With these results, the isospin eigenvalue problem is

$$\tfrac{1}{2}m^2\alpha(m-2)+[2N+(2N-1)m-2m^2]\alpha(m)+2(N-m)(N-m-1)\alpha(m+2)=T(T+1)\alpha(m). \quad \text{(II.6)}$$

If we now add $N(N-3)\alpha(m)$ to both sides of (II.6), then we obtain Eq. (5.8) with κ given by (5.17). Thus we have proven that the states (5.2) are isospin eigenstates and we have given an alternative derivation of the value of κ (5.17).

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Effective Interaction for Nuclear Matter

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The effective interaction in nuclear matter is defined as the long-range part of the two-body potential, which, in the Born approximation, gives the single-particle potential energy for the average momentum in the Fermi sea. For the Brueckner-Gammel-Thaler potential the effective interaction has been calculated, first for the free-particle propagator and then for the nuclear spectrum. The result shows that in the first case the separation distance ξ_0 is constant over a wide range of densities and does not lead to saturation. The nuclear separation distance $\xi(k_F)$ changes quite rapidly with the Fermi momentum k_F; for low densities it is very close to ξ_0, while for higher densities it becomes very much larger. At the density corresponding to $k_F=1.5$ F^{-1}, the long-range potential starts at $\xi=1.16$ F, and the rate of change of ξ with k_F is $(d\xi/dk_F)=0.8$ F^2. The minimum of the total energy per particle occurs at $k_F=1.35$ F^{-1} and is about -9 MeV. For $k_F=1.5$ F^{-1} the contributions of different partial waves are also calculated by a variational technique, and the results have been compared with previous calculations.

1. INTRODUCTION

NUCLEON-nucleon scattering at high energies suggests that nuclear forces become strongly repulsive at small distances. If these forces bind a system of many nucleons together, then at least for low densities, the interaction on the average must be attractive, and the effect of the repulsive part cancels only part of the effect of the attractive potential. Therefore it is possible to find an effective interaction which depends on the density and represents the remaining part of the attractive force.

Moszkowski and Scott[1] originally introduced the idea of separating the interaction in the two-body t matrix in such a way that the short-range part contributes nothing to the phase shift in each partial wave. Then the effective interaction, to the first order, is the Born approximation of the long-range part. Here we introduce an average separation distance for all of the states in the Fermi sea by the requirement that the expectation value of the short-range part of the many-body Hamiltonian should vanish. The long-range part which is the effective Hamiltonian will depend on the Fermi momentum (or density) of the system of nucleons. Thus the effect of the short-range part of the interaction may be replaced by a separation distance $\xi(k_F)$ and other physical quantities of interest, like the rearrangement energy and the compressibility, can be expressed in terms of the long-range interaction $\xi(k_F)$ and its derivatives with respect to k_F.[2] From the definition of $\xi(k_F)$ it is clear that its functional form depends on the shape and the strength of the short-range interaction and for different potentials it takes different forms. Here it is assumed that the two-nucleon interaction is given by the Brueckner-Gammel-Thaler (BGT) potential.[3] This potential is preferred over the semiphenomenological potentials of Breit[4] and Hamada and Johnson[5] for two reasons: (a) The BGT potential has a simple analytic form and (b) there are at least three other independent calculations of the binding energy of nuclear matter with this potential. It is therefore possible to compare our calculations with the results of other calculations.

Since the exact solution of the reaction matrix is quite complicated we calculate the separation distance by iteration. In Sec. 2 we find the integral equation for the reaction matrix and the condition that the solution of this equation must satisfy in order that the energy shift due to the perturbation be zero. In Sec. 3 we solve the reaction matrix by neglecting the exclusion principle and using the free-particle propagator e^F. We find that

[1] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960), hereafter referred to as MS.

[2] M. Razavy and S. J. Stack, Can. J. Phys. **43**, 605 (1965).

[3] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[4] K. E. Lassila, M. H. Hull, Jr., H. M. Rupple, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

[5] T. Hamada and I. D. Johnson, Nucl. Phys. **34**, 383 (1962).

the separation point ξ_0 is the boundary point of the solution of a set of differential equations. A more accurate value of ξ with the nuclear spectrum is found in Sec. 4, and from this value of ξ, the binding energy per particle is obtained. In the last section the complete reaction matrix is solved by variational method and the average potential energy per particle is found and compared with the value obtained by Brueckner and Masterson.[6]

2. THE EFFECTIVE INTERACTION

We consider a many-particle system interacting through two-body forces $v(r_{ij})$, and we divide the Hamiltonian of this system, H, into two parts H_0 and H^1:

$$H_0=\sum_i T_i+\tfrac{1}{2}\sum_{i,j} v(r_{ij})\theta(r_{ij}-\xi)\,, \tag{2.1}$$

$$H^1=\tfrac{1}{2}\sum_{i,j} v(r_{ij})\theta(\xi-r_{ij})\,, \tag{2.2}$$

where T_i is the kinetic-energy operator for the particle in the state i, θ is the step function, and ξ is a parameter which depends on the density of the system and will be determined later. If Φ_0 and Ψ_0 denote the ground-state eigenfunctions of H_0 and H, respectively, and E_0 and E the corresponding eigenvalues, then

$$E=E_0+\langle\Phi_0|H^1|\Psi_0\rangle \tag{2.3}$$

where we have used the normalization $\langle\Phi_0|\Psi_0\rangle=1$. The energy shift $\Delta E=\langle\Phi_0|H^1|\Psi_0\rangle$ is a function of the parameter ξ.

For small values of ξ when H^1 contains only the repulsive part of the potential, ΔE is positive, but when $\xi=\infty$ assuming the existence of a bound state for the system, ΔE must be negative. Thus if ΔE is a continuous function of ξ, it is possible to choose ξ so that

$$\Delta E(\xi)=0\,. \tag{2.4}$$

Equation (2.4) determines ξ as a function of the density, the range, and the strength of the interaction. To reduce the many-body equations (2.1)–(2.3) to equations involving two nucleons we first consider only two-body excitations and write the Hamiltonian for the two interacting nucleons as

$$h_0=T_i+T_j+v(r_{ij})\theta(r_{ij}-\xi)\,. \tag{2.5}$$

If we sum h_0 over all possible values of i and j we find $2H_0$ as given by (2.1). The effect of the rest of nucleons on the pair ij is represented in (2.5) by the presence of the cutoff ξ. Let $\varphi(mn)$ be the set of the eigenfunctions of h_0 with eigenvalues $e(m)+e(n)$. We expand the two-nucleon wave function in nuclear matter $\psi(mn)$ in terms of $\varphi(mn)$ to find the integral equation

$$\psi(mn)=\varphi(mn)+\sum_{k_a,k_b>k_F} \frac{\varphi(mn)\langle\varphi(ab)|v(r')\theta(\xi-r')|\psi(ab)\rangle}{e(m)+e(n)-e(a)-e(b)}\,. \tag{2.6}$$

The energy denominator in (2.6) may be modified so that the effect of higher order correlations is included in the determination of $\psi(mn)$. Then $e(a)+e(b)$ will no longer be the eigenvalues of h_0 and have little physical significance. These particle energies [$e(a)$ and $e(b)$] are sensitive to the form of the potential at short distances, and they should be calculated self-consistently from the reaction matrix. Once their forms as functions of the momenta of the particles k_a and k_b are known, then $\psi(mn)$ can be determined from (2.6). To find the separation distance ξ we put the diagonal elements of the reaction matrix equal to zero, i.e.,

$$\int_0^{\xi(mn)} \varphi(mn,r)v(r)\psi(mn,r)d^3r=0\,, \tag{2.7}$$

This relation is consistent with (2.4) but ξ now depends on the momenta $\mathbf{k}_m$ and $\mathbf{k}_n$ of the two nucleons. If $\mathbf{k}=\frac{1}{2}(\mathbf{k}_m-\mathbf{k}_n)$ is the relative momentum of this pair, we put the average of the matrix element given by (2.7) over the Fermi sea equal to zero, i.e., we find a $\xi(k_F)$ which satisfies the equation

$$\int_0^{\xi(k_F)}\int_0^{k_F} \frac{24k^2}{k_F{}^3}\left(1-\frac{2k}{3k_F}+\frac{k^3}{2k_F{}^3}\right)dk \times\varphi(\mathbf{k}\cdot\mathbf{r})v(r)\psi(\mathbf{k}\cdot\mathbf{r})d^3r=0\,. \tag{2.8}$$

It is also possible to evaluate ξ for the average momentum in the Fermi sea from Eq. (2.7). If ξ does not change very rapidly with k_F both methods will yield nearly the same result. Apart from the problem of self-consistent calculation of the particle energies $e(a)$ and $e(b)$, which has been discussed extensively in several papers,[7,8] ξ can be determined in the following way. We assume some value for ξ for a given density, from (2.5) we find h_0 and this in turn determines $\varphi(mn)$, $e(m)$, and $e(n)$. Then solving Eq. (2.6) we find the exact two-body wave function $\psi(mn)$, and using $\varphi(mn)$ and $\psi(mn)$ in (2.7) or (2.8) ξ can be calculated. This result should agree with the value of ξ used at the start of the calculation.

3. FIRST-ORDER CALCULATION

The short-range reaction matrix G^s defined by

$$\langle ij|G^s|mn\rangle=\langle\varphi(ij)|v(r)\theta(\xi-r)|\psi(mn)\rangle \tag{3.1}$$

[6] K. A. Brueckner and K. S. Masterson, Phys. Rev. **128**, 2267 (1962).

[7] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963), hereafter referred to as BBP.

[8] G. E. Brown, G. T. Shappert, and C. W. Wong, Nucl. Phys. **56**, 191 (1964).

satisfies the integral equation

$$G^s=v^s+v^s(Q/e^N)G^s, \tag{3.2}$$

which is obtained by substituting (2.6) in (3.1). The operator Q excludes the states inside the Fermi sea, and e^N is the energy denominator in (2.6) with the modified particle energies $e(a)$ and $e(b)$. An accurate value of the separation distance ξ satisfies Eq. (2.8) or

$$\int\langle\tfrac{1}{2}(\mathbf{k}_m-\mathbf{k}_n)|G^s|\tfrac{1}{2}(\mathbf{k}_m-\mathbf{k}_n)\rangle d\mathbf{k}_m d\mathbf{k}_n=0, \tag{3.3}$$

where the integration is over the Fermi sea. An approximate value for the separation distance may be found from the phase shifts and the long-range part of the potential ($r>1$F), if we make the following changes in Eq. (3.2): (a) Substitute the hole and particle energies in (2.6) and (3.2) by the kinetic energies, i.e., use a free particle propagator e^F. (b) Put $Q=1$ for all of the states m and n. (c) Use plane waves for the wave function $\varphi(mn)$:

$$\varphi(mn)=\exp[i(\mathbf{k}_m\cdot\mathbf{r}_m+\mathbf{k}_n\cdot\mathbf{r}_n)].$$

With these substitutions the diagonal elements of the reaction matrix G^{sF} can be written in terms of phase shifts. Denoting the relative momentum by $\mathbf{k}=\frac{1}{2}(\mathbf{k}_m-\mathbf{k}_n)$ we have

$$\langle k|G^{sF}|k\rangle=-(\pi/2k)\sum_{T,J}(2T+1)(2J+1)\times\delta_l^{T,J}(k,\xi_0)=0, \tag{3.4}$$

where $\delta_l^{T,J}(k,\xi_0)$ is the phase shift due to the short-range part of the potential, T is the isospin, and J the total angular momentum of the two nucleons. The quantity $\delta_l^{T,J}(k,\xi_0)$ is the boundary value (at $r=\xi_0$) of the solution of the first-order differential equation[9]

$$d\delta_l^{T,J}(k,r)/dr=-kr^2v_l^{T,J}(r)\times[\cos\delta_l^{T,J}(k,r)j_l(kr)-n_l(kr)\sin\delta_l^{T,J}]^2 \tag{3.5}$$

with the boundary condition at infinity

$$\delta_l^{T,J}(k,\infty)=\delta_l^{T,J}(k) \tag{3.6}$$

and the subsidiary condition (3.4).

We have integrated Eq. (3.5) for the states $^1S_0$, $^1P_1$, $^3P_0$, $^3P_1$, $^3P_2$, $^1D_2$, and $^3D_2$ with the BGT[3] potential for the average momentum $\bar{k}=(\sqrt{0.3})k_F$ for different k_F's ($k_F=1.1$–1.7 F^{-1}). The phase function $\delta_l^{T,J}(\bar{k},r)$ as a function of r is plotted in Figs. 1 and 2. For a better accuracy we have integrated Eq. (3.5) from the core to infinity. The common separation distance ξ_0 obtained in this way is practically constant when k_F is between 1.1 and 1.7 F^{-1}, and is equal to 1.07 F. In this calculation we have neglected the coupling between the $^3P_1$ and $^3F_1$ states. The $^3D_1$ and $^3D_3$ waves have not been included and their effects are assumed to be small. For the coupled triplet states $^3S_1-^3D_1$ we have used the factorization method[10] to find the separation point. In the $^3S_1$ state the separation distance increases slowly with k_F. This variation is shown in Table I,

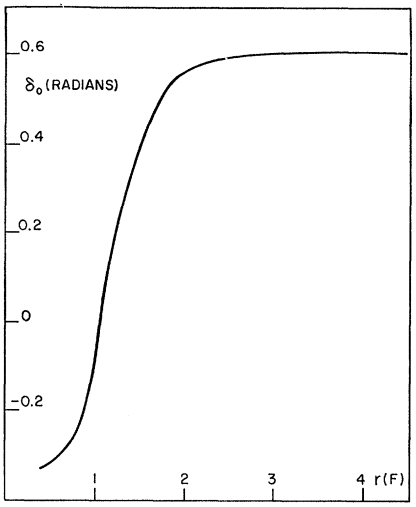

FIG. 1. The $^1S_0$ phase function for the BGT potential.

TABLE I. First-order values of the separation distance ξ_0, for the $^3S_1$ state for the average momentum in the Fermi sea $\bar{k}=(\sqrt{0.3})k_F$.

k_F (F^{-1})	1.1	1.2	1.3	1.4	1.5	1.6	1.7
ξ_0 (F)	0.97	0.98	0.99	1.01	1.02	1.05	1.09

where $\xi_0(\bar{k})=\xi_0(k_F)$ with $\bar{k}=(\sqrt{0.3})k_F$ is given for different values of k_F. Thus neglecting the effect of the higher partial waves, the average value of ξ_0 is about 1 F. In the next section it will be shown that this value is very close to the exact ξ when the density is low ($k_F<1.5$ F^{-1}).

The diagonal elements of the long-range part of the potential (to the first order) are the matrix elements of the potential $v(r)\theta(r-\xi_0)$ with the plane waves

$$\langle k|v(r)\theta(r-\xi_0)|k\rangle=\tfrac{1}{2}\pi\sum_{T,J}(2J+1)(2T+1)\times\int_{\xi_0}^{\infty}j_l^2(kr)v_l^{T,J}(r)r^2dr. \tag{3.7}$$

The average of the matrix element (3.7) over the Fermi sea is the effective interaction to the first order. To compare the result of our calculation with that of Moszkowski and Scott (MS)[1] we have calculated the single-particle potential energy $\bar{U}$ by multiplying the matrix element (3.7) for the average momentum by the the density $\rho=2k_F^3/3\pi^2$ to give

$$\bar{U}=(2k_F^3/3\pi^2)\langle\bar{k}|v(r)\theta(r-\xi_0)|\bar{k}\rangle, \tag{3.8}$$

[9] F. Calogero, Nuovo Cimento **27**, 261 (1963).

[10] M. Razavy, Phys. Rev. **130**, 1091 (1963).

TABLE II. The contributions of different partial waves to the average energy per particle (in units of MeV).

k_F (F^{-1})	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	$^3D_1$	$^3D_2$	$^3D_3$	Total	MS
1.1	−9.68	−5.81	2.35	−3.37	5.08	−3.99	−1.03	0.29	−1.50	−0.43	−17.87	−17.2
1.2	−11.50	−6.98	3.17	−4.36	6.75	−5.46	−1.67	0.37	−2.32	−0.64	−22.35	−21.4
1.3	−13.80	−8.18	4.14	−5.47	8.70	−7.24	−2.56	0.60	−3.41	−0.90	−26.84	−26.2
1.4	−15.30	−9.56	5.28	−6.66	10.90	−9.35	−3.77	0.84	−4.80	−1.19	−32.69	−31.5
1.5	−17.10	−10.26	6.47	−7.93	13.46	−11.80	−5.35	1.12	−6.50	−1.50	−38.33	−37.3
1.6	−18.80	−11.94	7.80	−9.20	16.20	−14.60	−7.34	1.47	−8.60	−1.84	−45.39	−43.7
1.7	−20.30	−10.81	9.29	−10.50	19.36	−17.70	−9.80	1.86	−11.08	−2.17	−53.04	−50.7

where $\bar{k}=(\sqrt{0.3})k_F$. The results are given for each partial wave separately in Table II. The total contribution to the single-particle potential energy in our computation (−38.3 MeV) is almost identical to those of Azziz and Signell[11] (−38 MeV) and MS[1] (−37.3 MeV). However, our calculation gives different results in each partial wave, and this is especially noticeable for $^1S_0$ and $^3P_2$ waves. In our work there is no sign of saturation in the range of densities corresponding to the Fermi momenta $k_F=1.1$–1.7 F^{-1}; a result identical to that of MS.

4. SECOND-ORDER CALCULATION

To find the exact separation distance it is necessary to obtain the correct energy denominator e^N for the G^s matrix. This quantity e^N consists of two parts; the hole energies $e(m)+e(n)$ and the particle energies $e(a)+e(b)$. The hole energies are related to the long-range part of the potential that has been found in Sec. 3, and only particle energies remains to be calculated. As we mentioned earlier, these particle energies should be obtained self-consistently, and the result is very sensitive to the interaction at very short distances. Here, instead of calculating these intermediate energies accurately, we extend the method of separating the potential and compute the single-particle energy in the first Born approximation. However, we use the result of the work of Rajaraman[12] and assume that the potential energy in the intermediate states is give by

$$1\times(\text{contribution from even-}l\text{ states only}).$$

If $U(k_b)$ denotes the potential energy of a particle of momentum k_b ($k_b>k_F$), and $\xi_e(k)$ is the common separation distance for even states only [$\mathbf{k}=\frac{1}{2}(\mathbf{k}_b-\mathbf{k}_n)$], then

$$U(k_b)=\frac{2}{\pi^2}\sum_{\text{even }l}(2l+1)\int_{k_n<k_F}d\mathbf{k}_n\int_{\xi_e(k)}^{\infty}j_l^2(kr)v(r)r^2dr. \tag{4.1}$$

For mathematical simplicity, we fit $U(k_b)$ by a quadratic function of k_b,

$$U(k_b)=\left(\frac{1}{m^*}-1\right)\frac{k_b^2}{2}+\Delta, \tag{4.2}$$

where m^* is the effective-mass parameter and Δ is a constant. For the potential energy of holes $U(k_m)$ ($k_m<k_F$) we assume that $U(k_m)$ has the following form:

$$U(k_m)=\left(\frac{1}{m^*}-1\right)\tfrac{1}{2}k_m^2+\Delta-\tfrac{1}{2}\left(\frac{a+bk_m^2}{1+ck_m^2}\right). \tag{4.3}$$

Here m^* and Δ have the same numerical values as in (4.2) and a, b, and c are constants. The energy denominator e^N in Eq. (3.2) may be expressed in terms of these constants, namely

$$e^N(k,k',P)=e(m)+e(n)-e(a)-e(b)=\frac{k^2-k'^2}{m^*}$$
$$-\left\{\frac{a+(b+ca)(\tfrac{1}{4}P^2+k^2)+cb(\tfrac{1}{4}P^2-k^2)^2}{1+2c(\tfrac{1}{4}P^2+k^2)+c^2(\tfrac{1}{4}P^2-k^2)^2}\right\}, \tag{4.4}$$

where $\mathbf{k}$ and $\mathbf{k}'$ are the relative momenta in the initial and in the intermediate states and $\mathbf{P}$ is the momentum of the center of mass. The effect of P in e^N is quite small, and we put $P=0$ to simplify Eq. (4.4). The short-range potential $v(r)\theta(\xi-r)$ has large Fourier components

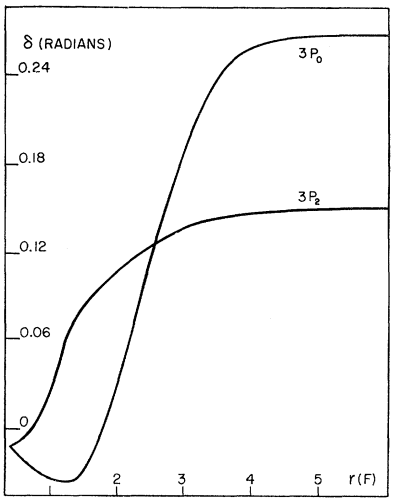

FIG. 2. The $^3P_0$ and $^3P_2$ phase functions for the BGT potential.

[11] N. Azziz and P. Signell, Nucl. Phys. **59**, 444 (1964).

[12] R. Rajaraman, Phys. Rev. **129**, 265 (1963). See also H. A. Bethe, *ibid.* **138**, B804 (1965).

for $k>k_F$, and will scatter the nucleons outside the Fermi sea, so that we may put $Q=1$ in Eq. (3.2) without changing the diagonal elements of G^s appreciably. The result can be corrected in a simple way.[8] Approximating $\varphi(mn)$ by a plane wave, it is not difficult to show that the wave function satisfies the following partial differential equation[7]:

$$[\nabla^2-\gamma^2(k)][\psi(\mathbf{r})-\exp(i\mathbf{k}\cdot\mathbf{r})] =m^*v(r)\theta(\xi-r)\varphi(r), \quad (4.5)$$

where

$$\gamma^2(k)=m^*(a+bk^2)/(1+ck^2)-k^2.$$

The partial-wave expansion reduces (4.5) to a set of total differential equations:

$$\left[\frac{d^2}{dr^2}-\left(\frac{l(l+1)}{r^2}+\gamma^2\right)\right][u_l(r)-rj_l(kr)] =m^*v(r)\theta(\xi-r)u_l(r), \quad (4.6)$$

with boundary conditions

$$u_l(c)=0 \quad \text{and} \quad \left[\frac{d(u_l-rj_l)/dr}{u_l-rj_l}\right]_{r=\xi} =\left[\frac{d[rh_l^{(1)}(i\gamma r)]/dr}{rh_l^{(1)}(i\gamma r)}\right]_{r=\xi}, \quad (4.7)$$

where ξ should be determined from the condition

$$\tfrac{1}{2}\pi\sum_{T,J}(2T+1)(2J+1)\int_0^{\xi} rj_l(kr)v_l^{T,J}(r)u_l(r)dr=0. \quad (4.8)$$

The exact solution of the differential equation (4.6) subject to the conditions (4.7) and (4.8) is difficult, because ξ is unknown. However, we can solve the problem approximately by taking ξ_0 instead of ξ to be one of the boundaries of Eq. (4.6), and take $u_l(r)\approx rj_l(kr)$ in the interval ξ_0 to ξ. The result of our calculation shows that at least for low densities ξ is close to ξ_0, so that this is a valid approximation. Equation (4.8) in this case takes the form

$$\sum_{T,J}(2J+1)(2T+1)\left\{\int_0^{\xi_0} rj_l(kr)v_l^{T,J}(r)u_l(r)dr +\int_{\xi_0}^{\xi} r^2j_l^2(kr)v_l^{T,J}(r)dr\right\}=0. \quad (4.9)$$

The first integral in (4.9) may be written in terms of the distortion of the wave function $\chi_l(r)=rj_l(kr)-u_l(r)$:

$$\int_0^{\xi_0} rj_l(kr)v_l^{T,J}(r)u_l(r)dr=\left(\frac{\gamma^2+k^2}{m^*}\right)\int_0^{\xi_0} rj_l(kr)\chi_l(r)dr +\left[rj_l(kr)\frac{d}{dr}\chi_l(r)-\chi_l(r)\frac{d}{dr}[rj_l(kr)]\right]_{r=\xi_0}. \quad (4.10)$$

From Eqs. (4.5), (4.9), and (4.10) the separation distance ξ can be determined. In our numerical work we have first calculated the particle energies from the integral

$$U(k)=\frac{8k_F^3}{3\pi^2}\sum_{\text{even } l}(2l+1)\int_{\xi_e(k)}^{\infty} j_l^2(kr)v_l(r)r^2dr \quad (4.11)$$

and then fitted $U(k)$ with a quadratic function of k. The results for two values of k, namely $k=1.5k_F$ and $k=2k_F$ together with the separation distances for even states $\xi_e(k)$ are given in Table III. From (4.2) we have

TABLE III. The separation distance and the potential energy in the intermediate states. The symbols used are defined in Eq. (4.1).

	$k=1.5k_F$		$k=2k_F$	
k_F (F^{-1})	ξ_e (F)	U (MeV)	ξ_e (F)	U (MeV)
1.3	1.03	−20.5	1.06	−18.3
1.4	1.03	−25.0	1.08	−21.7
1.5	1.04	−30.1	1.10	−25.5
1.6	1.04	−35.5	1.12	−29.2

calculated m^* and Δ and using the results obtained in the preceding section we have found $U(k_m)$ from (4.3) and e^N from (4.4). The constants of Eq. (4.3) are given in Table IV. Finally, using the factorization method[10]

TABLE IV. The effective mass and other constants of the energy denominator e^N [Eqs. (4.2) and (4.3)].

k_F (F^{-1})	γ^2 (F^{-2})	m^*	a (F^{-2})	b	c (F^2)
1.3	0.88	0.982	2.49	−0.009	1.55
1.4	1.07	0.956	3.25	−0.011	1.55
1.5	1.17	0.946	4.00	−0.014	1.55
1.6	1.42	0.936	4.84	−0.017	1.55

we have integrated (4.6), and from (4.9) and (4.10) have found the new separation distance. With the new ξ we have calculated $\langle\bar{k}|v(r)\theta(r-\xi)|\bar{k}\rangle$ and $\bar{U}$ from Eqs. (3.7) and (3.8). In Table V and Fig. 3 the separation distances ξ for different values of k_F are given. In the same table we have given the corrections to the first-order contribution of each partial wave to the average single-particle potential energy. In this calculation the $^3S_1$ state is again separated from the rest of the partial waves. Hence we have given two separation distances $\xi(^3S_1)$ and $\xi(^1S_0-{}^3D_2)$, but it seems that at least for the range of the k_F's that we have considered they are very close and there is no need for further computation to obtain a single value of ξ for a given density. The corrected separation distance changes the contributions of different partial waves to the effective interaction. These changes are very significant in the $^3P_1$ and $^1D_2$ states but they have opposite signs and partly cancel each other. The self-consistency for the effective interaction can be verified by using the cor-

TABLE V. The separation distance ξ (k_F) and the correction to the single-particle potential energy (in units of MeV).

k_F (F^{-1})	ξ ($^1S_0-^3D_2$)	ξ ($^3S_1$)	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	$^3D_2$	Total
1.3	1.05	1.05	1.15	0.76	0.21	0.64	2.11	1.22	−5.21	−1.1	−0.43
1.4	1.10	1.10	2.19	1.43	0.36	1.08	3.77	1.96	−7.35	−1.67	1.78
1.5	1.17	1.14	3.20	1.90	0.49	1.87	6.37	2.81	−9.59	−2.02	4.86
1.6	1.28	1.23	4.69	3.50	0.77	4.61	20.95	4.26	−11.8	−2.83	24.1

rected ξ to find $U(k_m)$, e^N, and $\psi(r)$ and to determine ξ again, which should be the same as before. As ξ_0 and ξ are not very different in our calculation and as we have not used a self-consistent particle energy $U(k_b)$, we will not try to establish this self-consistency.

Once the corrected value of ξ has been found we can use ordinary perturbation method to calculate hole energies. The unperturbed Hamiltonian in this case is the kinetic energy and the long-range part of the potential $v^l=v(r)\theta(r-\xi)$is the perturbation. Using the Raleigh-Schrödinger expansion, the second-order correction to the contribution of the long-range potential is Δv^l

$$\Delta v^l=\frac{1}{(2\pi)^3}\int|\langle k|v|k'\rangle|^2\frac{Q(k,k')d\mathbf{k}'}{e^F(k,k')} \tag{4.12}$$

or in terms of the partial waves

$$\langle k|\Delta v^l|k\rangle=\sum_{J,T}(2T+1)(2J+1)\int_0^\infty\frac{Q(\mathbf{k}',\mathbf{P})}{k^2-k'^2}k'^2dk'$$
$$\times\left[\int_{\xi(k)}^\infty j_l(k'r)v(r)j_l(kr)r^2dr\right]^2. \tag{4.13}$$

This correction changes $\bar{U}$ to $(\bar{U}+\Delta\bar{U},)$ where

$$\Delta\bar{U}=(2k_F^3/3\pi^2)\langle\bar{k}|\Delta v^l|\bar{k}\rangle \tag{4.14}$$

with $\bar{k}=(\sqrt{0.3})k_F$. To simplify the above integrals we substitute $Q(\mathbf{k}',\mathbf{P})$ by its average value over the angular variables of k' [1,8]:

$$\bar{Q}=0 \quad \text{if} \quad k'^2+\tfrac{1}{4}P^2<k_F^2,$$
$$\bar{Q}=1 \quad \text{if} \quad k'-\tfrac{1}{2}P>k_F, \tag{4.15}$$
$$\bar{Q}=(k'P)^{-1}(k'^2+\tfrac{1}{4}P^2-k_F^2) \quad \text{otherwise},$$

and we also use the average of P^2 in (4.14), i.e.

$$\bar{P}^2=(12/5)k_F^2(1-k/k_F)[1+k^2/3k_F(2k_F+k)]. \tag{4.16}$$

The results of the numerical integration of (4.12) and $\Delta\bar{U}$ Eq. (4.14) are given in Table VI. Finally in Table VII the total energy per particle as a function of the Fermi momentum is given. Figure 4 shows the total energy per particle as a function of k_F^2.

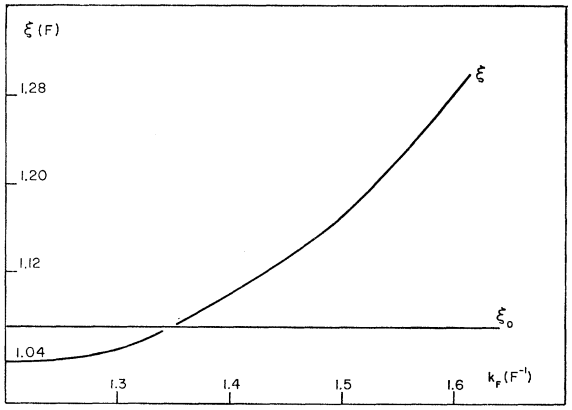

FIG. 3. Variation of the first-order separation distance ξ_0 and the nuclear separation distance ξ with the Fermi momentum.

5. VARIATIONAL METHOD

In calculating the binding energy we have made several approximations at different stages of the computations, but we believe that, except for the crude estimate of the particle energies, the other approximations are justified. As an alternative way of finding the binding energy we have applied Schwinger's variational method to determine the reaction matrix. The agreement between the results of these two methods is very good and indicates the validity of the approximations made in Sec. 4. In the following calculation the hard

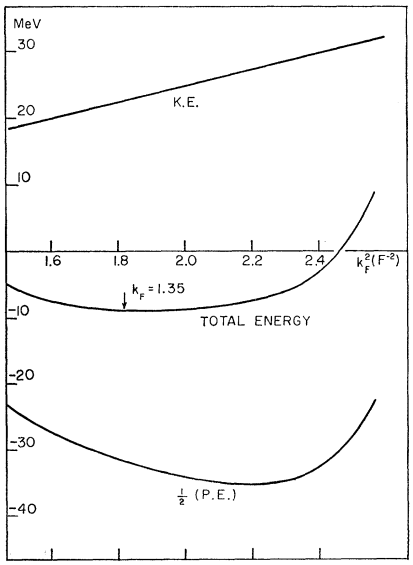

FIG. 4. The total energy per particle is shown as a function of k_F^2.

TABLE VI. Second-order contributions of the long-range interaction to the single-particle potential energy (in units of MeV).

k_F (F^{-1})	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	$^3D_2$	Total
1.3	−0.8	−0.4	−0.1	−0.1	−0.3	−0.3	−0.1	−0.1	−2.2
1.4	−0.6	−0.3	−0.1	−0.1	−0.3	−0.3	−0.1	−0.2	−2.0
1.5	−0.5	−0.2	−0.1	−0.1	−0.2	−0.2	−0.1	−0.2	−1.6
1.6	−0.3	−0.1	−0.1	−0.0	−0.1	−0.1	−0.1	−0.2	−1.0

TABLE VII. The total energy per particle (MeV).

k_F (F^{-1})	$^1S_0$	$^3S_1$	$^1P_1$	$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	$^3D_1$	$^3D_2$	$^3D_3$	BE/A
1.3	−13.4	−7.8	4.2	−4.9	10.5	−6.3	−7.8	0.6	−4.6	−0.9	−8.4
1.4	−13.7	−8.4	5.5	−5.6	14.4	−7.7	−11.2	0.8	−6.6	−1.2	−8.4
1.5	−14.3	−8.6	6.8	−6.1	19.6	−9.2	−15.0	1.1	−8.8	−1.5	−7.1
1.6	−14.4	−8.6	8.5	−4.6	36.0	−10.5	−19.2	1.5	−11.6	−1.8	9.4

core is separated from the rest of the potential and the core contribution is calculated from the result given by Bethe *et al.*[13] Thus

$$\bar{U}=4\pi c\rho(k_F)\{\tfrac{1}{3}(x^2+y^2)+(x/8)(1-y^2/x^2)j_1(2x)+(1+y)[1-\tfrac{1}{4}j_0(2x)]+(1+y^{-1})\times[\tfrac{1}{3}x^2+\tfrac{1}{8}xj_1(2x)]\}, \quad (5.1)$$

where $x=(\sqrt{0.3})k_Fc$ and $y=\gamma(\bar{k})c$. While the form of $\bar{U}$ is identical with that obtained by the reference spectrum, $\gamma^2(k)$ in (5.1) has a different dependence on k from the $\gamma^2(k)$ given by BBP. The reason that the hard-core contribution has the same form is that in both cases the energy denominator $e(k,k',P)$ is assumed to be a quadratic function of k', and the exclusion operator Q is set equal to unity.

Now we choose $\xi=\infty$, then $\varphi(mn)$ will be a plane wave and the perturbation will consist of the complete (outer) potential. The two-nucleon wave function in nuclear matter $\psi(mn)$ satisfies the integral equation

$$\psi_l(kr)=j_l(kr)+\int_0^\infty G_l(r,r')v(r')\psi_l(r')r'^2dr', \quad (5.2)$$

where the Green's function $G_l(r,r')$ is given by

$$G_l(r,r')=\frac{2}{\pi}\int_0^\infty j_l(k'r')\frac{Q(k',P)}{e(k,k',P)}j_l(k'r)k'^2dk'. \quad (5.3)$$

At $r=c$, $\psi(kr)$ is equal to zero. Imposing this condition on (5.2) we find the following equation:

$$\psi_l(kr)=f_l(r)+\int_c^\infty K_l(r,r')v(r')\psi_l(r')r'^2dr', \quad (5.4)$$

where $K_l(r,r')$ and $f_l(r)$ are related to $G_l(r,r')$ and $j_l(kr)$ by

$$K_l(r,r')=G_l(r,r')-[1/G_l(c,c)]G_l(r,c)G_l(c,r') \quad (5.5)$$

and

$$f_l(r)=j_l(kr)-j_l(kc)G_l(c,r)/G_l(c,c). \quad (5.6)$$

Now we consider two quadratic functionals of ψ_l,

$$P[\psi_l]=|\langle f_l|v|\psi_l\rangle|^2, \quad (5.7)$$

$$Q[\psi_l]=\langle\psi_l|v|\psi_l\rangle-\langle\psi_l|v(r)K(r,r')v(r')|\psi_l\rangle$$
$$=\int_c^\infty v(r)\psi_l^2(kr)r^2dr-\frac{2}{\pi}\int_0^\infty\frac{\bar{Q}(k,k',P)}{e(k,k',P)}k'^2dk'\times\left\{\int_c^\infty f_l(k'r)v(r)\psi_l(r)r^2dr\right\}\times\left\{\int_c^\infty j_l(k'r')v(r')\psi_l(r')r'^2dr'\right\}. \quad (5.8)$$

The extremum of $J[\psi_l]=P[\psi_l]/Q[\psi_l]$ is obtained when ψ_l satisfies the integral equation (5.4) and we have extremum of

$$J[\psi_l]=\langle f_l|v|\psi_l\rangle. \quad (5.9)$$

The quantity $\langle f_l|v|\psi_l\rangle$ is the reaction matrix for the outer potential and consists of two parts,

$$\langle f_l|v|\psi_l\rangle=\langle j_l|v|\psi_l\rangle-[j_l(kc)/G_l(c,c)]\times\langle G_l(c,r)|v(r)|\psi_l(r)\rangle. \quad (5.10)$$

The first term $\langle j_l|v|\psi_l\rangle$ in Eq. (5.10) is the familiar form of the reaction matrix for the lth partial wave, but the second term represents the effect of distortion of the wave function caused by the hard core. If we assume that the energy denominator $e(k,k',P)$ is a quadratic function of k' and put $P=0$ and $\bar{Q}=1$, then we can write

$$e(k,k',P)=-(1/m^*)[\gamma^2(k)+k'^2] \quad (5.11)$$

[13] Equation (5.34) of Ref. 7.

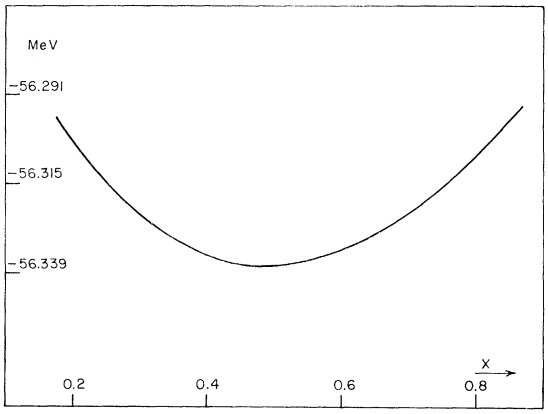

FIG. 5. The quantity $J[\psi]$ is plotted as a function of the variational parameter x for the $^1S_0$ wave. The reaction matrix corresponds to the minimum of $J[\psi]$.

and substituting in (5.3) we find

$$G(r,r')=\frac{-2m^*}{\pi}\int_0^\infty \frac{j_l(k'r)j_l(k'r')}{k'^2+\gamma^2}k'^2dk'$$
$$=m^*\gamma j_l(i\gamma r)h_l^{(1)}(i\gamma r'), \quad r<r',$$
$$=m^*\gamma j_l(i\gamma r')h_l^{(1)}(i\gamma r), \quad r>r'. \tag{5.12}$$

From Eqs. (5.10) and (5.12) it follows that

$$\langle f_l|v|\psi_l\rangle=\int_c^\infty\left[j_l(kr)-j_l(kc)\frac{h_l^{(1)}(i\gamma r)}{h_l^{(1)}(i\gamma c)}\right]\times v(r)\psi_l(r)r^2dr, \tag{5.13}$$

which is identical with Eq. (9.11) of BBP for the contribution of the outer potential.

In the numerical computation, we have tried to find the minimum of $J[\psi_l]$ using (5.7), (5.8), and (5.9). The energy denominator in the Green's function $e(k,k',P)$ is given by (4.4) and $\bar{Q}$ by (4.15). The trial function $\psi_l(r)$ is chosen to be

$$\psi_l(r)=j_l(kr)-[h_l(ixr)/h_l(ixc)]j_l(kc), \tag{5.14}$$

where x is a parameter. This trial function is the exact solution of the integral equation (5.2) if the Green's function is assumed to be given by (5.12), with $\gamma=x$. For $x=\gamma$ the quantity $\langle f_l|v|\psi_l\rangle$ is related to the modified Born approximation (MBA) of BBP, since then from (5.13) and (5.14) we have

$$\langle f_l|v|\psi_l\rangle=\int_c^\infty\left[j_l(kr)-j_l(kc)\frac{h_l^{(1)}(i\gamma r)}{h_l^{(1)}(i\gamma c)}\right]^2\times v(r)r^2dr=\text{MBA}. \tag{5.15}$$

The results of the calculation are given in Table VIII for different partial waves for $k_F=1.5F^{-1}$. For comparison the values computed from (5.15) and the results of the calculation of Brueckner and Masterson are also given. Figure 5 shows the variation of $J[\psi]$ as a function of x for the $^1S_0$ wave.

TABLE VIII. The results of the variational calculation of the reaction matrix ($\frac{1}{2}\bar{U}$ in units of MeV).

State	Core	Variational	Total	MBA	BM
$^1S_0$	13.5	−28.2	−14.7	−18.2	−14.9
$^1P_1$	0.4	6.8	7.2	6.7	7.3
$^3P_0$	0.4	−7.5	−7.1	−7.1	−6.8
$^3P_1$	1.1	15.3	16.4	15.5	16.9
$^3P_2$	1.8	−17.6	−15.8	−15	−16.6

6. CONCLUSION

We have constructed an effective interaction for nuclear matter, which is a function of the density of the system, and in the first Born approximation gives an accurate value of the potential energy of a nucleon in the Fermi sea. That the contributions of higher terms in the plane-wave expansion of the long-range wave function $\varphi(k,r)$ are very small can be seen by comparing the values given in Tables II and VI. The small difference between the wave function φ and the plane wave is a result of the exclusion principle.[14] The smallness of the second and higher order Born approximations suggests that at least for low densities the single-particle potential energy, which, in the coordinate space is nonlocal, can be replaced by a long-range local potential which is a function of the density. Then an expansion of the reaction matrix in powers of this potential converges and saturation results from the strong variation of the effective Hamiltonian with the density of nuclear matter. With this method the binding energy per particle for the BGT potential is found to be about 9 MeV, which is lower than the value of 15 MeV calculated by Moszkowski and Scott and by Brueckner and Gammel, but it is closer to the value of 7.5 MeV obtained recently by Coon and Dabrowski.[15] The equilibrium density corresponds to $k_F=1.34$ F^{-1} in our calculation as compared to $k_F=1.5$ F^{-1} of MS and of Brueckner and Gammel. Finally, from the result of the preceding section it is evident that for the potentials with hard core, Schwinger's variational method for the calculation of the reaction matrix is accurate, even for the S-wave contribution. The energy denominator for the first iteration of the variational method may be calculated by Born approximation of the long-range part or directly from the phase shifts.

ACKNOWLEDGMENTS

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[14] H. A. Bethe, Phys. Rev. **103**, 1353 (1956).
[15] S. Coon and J. Dabrowski, Phys. Rev. **140**, B287 (1965).