if $F(x)$ has limited total fluctuation when $x\geq 0$, and if γ is such that $\nu\gamma\to\infty$ when $\nu\to\infty$, then

$$\nu^{\mu}\int_0^{\gamma}x^{\mu-1}F(x)\sin\nu x\,dx\to F(0+)\Gamma(\mu)\times\sin(\mu\pi/2),(-1<\mu<1). \tag{B11}$$

If $0<\mu<1$, the sine function may be replaced by cosine throughout. Bromwich's theorem gives directly only the leading term in the asymptotic series, but the higher terms may be obtained by differentiation. Assuming

$$F(x)=F(0)+xF'(0),$$

[24] This technique is due to Tiemann, Ref. 3.
[25] We are indebted to C. K. Iddings for suggesting this solution.
[26] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (The Macmillan Company, New York, 1944), 2nd edition, p. 230.

we need know only that

$$\int_0^1 x^{-1/2}\begin{pmatrix}\sin\beta x\\ \cos\beta x\end{pmatrix}dx \to (\pi/2\beta)^{1/2},$$
$$\int_0^1 x^{-1/2}\times x\begin{pmatrix}\cos\beta x\\ -\sin\beta x\end{pmatrix} \to -(2\beta)^{-1}(\pi/2\beta)^{1/2}. \quad \text{(B12)}$$

It is found that whenever application of Bromwich's theorem is successful, integration by parts fails, and vice versa. However, the integral for $f_2^{(2)}$ contains both types of behavior, so each type must be isolated. The equations defining $f_2^{(2)}$ are as above [i.e., (B7)–(B9)], but here $F(\theta,\varphi)$ is given by (to order $1/\beta$)

$$F(\theta,\varphi)=\{e^{iL_0(\mathbf{x})}iL_2(\mathbf{x})\}_{r=R}. \quad \text{(B13)}$$

The results are

$$f_2^{(1)}=[2m(4\pi eq_2)/Q^2\beta]e^{iL_0(0,0,R)}e^{i\psi}P_2(\mu_Q)\times[\sin\alpha+\beta^{-1}(3\cos\alpha-8\gamma\sin\alpha\sin^3(\tfrac{1}{2}\Theta))-i\gamma\beta^{-1}\cos\alpha(5-8\sin^3(\tfrac{1}{2}\Theta))], \quad \text{(B14)}$$

$$f_2^{(2)}=[4m(4\pi eq_2)^2/15\beta Q^2]e^{iL_0(0,0,R)}e^{i\psi}P_2(\mu_Q)\times\{P_2(\mu_Q)[\sin(\tfrac{1}{2}\Theta)(1-4\sin^2(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta))e^{i\alpha}+i\sin\alpha(1+[1+\sin(\tfrac{1}{2}\Theta)]^{-1})]+P_2(\mu_T)[2\sin(\tfrac{1}{2}\Theta)\cos^2(\tfrac{1}{2}\Theta)[1+2\sin^2(\tfrac{1}{2}\Theta)]e^{i\alpha}+i\sin\alpha([1-\sin(\tfrac{1}{2}\Theta)][1+\sin(\tfrac{1}{2}\Theta)]^{-1})]\}. \quad \text{(B15)}$$

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Excitation Functions for Simple Nuclear Reactions of π^- Mesons with Ar^{40}†

CHARLES O. HOWER AND SHELDON KAUFMAN
Chemistry Department and Princeton-Pennsylvania Accelerator, Princeton University, Princeton, New Jersey

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Cross sections have been measured for the reactions $Ar^{40}(\pi^-,\pi^-p)Cl^{39}$ and $Ar^{40}(\pi^-,\pi^-pn)Cl^{38}$ in the energy range 500–1100 MeV. The excitation function of the former reaction has peaks at about 600 and 900 MeV pion kinetic energy, corresponding to known resonances in the π^--p scattering cross section. This implies that a quasifree pion-proton collision occurs as the initial step of this simple reaction. The (π^-,π^-pn) excitation function, in contrast, is essentially independent of energy in this energy range, indicating a mechanism or combination of mechanisms sufficiently complex to average out the structure in the π^--p and π^--n cross sections. The effect of the proton momentum distribution on the π^--p resonances is calculated, and by comparison with the (π^-,π^-p) cross section a momentum distribution is deduced which is in agreement with the results of $(p,2p)$ quasifree scattering experiments. Cross sections for the reactions $Ar^{40}(\pi^-,\pi^-2p)S^{38}$, $Ar^{40}(p,2p)Cl^{39}$, and $Ar^{40}(p,2pn)Cl^{38}$ at a single energy were also measured.

I. INTRODUCTION

THE use of energetic π mesons as bombarding particles in the study of nuclear reactions has several advantages. The presence of three charge states of the meson permits the study of charge-exchange reactions in which either one or two units of charge are transferred. There is a clear distinction between the incident particle and the particles in the target nucleus, so that there is no ambiguity as to whether the incident particle is absorbed or re-emitted. Finally, the pion-nucleon total cross sections exhibit pronounced resonances, in contrast to the relatively smooth nucleon-nucleon cross sections. This last feature is the basis of the present experiments.

The Serber model[1] of high-energy nuclear reactions assumes that the initial step of a reaction is an interaction of the projectile with a single nucleon in the nucleus. The extent to which the type of initial interaction affects the probability of forming a particular product can be studied by bombarding the same nucleus with different projectiles, especially when there are large differences in the projectile-nucleon cross sections. An alternative experiment is to use a projectile which has widely different cross sections with protons and neutrons, in order to learn what effect the initial interaction has.

The class of reactions (x,xN), where x is a bombarding particle and N is a nucleon, is one in which the initial interaction can be important. The clean knockout mechanism,[2] in which x collides with N and both particles leave the nucleus, which then has insufficient excitation energy to evaporate additional particles, would be sensitive to the x-N scattering cross section. On the other hand, a mechansim in which particle x scattered inelastically from the nucleus and the nucleus de-excited by evaporating nucleon N would not be,

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[2] J. R. Grover and A. A. Caretto, Jr., Ann. Rev. Nucl. Sci. 14, 51 (1964).