Eigenstates of the $J=0$, $T=1$, Charge-Independent Pairing Hamiltonian. I. Seniority-Zero States*

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Exact equations for the seniority-zero states of $2N$ nucleons in an arbitrary external potential well and interacting through a $J=0$, $T=1$, charge-independent pairing interaction are derived. Exact solutions of these equations are exhibited for the ground states and some excited states of these systems. These states are characterized by having wave functions that are totally symmetric functions of the isospins of the pairs of particles and have total isospin $T=0,2,\cdots, N$ for N even and $T=1,3,\cdots, N$ for N odd. The ground states are included in the sets of states with $T=0$ or 1 and the states of the pairing Hamiltonian with a single type of nucleon belong to the charge multiplet with $T=N$. The calculation of the energies and wave functions of these states is reduced to the solution of N coupled, nonlinear, algebraic equations in N unknowns. An explicit expression is given for the occupation probabilities of the levels of the single-particle well in these states.

I. INTRODUCTION

THE pairing model[1] is a model of the nucleus that includes some of the correlations between nucleons that would arise from a short-range residual nucleon-nucleon force. In this model, the residual interaction is approximated by a pairing force between identical nucleons. One objection to this model is that it does not include any neutron-proton interactions.[1] That is, neutrons may interact with neutrons and protons may interact with protons but neutrons may not interact with protons in the model. A natural way to remove this objection and introduce neutron-proton interactions into the pairing model is to consider its charge-independent generalization. In this charge-independent pairing model, neutrons and protons are on an equal footing and any two nucleons that are coupled to angular momentum $J=0$ and isospin $T=1$ interact through a pairing force. While this model may not be a realistic model of the nucleus, due to its neglect of any $T=0$ forces between nucleons, it does represent an interesting many-body problem that is more realistic than the pairing model and yet can still, as we shall show, be treated exactly. In addition to this, the techniques that we will develop in this paper to treat the relatively simple $J=0$, $T=1$, charge-independent pairing Hamiltonian can, with only slight modifications, be applied to the far more complicated $L=0$, $T=0$, 1 charge-spin-independent pairing Hamiltonian.[2,3] Thus, the simpler charge-independent pairing Hamiltonian provides a useful illustration of these techniques.

There have been two approaches to the problem of calculating the eigenstates of the charge-independent pairing Hamiltonian. In the first approach, the methods of group theory have been used to show that the states of this Hamiltonian for a configuration j^n may be classified by using the five-dimensional rotation group.[3-5] A five-dimensional quasispin formalism may then be used to treat the eigenstates of this Hamiltonian when the interaction spans several single-particle levels.[6] However, this approach does not give analytical expressions for the eigenstates and one still has to diagonalize the Hamiltonian matrix numerically. The second approach to the eigenstates of this Hamiltonian has been through generalizations of the BCS-Bogoliubov, Valatin formalism that include neutron-proton correlations.[7-11] However, the accuracy of these formalisms is generally difficult to assess.[12,13]

With this paper, we initiate a third approach to these eigenstates. Our methods are closely related to those which have been successfully used on the pairing model.[14,15] We will first derive a Schrödinger equation for the seniority-zero states of the Hamiltonian. We will then exhibit analytical expressions for certain *exact* solutions of this equation. The states that we solve the Schrödinger equation exactly for are those states whose wave functions are totally symmetric in the isospins of the pairs of particles. For $2N$ particles, these states have total isospin $T=0, 2, \cdots, N$, for N even and $T=1, 3, \cdots, N$, for N odd. For each value of T, the solution of our equations yields a whole class of states whose wave functions are totally symmetric in the isospins of the pairs of particles and whose total isospin is the given value of T. The number of states in each class depends upon the single-particle spectrum. The lowest state with $T=0$ or 1 is the ground state and the states of the pair-

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[1] A. M. Lane, *Nuclear Theory* (W. A. Benjamin, Inc., New York, 1964), Part I and the references cited therein.
[2] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) 84, 673 (1964).
[3] M. Ichimura, Progr. Theoret. Phys. (Kyoto) 32, 757 (1964).
[4] B. H. Flowers and S. Szpikowski, Proc. Phys. Soc. (London) 84, 193 (1964).
[5] J. C. Parikh, Nucl. Phys. 63, 214 (1965).
[6] K. T. Hecht, Phys. Rev. 139, B794 (1965).
[7] M. Baranger, Phys. Rev. 122, 992 (1961).
[8] B. Bredmond and J. G. Valatin, Nucl. Phys. 41, 640 (1963).
[9] M. Ichimura, Progr. Theoret. Phys. (Kyoto) 31, 575 (1964).
[10] B. H. Flowers and M. Vujicic, Nucl. Phys. 49, 586 (1963).
[11] A. Goswami, Nucl. Phys. 60, 228 (1964).
[12] R. W. Richardson, Phys. Letters 14, 325 (1965).
[13] R. W. Richardson, Phys. Rev. 141, 949 (1966).
[14] R. W. Richardson and N. Sherman, Nucl. Phys. 52, 221 (1964).
[15] R. W. Richardson, J. Math. Phys. 6, 1034 (1965).

ing Hamiltonian with only one type of nucleon belong to the charge multiplet with $T=N$. The calculation of the wave functions and energies of these states is reduced to the solution of N-coupled algebraic equations and the calculation of an eigenvector of a tridiagonal matrix for which the eigenvalue is known explicitly. We call the N unknowns the pair energies and the equations that they satisfy are simple generalizations of the previously published[14,15] equations for the exact eigenstates of the pairing Hamiltonian with one type of nucleon present. In fact, the equations that we derive here only differ from these equations by the replacement of a numerical factor 2 by

$$\kappa=[N(N-3)+T(T+1)]/N(N-1).$$

Note that for $T=N$, $\kappa=2$ and we obtain the equations for the states of a single type of nucleon. Once the equations for the pair-energies have been solved, we will show that the occupation probabilities of the levels of the single-particle well may be obtained by the solution of an $N\times N$ system of linear algebraic equations. We only treat the seniority-zero states which are totally symmetric in the isospins of the pairs in this paper. Subsequent papers will deal with seniority-one and seniority-two states and states of different isospin symmetry.

In Sec. II, we define our notation and discuss the Hamiltonian together with the seniority operators and the total isospin operator which commute with the Hamiltonian. Section III is devoted to the derivation of a Schrödinger equation for the seniority-zero states of this Hamiltonian. In order to gain some familiarity with this equation, we discuss the special case of four nucleons in some detail in Sec. IV. In Sec. V, the Schrödinger equation is solved for the states with total isospin symmetry for arbitrary N, Finally, in Sec. VI, we derive a simple expression for the occupation probabilities of the single-particle levels of the external potential well.

II. THE CHARGE-INDEPENDENT PAIRING HAMILTONIAN

After defining the charge-independent pairing Hamiltonian, we will discuss the seniority operators which commute with it and indicate their interpretation. Isospin operators, which are useful in writing down the commutation rules of the various operators that appear in the Hamiltonian, are then defined. And, the section closes with a listing of those commutation rules that will be needed in the sections to follow.

In order to write down the charge-independent pairing Hamiltonian, we first assume that the nucleons are contained in a potential well. Throughout this work, this potential well will remain arbitrary. To emphasize this arbitrariness, we choose the uncommitted symbol f to label the levels of this well apart from their assumed time reversal and charge degeneracy. That is, a complete set of single-particle quantum numbers is denoted by (f,σ,q), where $\sigma=\pm$ indicates states that are conjugate with respect to time reversal and $q=p$ or n for proton or neutron. For example, for the spherical shell model, $f=(n,l,j,|m|)$, $\sigma=m/|m|$, and $q=p$ or n. The Hamiltonian for a system of noninteracting nucleons in this potential well is then, in second quantized form,

$$K=\sum_f 2\epsilon_f N_f, \tag{2.1}$$

where the energy of level f of the potential well has been denoted by ϵ_f and where

$$N_f=N_{fp}+N_{fn}, \tag{2.2}$$

with

$$N_{fq}=\tfrac{1}{2}(a_{f+q}{}^\dagger a_{f+q}+a_{f-q}{}^\dagger a_{f-q}), \tag{2.3}$$

and where $a_{f\sigma q}{}^\dagger$ and $a_{f\sigma q}$ are nucleon creation and annihilation operators satisfying the usual Fermi anticommutation rules

$$[a_{f\sigma q},a_{f'\sigma' q'}{}^\dagger]_+=\delta_{ff'}\delta_{\sigma\sigma'}\delta_{qq'}. \tag{2.4}$$

Throughout this work, we will choose the phases of our single-particle states so as to eliminate the ubiquitous phase factor $(-)^{j-m}$ that appears in many pairing-model calculations.[2-6] Thus, the states $(f,\pm,q)$ are the time-reversed images of the states $(f,\mp,q)$.

The charge-independent pairing Hamiltonian is obtained by adding to K the interaction operator $-gV$, where

$$V=\sum_{t=0,\pm}\sum_{ff'} b_{ft}{}^\dagger b_{f't}, \tag{2.5}$$

where

$$\begin{aligned} b_{f+}&=a_{f-p}a_{f+p},\\ b_{f0}&=(1/\sqrt{2})(a_{f-n}a_{f+p}+a_{f-p}a_{f+n}),\\ b_{f-}&=a_{f-n}a_{f+n}. \end{aligned} \tag{2.6}$$

The operators $b_{ft}{}^\dagger$ are given by the Hermitian conjugates of (2.6). Note that $b_{ft}{}^\dagger$ creates a pair of nucleons in the level f of the potential well with total isospin $T=1$ and z component of the isospin $T_0=t$. (We use $\pm$ to denote $t=\pm1$.) Also note that we have not carried out the usual angular-momentum coupling to $J=0$ in our pair operators (2.6). We reserve this coupling until last and it is accomplished by sums over f such as those that appear in (2.5). Finally, it should be pointed out that the sums on f in (2.5) are over a finite range of values of f and that the specification of this range is part of the definition of V. In what follows, all sums on f will be restricted to this range since those particles that occupy levels outside this range are not interacting and therefore not interesting. Thus, the charge-independent pairing Hamiltonian is given by

$$H=K-gV, \tag{2.7}$$

where K is given by (2.1) and V by (2.5) and all sums on f are over a specified finite set of values.

The set of seniority operators defined by

$$\nu_f=\nu_{fp}+\nu_{fn}, \tag{2.8}$$

where

$$\nu_{fq}=a_{f+q}^{\dagger}a_{f+q}-a_{f-q}^{\dagger}a_{f-q} \tag{2.9}$$

commute with the Hamiltonian (2.7) and therefore represent constants of the motion. The eigenvalues of ν_f are 0, ±1, and ±2. The interpretation of these eigenvalues is given in Table I in terms of the allowable

TABLE I. The allowable occupations of the level f as determined by the value of ν_f. Each bracket contains an allowable set of values of σq that must be occupied to be consistent with the given value of ν_f. The symbol (−) represents the level f being unoccupied.

ν_f	Allowable occupations
2	$(+p,+n)$
1	$(+p)$, $(+n)$, $(+p,+n,-n)$, $(+p,-p,+n)$
0	$(-)$, $(+p,-p)$, $(+n,-n)$, $(+p,-n)$, $(-p,+n)(+p,-p,+n,-n)$
−1	$(-p)$, $(-n)$, $(-p,+n,-n)$, $(+p,-p,-n)$
−2	$(-p,-n)$

occupations of the fourfold degenerate (two values of σ and two values of q) level labeled f. The total seniority of a state is defined by

$$\nu=\sum_f|\nu_f|\,. \tag{2.10}$$

The states that we shall treat in this paper have $\nu=0$ and therefore $\nu_f=0$ for all f. The allowable occupations of the single-particle levels can be read on the corresponding line in Table I.

We define the isotopic-spin operator for the level f, $\mathbf{T}_f$ as the operator whose components are

$$\begin{aligned}T_{f+}&=a_{f+p}^{\dagger}a_{f+n}+a_{f-p}^{\dagger}a_{f-n}\,,\\ T_{f-}&=a_{f+n}^{\dagger}a_{f+p}+a_{f-n}^{\dagger}a_{f-p}\,,\end{aligned} \tag{2.11}$$

and

$$T_{f0}=N_{fp}-N_{fn}\,.$$

The total isotopic-spin operator is

$$\mathbf{T}=\sum_f\mathbf{T}_f\,. \tag{2.12}$$

As a result of the charge independence and charge conservation of the Hamiltonian, $\mathbf{T}^2$ and T_0 commute with it and therefore represent constants of the motion.

TABLE II. The commutators $[A,B]$. All operators are implicitly labeled by the quantum numbers f. All commutators of operators labeled with different values of f vanish.

A \ B	$b_+^{\dagger}$	$b_0^{\dagger}$	$b_-^{\dagger}$
b_+	$1-2N_p$	$-(1/\sqrt{2})T_-$	0
b_0	$-(1/\sqrt{2})T_+$	$1-N$	$-(1/\sqrt{2})T_-$
b_-	0	$-(1/\sqrt{2})T_+$	$1-2N_n$
N_p	$b_+^{\dagger}$	$\frac{1}{2}b_0^{\dagger}$	0
N	$b_+^{\dagger}$	$b_0^{\dagger}$	$b_-^{\dagger}$
N_n	0	$\frac{1}{2}b_0^{\dagger}$	$b_-^{\dagger}$
T_+	0	$\sqrt{2}b_+^{\dagger}$	$\sqrt{2}b_0^{\dagger}$
T_0	$b_+^{\dagger}$	0	$-b_-^{\dagger}$
T_-	$\sqrt{2}b_0^{\dagger}$	$\sqrt{2}b_-^{\dagger}$	0

In the next section, we will need certain commutators of the N's, b's, and T's. These may be calculated from the definitions (2.2), (2.6), and (2.11) and the anticommutation relations (2.4). Those commutators that are necessary for the calculation of the seniority-zero states are listed in Table II.

This completes the specification of the Hamiltonian and the operators which make it up. We now turn to the calculations of the seniority-zero eigenstates of this Hamiltonian.

III. EQUATION FOR THE SENIORITY-ZERO EIGENSTATES

For the seniority-zero states of the charge-independent pairing Hamiltonian, the allowable occupations of the single-particle levels may be read off the $\nu_f=0$ line of Table I. In addition to these restrictions, there is a restriction due to the fact that all pairs are eventually coupled to $J=0$ and therefore $T=1$. Thus, only the $T=1$ component of the two possible occupations with one neutron and one proton is needed. A complete set of $2N$-particle seniority-zero states is therefore given by

$$b_{f_1t_1}^{\dagger}\cdots b_{f_Nt_N}^{\dagger}|0\rangle\,, \tag{3.1}$$

where $|0\rangle$ is the vacuum state of our specified set of single-particle levels. We may now expand an arbitrary $2N$-particle seniority-zero state in the set of states (3.1), i.e.,

$$|\psi\rangle=\sum_{f_1t_1\cdots f_Nt_N}\psi(f_1t_1\cdots f_Nt_N)b_{f_1t_1}^{\dagger}\cdots b_{\sigma_Nt_N}^{\dagger}|0\rangle\,, \tag{3.2}$$

where the wave function $\psi(f_1t_1\cdots f_Nt_N)$ must be a totally symmetric function of the variables f_it_i, i.e., it must be symmetric under the interchanges $f_it_i\leftrightarrows f_jt_j$. In this section we will develop an equation for the wave function $\psi(f_1t_1\cdots f_Nt_N)$ which, when satisfied, will make the state $|\psi\rangle$ an eigenstate of the Hamiltonian (2.7).

The wave function $\psi(f_1t_1\cdots f_Nt_N)$ is to be determined so that the state $|\psi\rangle$ satisfies the Schrödinger equation

$$H|\psi\rangle=E|\psi\rangle\,. \tag{3.3}$$

We therefore consider the effect of H on the state (3.2), i.e.,

$$\begin{aligned}H|\psi\rangle&=\sum_{f_1t_1\cdots f_Nt_N}\psi(f_1t_1\cdots f_Nt_N)H\prod_{k=1}^{N}b_{fkt_k}^{\dagger}|0\rangle\\ &=\sum_{f_1t_1\cdots f_Nt_N}\psi(f_1t_1\cdots f_Nt_N)\{\sum_{i=1}^{N}(\prod_{k\neq i}b_{\sigma kt_k}^{\dagger})[H,b_{f_it_i}^{\dagger}]\\ &\quad+\tfrac{1}{2}\sum_{i,j=1}^{N}{}'(\prod_{k\neq i,j}b_{fkt_k}^{\dagger})[[H,b_{f_it_i}^{\dagger}],b_{f_jt_j}^{\dagger}]\}|0\rangle\,,\end{aligned} \tag{3.4}$$

where here and elsewhere the prime on the sum on i and j excludes the values $i=j$. This last expression results from some commutator algebra and the fact that the triple commutator of H with three $b^{\dagger}$'s vanishes. We are

TABLE III. The matrix $C(t_1t_2;t_1't_2')$.

t_1t_2 \ $t_1't_2'$	++	+0	+−	0+	00	0−	−+	−0	−−
++	2	−	−	−	−	−	−	−	−
+0	−	1	−	1	−	−	−	−	−
+−	−	−	−	−	1	−	−	−	−
0+	−	1	−	1	−	−	−	−	−
00	−	−	1	−	1	−	1	−	−
0−	−	−	−	−	−	1	−	1	−
−+	−	−	−	−	1	−	−	−	−
−0	−	−	−	−	−	1	−	1	−
−−	−	−	−	−	−	−	−	−	2

thus led to the consideration of the single and double commutators of H with one and two $b^\dagger$'s.

The commutators of H with the $b^\dagger$'s can be evaluated using (2.7) for H and the commutators listed in Table II. The results are

$$[H,b_{f+}^\dagger]=2\epsilon_f b_{f+}^\dagger-g\sum_{f'}[b_{f'+}^\dagger(1-2N_{fp})-(1/\sqrt{2})b_{f'0}^\dagger T_{f+}],$$
$$[H,b_{f0}^\dagger]=2\epsilon_f b_{f0}^\dagger-g\sum_{f'}[-(1/\sqrt{2})b_{f'+}^\dagger T_{f-}+b_{f'0}^\dagger(1-N_f)-(1/\sqrt{2})b_{f'-}^\dagger T_{f+}], \quad (3.5)$$

and

$$[H,b_{f-}^\dagger]=2\epsilon_f b_{f-}^\dagger-g\sum_{f'}[-(1/\sqrt{2})b_{f'0}^\dagger T_{f-}+b_{f'-}^\dagger(1-2N_{fn})].$$

These expressions simplify greatly when they are applied to the vacuum as in (3.4). We then have

$$[H,b_{ft}^\dagger]|0\rangle=(2\epsilon_f b_{ft}^\dagger-g\textstyle\sum_{f'} b_{f't}^\dagger)|0\rangle. \quad (3.6)$$

The double commutator of H with two $b^\dagger$'s can be evaluated using (3.5) and Table II. We write these double commutators as

$$[[H,b_{f_1t_1}^\dagger],b_{f_2t_2}^\dagger]=g\delta_{f_1f_2}\sum_{t_1',t_2'}C(t_1t_2;t_1't_2')\sum_f b_{ft_1'}^\dagger b_{ft_2'}^\dagger. \quad (3.7)$$

The matrix $C(t_1t_2;t_1't_2')$, which is defined by (3.7), is given in Table III. We note in passing that C has the symmetries

$$C(t_1t_2;t_1't_2')=C(t_2t_1;t_1't_2')=C(t_1t_2;t_2't_1')=C(t_2t_1;t_2't_1')$$

and

$$C(t_1t_2;t_1't_2')=C(t_1't_2';t_1t_2).$$

If we substitute (3.6) and (3.7) into (3.4) and relabel some of the summation indices, we obtain

$$H|\psi\rangle=\sum_{f_1t_1\cdots f_Nt_N}[(2\epsilon_{f_1}+\cdots+2\epsilon_{f_N})\psi(f_1t_1\cdots f_Nt_N)-g\sum_{i=1}^N\sum_f\psi(f_1t_1\cdots f_{i-1}t_{i-1},ft_i,f_{i+1}t_{i+1}\cdots f_Nt_N)$$
$$+\frac{g}{2}\sum_{i,j=1}^{N}{}'\sum_{t_i't_j'}C(t_it_j;t_i't_j')\psi(f_1t_1\cdots f_{i-1}t_{i-1},f_jt_i',f_{i+1}t_{i+1}\cdots f_{j-1}t_{j-1},f_jt_j',f_{j+1}t_{j+1}\cdots f_Nt_N)]\prod_{k=1}^N b_{f_kt_k}^\dagger|0\rangle. \quad (3.8)$$

This leads to the equation

$$(2\epsilon_{f_1}+\cdots+2\epsilon_{f_N}-E)\psi(f_1t_1\cdots f_Nt_N)-g\sum_{i=1}^N\sum_f\psi(f_1t_1\cdots ft_i\cdots f_Nt_N)$$
$$+\frac{g}{2}\sum_{i,j=1}^{N}{}'\sum_{t_i't_j'}C(t_it_j;t_i't_j')\psi(f_1t_1\cdots f_jt_i'\cdots f_jt_j'\cdots f_Nt_N)=0 \quad (3.9)$$

for the wave function ψ. The reader who is familiar with the work on the exact eigenstates of the pairing Hamiltonian with one type of nucleon given in Refs. 14 and 15 will recognize that Eq. (3.9) is the same equation as is given there when $t_1=t_2=\cdots=t_N=\pm$. It should be pointed out that (3.9) is more than just the Schrödinger eigenvalue problem. It is the Schrödinger eigenvalue problem plus a definition of certain unphysical components of ψ which violate the Pauli principle. This definition is introduced when we ignore the fact that the state

$$\prod_{k=1}^N b_{f_kt_k}^\dagger|0\rangle$$

may be zero for some values of $f_1t_1\cdots f_Nt_N$.

Equation (3.9) is an equation for all the seniority-zero states of $2N$ nucleons which are interacting through charge-independent pairing forces. In the next section, we will discuss the solutions of (3.9) for $N=2$ in some detail. This material is meant as an introduction to Sec. V, where we exhibit some solutions of (3.9) for arbitrary N.

IV. SENIORITY-ZERO STATES OF FOUR NUCLEONS

We will now solve Eq. (3.9) for the seniority-zero, $T=0$, 1 and 2 states of four nucleons. We treat this simplest nontrivial example of Eq. (3.9) in some detail as an introduction to the structure of this equation. We will only treat the $T_0=0$ states since the states with

other values of T_0 can be obtained by applying the isospin raising or lowering operators T_+ or T_- to these $T_0=0$ states.

Equation (3.9), for $N=2$, is

$$(2\epsilon_{f_1}+2\epsilon_{f_2}-E)\psi(f_1t_1,f_2t_2)$$
$$-g\sum_f[\psi(ft_1,f_2t_2)+\psi(f_1t_1,ft_2)]+\tfrac{1}{2}g\sum_{t_1't_2'}C(t_1t_2;t_1't_2')$$
$$\times[\psi(f_1t_1',f_1t_2')+\psi(f_2t_1';f_2t_2')]=0. \quad (4.1)$$

Since we are considering those states with $T_0=t_1+t_2=0$, there are only three allowable values of (t_1t_2), i.e., $t_1t_2=+-$, $-+$, or 00. Using the values of the matrix C given in Table III, we obtain from (4.1) the three coupled equations

$$(2\epsilon_1+2\epsilon_2-E)\psi(1+,2-)$$
$$-g\sum_f[\psi(f+,2-)+\psi(1+,f-)]$$
$$+\tfrac{1}{2}g[\psi(10,10)+\psi(20,20)]=0,$$
$$(2\epsilon_1+2\epsilon_2-E)\psi(1-,2+)$$
$$-g\sum_f[\psi(f-,2+)+\psi(1-,f+)]$$
$$+\tfrac{1}{2}g[\psi(10,10)+\psi(20,20)]=0,$$

and

$$(2\epsilon_1+2\epsilon_2-E)\psi(10,20)-g\sum_f[\psi(f0,20)+\psi(10,f0)]$$
$$+\tfrac{1}{2}g[\psi(1+,1-)+\psi(1-,1+)+\psi(10,10)$$
$$+\psi(2+,2-)+\psi(2+,2+)+\psi(20,20)]=0, \quad (4.2)$$

where we have denoted f_1 and f_2 by 1 and 2. Let us introduce the functions

$$\psi_0(12)=\psi(10,20),$$
$$\psi_1(12)=\psi(1+,2-)-\psi(1-,2+),$$

and

$$\psi_2(12)=\psi(1+,2-)+\psi(1-,2+). \quad (4.3)$$

We can obtain equations for these new functions by taking linear combinations of Eqs. (4.2). These equations are

$$(2\epsilon_1+2\epsilon_2-E)\psi_0(12)-g\sum_f[\psi_0(f2)+\psi_0(1f)]$$
$$+\tfrac{1}{2}g[\psi_0(11)+\psi_2(11)+\psi_0(22)+\psi_2(22)]=0, \quad (4.4a)$$
$$(2\epsilon_1+2\epsilon_2-E)\psi_2(12)-g\sum_f[\psi_2(f2)+\psi_2(1f)]$$
$$+g[\psi_0(11)+\psi_0(22)]=0, \quad (4.4b)$$

and

$$(2\epsilon_1+2\epsilon_2-E)\psi_1(12)-g\sum_f[\psi_1(f2)+\psi_1(1f)]=0. \quad (4.4c)$$

The solution of these equations splits into two cases. In the first case $\psi_0=\psi_2=0$ and $\psi_1\neq0$ and in the second case $\psi_0\neq0$, $\psi_2\neq0$, and $\psi_1=0$. This separation is just the separation into those states that are charge-antisymmetric and charge-symmetric, respectively. Since the wave function must be symmetric in the variables f_it_i, charge antisymmetry (symmetry) implies antisymmetry (symmetry) with respect to the variables f_1f_2.

The charge antisymmetric states are the simplest solutions of Eqs. (4.4). For these states, $\psi_0=\psi_2=0$ and ψ_1 is an antisymmetric solution of Eq. (4.4c). It is easily verified that the unnormalized solutions are

$$E=E_1+E_2,$$
$$\psi_1(f_1f_2)=A(2\epsilon_{f_1}-E_1)^{-1}(2\epsilon_{f_2}-E_2)^{-1}, \quad (4.5)$$

where A is an antisymmetrizer operating on 1 and 2, and E_1 and E_2 satisfy

$$1/g=\textstyle\sum_f 1/(2\epsilon_f-E_i), \quad i=1,2 \quad (4.6)$$

and

$$E_1\neq E_N. \quad (4.7)$$

This last requirement is necessary for a nonzero antisymmetric state. It is readily verified that these states have total isospin $T=1$.

The charge-symmetric states are the most interesting states and the calculation of these states introduces the methods that we will use to calculate the isospin-symmetric states of $2N$ nucleons in the next section. These states are characterized by $\psi_0\neq0$, $\psi_2\neq0$, and $\psi_1=0$, where ψ_0 and ψ_2 are to be determined by Eqs. (4.4a) and (4.4b). Let us look for solutions of (4.4a) and (4.4b) of the form

$$\psi_0(12)=\alpha_0\psi(12), \quad \psi_2(12)=\alpha_2\psi(12), \quad (4.8)$$

where α_0, α_2, and the symmetric function ψ are to be determined. Compatability of Eqs. (4.4a), (4.4b), and (4.8) leads to the eigenvalue problem

$$(1-\kappa)\alpha_0+\alpha_2=0, \quad 2\alpha_0-\kappa\alpha_2=0 \quad (4.9)$$

for α_0 and α_2. The remaining equation for ψ is

$$(2\epsilon_1+2\epsilon_2-E)\psi(12)-g\sum_f[\psi(f2)+\psi(1f)]$$
$$+\tfrac{1}{2}\kappa g[\psi(11)+\psi(22)]=0. \quad (4.10)$$

We therefore have two eigenvalue problems to solve. One for the eigenvector α and its associated eigenvalue κ and one for the function ψ and its associated eigenvalue E. Note that the coupling between these two problems is only through the presence of κ in the equation for ψ.

The solutions of (4.9) are

$$\kappa=-1, \quad \alpha_0=-\tfrac{1}{2}\alpha_2, \quad T=0$$

and (4.11)

$$\kappa=2, \quad \alpha_0=\alpha_2, \quad T=2,$$

where we have indicated the corresponding value of the total isospin of the state T. Equation (4.10) may be solved using the same techniques that were used to treat pairing forces between identical nucleons[15,16] and which we reproduce in Appendix I. The results are

$$E=E_1+E_2$$
$$\psi(f_1f_2)=S(2\epsilon_{f_1}-E_1)^{-1}(2\epsilon_{f_2}-E_2)^{-1}, \quad (4.12)$$

where S is a symmetrizer operating on 1 and 2, and where E_1 and E_2 satisfy

$$1/g+\kappa/(E_2-E_1)=\textstyle\sum_f 1/(2\epsilon_f-E_1)$$
$$1/g+\kappa/(E_1-E_2)=\textstyle\sum_f 1/(2\epsilon_f-E_2) \quad (4.13)$$

and

$$E_1 \neq E_2. \tag{4.14}$$

Thus, the $T=0$ and 2 states may be calculated by solving (4.13) using the values of κ given in (4.11).

In calculating the above eigenstates, we have first calculated the eigenstate of H and then verified that it is a state with good isospin. For the states that we are calculating, this procedure is simpler than the more natural approach of constructing states of a given isospin and then diagonalizing the Hamiltonian within the subspace of these states. In the next section, we will construct the states of $2N$ nucleons whose wave functions are totally symmetric in the variables $t_1 \cdots t_N$. Our results will be a natural generalization of Eqs. (4.8)–(4.14).

V. ISOSPIN-SYMMETRIC SENIORITY-ZERO STATES OF $2N$ NUCLEONS

We will now generalize the results of Sec. IV and obtain expressions for the isospin symmetric seniority-zero eigenstates of $2N$ nucleons. For simplicity, we will first treat the cases with N even and then we will indicate the necessary modification for those cases with N odd. As in Sec. IV, we will lose no generality by treating only those states for which $T_0=0$.

The wave functions of the isospin-symmetric states are totally symmetric functions of the variables $t_1 \cdots t_N$. However, since each of these variables must take on one of the three possible values 0 or $\pm$, the wave function can only depend on $n_\pm$, the number of the t_i that equal $\pm$, and n_0, the number of the t_i that equal zero. However, these numbers are restricted by the facts that the total number of variables is N, i.e.,

$$n_+ + n_- + n_0 = N$$

and the total z component of the isospin is T_0, i.e.,

$$n_+ - n_- = T_0,$$

Therefore, having specified N and T_0, the isospin dependence of the wave function is determined by only one number which we choose to be

$$n_+ + n_- = \sum_{i=1}^{N} |t_i| . \tag{5.1}$$

Following the example of Sec. IV, we look for solutions of Eq. (3.9) which are of the form

$$\psi(f_1 t_1 \cdots f_N t_N) = \delta(\sum_{i=1}^{N} t_i, 0)\alpha(\sum_{i=1}^{N} |t_i|)\psi(f_1 \cdots f_N), \tag{5.2}$$

where δ is a Kronecker delta which insures that

$$T_0 = \sum_{i=1}^{N} t_i = 0$$

and α and ψ are to be determined from Eq. (3.9). Note that ψ must be a totally symmetric function of its arguments.

We may project the equations for the isospin symmetric states out of Eqs. (3.9) by summing these equations on $t_1 \cdots t_N$ subject to the conditions

$$\sum_{i=1}^{N} t_i = 0, \quad \sum_{i=1}^{N} |t_i| = m, \tag{5.3}$$

where $m=0, 2, \cdots, N$ (recall that N has been assumed even). In order to do this summation, we need the combinatorial result

$$\sum_{t_1 \cdots t_N} \delta(\sum_{i=1}^{N} |t_i|, m)\delta(\sum_{i=1}^{N} t_i, T_0) = N!/[(N-m)!(\tfrac{1}{2}m+\tfrac{1}{2}T_0)!(\tfrac{1}{2}m-\tfrac{1}{2}T_0)!]. \tag{5.4}$$

We now multiply Eq. (3.9) by $\delta(\sum|t_i|, m)\delta(\sum t_k, 0)$ and sum on $t_1 \cdots t_N$. Using the assumed form of ψ, (5.2), the first two terms of (3.9) have a common factor of

$$\sum_{t_1 \cdots t_N} \delta(\sum|t_i|, m)\delta(\sum t_i, 0)\alpha(\sum|t_k|) = \frac{N!}{(N-m)![(\frac{1}{2}m)!]^2}\alpha(m), \tag{5.5}$$

where we have used (5.3). For the last term of (3.9), we need, for example, for $i=1$ and $j=2$

$$\sum_{t_1 \cdots t_N} \delta(\sum|t_i|, m)\delta(\sum t_i, 0) \sum_{t_1' t_2'} C(t_1 t_2; t_1' t_2')\alpha(\sum|t_i| + |t_1'| + |t_2'| - |t_1| - |t_2|)$$

$$= \frac{(N-2)!}{(N-m)![(\frac{1}{2}m)!]^2}\{\tfrac{1}{2}m^2\alpha(m-2) + [N(N-1) + (2N-1)m - 2m^2]\alpha(m) + 2(N-m)(N-m-1)\alpha(m+2)\}, \tag{5.6}$$

where we have used Table III for the matrix C to perform the sums on t_1, t_2, t_1', and t_2' and (5.4) to perform the sums on $t_3 \cdots t_N$. Collecting these results, we have from (3.9)

$$\frac{(N-2)!}{(N-m)![(\frac{1}{2}m)!]^2}\{N(N-1)\alpha(m)[(2\epsilon_{f_1} + \cdots + 2\epsilon_{f_N} - E)\psi(f_1 \cdots f_N) - g\sum_{i=1}^{N}\sum_{f}\psi(f_1 \cdots f_{i-1} f f_{i+1} \cdots f_N)]$$
$$+ \tfrac{1}{2}g[\tfrac{1}{2}m^2\alpha(m-2) + [N(N-1) + (2N-1)m - 2m^2]\alpha(m) + 2(N-m)(N-m-1)\alpha(m+2)]$$
$$\times \sum_{i,j=1}^{N}{}' \psi(f_1 \cdots f_{i-1} f_j f_{i+1} \cdots f_N)\} = 0, \tag{5.7}$$

for $m=0, 2, \cdots, N$. The compatability of Eqs. (5.7) leads to the eigenvalue problem

$$\tfrac{1}{2}m^2\alpha(m-2)+[N(N-1)+(2N-1)m-2m^2]\alpha(m) +2(N-m)(N-m-1)\alpha(m+2)=N(N-1)\kappa\alpha(m) \quad (5.8)$$

for $\alpha(m)$ and its corresponding eigenvalue which, for convenience, we have denoted by $N(N-1)\kappa$. When (5.8) is satisfied, we obtain the equation

$$(2\epsilon_{f_1}+\cdots+2\epsilon_{f_N}-E)\psi(f_1\cdots f_N) -g\sum_{i=1}^{N}\sum_{f}\psi(f_1\cdots f_{i-1}ff_{i+1}\cdots f_N) +\tfrac{1}{2}\kappa g\sum_{i,j=1}^{N}{}'\psi(f_1\cdots f_{i-1}f_jf_{i+1}\cdots f_N)=0 \quad (5.9)$$

for the function ψ from Eq. (5.7). Equation (5.9) can be solved by the methods of Ref. 14 which are reproduced in Appendix I. The result is that the energy is given by

$$E=\sum_{i=1}^{N}E_i. \quad (5.10)$$

and the wave function ψ is given by

$$\psi(f_1\cdots f_N)=S\prod_{i=1}^{N}(2\epsilon_{f_i}-E_i)^{-1} \quad (5.11)$$

where S is a symmetrizer and the pair energies E_i satisfy the coupled system of equations

$$1/g+\kappa\textstyle\sum_j{}'\,1/(E_j-E_i)=\sum_f 1/(2\epsilon_f-E_i), \quad i=1\cdots N, \quad (5.12)$$

subject to the conditions

$$E_i\neq E_j, \quad \text{all} \quad i\neq j. \quad (5.13)$$

For simplicity, all of the above has been derived under the assumption that N is even. The only modification needed to include the case of N odd is to change the upper bound of m from N to $N-1$. Thus, for N odd, $m=0, 2, \cdots, N-1$.

Thus far, we have reduced the problem to calculating the isospin-symmetric states to the diagonalization of a tridiagonal matrix (5.5) and the solution of the system of coupled nonlinear algebraic equations (5.12). We can, however, proceed somewhat further and derive an explicit expression for the eigenvalue κ. This expression proceeds from the observation that the eigenvalue problem (5.8) must be related to the problem of calculating isospin eigenstates. We should therefore be able to give an explicit expression for κ in terms of T, the total isospin of the state. This may be done by a direct calculation of the isospin-symmetric eigenstates of $\mathbf{T}^2$ We do this in Appendix II, where we also verify that our states are isospin eigenstates. However, an expression for κ is most easily derived by comparing the strong coupling limit of Eqs. (5.12) with the known results for a single j shell.[3–5] In the strong-coupling limit, the energies of the isospin-symmetric states must go over to the energies of the states of a single j shell since it is these states that survive the strong coupling limit. The energies of the seniority-zero states of $2N$ nucleons in a single j shell of pair degeneracy Ω, i.e., $\Omega=\sum_{f1}$, are given by[3–5]

$$E/g=-N\Omega+\tfrac{1}{2}N(N-3)+\tfrac{1}{2}T(T+1). \quad (5.14)$$

We obtained the strong coupling limit of Eq. (5.12) by setting $\epsilon_f=0$, i.e.,

$$1/g+\kappa\textstyle\sum_j{}'\,1/(E_j-E_i)=\Omega/(-E_i), \quad i=1\cdots N. \quad (5.15)$$

Multiplying (5.15) by E_i and summing on i, we get

$$E/g=-N\Omega+\kappa N(N-1)/2. \quad (5.16)$$

Comparing (5.14) and (5.16) we then obtain

$$\kappa=[N(N-3)+T(T+1)]/N(N-1), \quad (5.17)$$

which exhibits the explicit dependence of κ on N and T. It is readily verified (see Appendix II) that (5.17) is the eigenvalue of (5.8) for states of natural isospin, i.e., even N and T or odd N and T.

The effect of the conditions (5.13) on the solution of Eqs. (5.12) can be investigated using the methods of Ref. 15. The results of this investigation are that when K of the pair energies are equal and violate the conditions (5.13), then the following conditions must be satisfied: (1) The value of the K equal pair energies must be equal to one of the values of $2\epsilon_f$ which appears on the right-hand side of (5.12). We will call this value $2\epsilon_0$. (2) The pair degeneracy Ω_0 of the level whose energy is ϵ_0 must satisfy $2\Omega_0=\kappa(K-1)$. (3) When the first two conditions are satisfied, the interaction strength must still satisfy a Kth degree algebraic equation. Condition (1) insures that the poles on the left-hand side of Eq. (5.12) are located at the same places as those on the right-hand side, and condition (2) insures that the residues at these poles are the same. Condition (3) is essentially the vanishing of a determinant which then allows for nonzero solutions of Eqs. (5.12). Thus, the conditions (5.13) are automatically satisfied except possibly for some isolated values of g. It should be noted that condition (2) is quite restrictive since, for most values of κ, there is no value of K such that K is less than or equal to N and $\kappa(K-1)$ is an even integer. For these values of κ, the conditions (5.13) are satisfied for all values of g and all single-particle spectra.

To summarize, we have the $T_0=0$, isospin-symmetric states of $2N$ nucleons given by

$$|\psi\rangle=\sum_{f_1t_1\cdots f_Nt_N}\psi(f_1t_1\cdots f_Nt_N)b_{f_1t_1}{}^{\dagger}\cdots b_{f_Nt_N}{}^{\dagger}|0\rangle, \quad (3.2)$$

where

$$\psi(f_1t_1\cdots f_Nt_N)=\delta(\textstyle\sum t_i,0)\alpha(\sum|t_i|)\psi(f_1\cdots f_N). \quad (5.2)$$

The isospin-dependent part of the wave function $\alpha(m)$ satisfies

$$\tfrac{1}{2}m^2\alpha(m-2)+[N(N-1)+(2N-1)m-2m^2]\alpha(m)$$
$$+2(N-m)(N-m-1)\alpha(m+2)=N(N-1)\kappa\alpha(m), \quad (5.8)$$

where

$$\kappa=[N(N-3)+T(T+1)]/N(N-1). \quad (5.17)$$

The isospin-independent part of the wave function ψ is given (up to a normalization constant) by

$$\psi(f_1\cdots f_n)=S\prod_{i=1}^{N}(2\epsilon_{f_i}-E_i)^{-1}, \quad (5.11)$$

where S is a symmetrizer and the pair energies E_i satisfy

$$1/g+\kappa\textstyle\sum_j' 1/(E_j-E_i)=\sum_f 1/(2\epsilon_f-E_i), \quad i=1\cdots N \quad (5.12)$$

subject to the conditions

$$E_i\neq E_j, \quad \text{for} \quad i\neq j \quad (5.13)$$

which are always satisfied except possibly on a discrete set of values of g. The energies of these states are just the sum of the pair energies

$$E=\sum_{i=1}^{N} E_i. \quad (5.10)$$

The total isospin T takes on all even (odd) values less than or equal to N for N even (odd).

The specification of which state one is solving for when one solves Eqs. (5.12) may be done by giving the $g=0$ limits of the pair-energies. This method has proven to be practical[13] in the numerical solution of (5.12) for $\kappa=2$ and it should be practical for other values of κ. However, one must be careful to specify these limits in such a way that they are consistent with the character of the isospin state. For example, consider the lowest states of a given isospin of 8 nucleons in a one-dimensional harmonic well for which $\epsilon_n=n$, $n=1\cdots\Omega$ and $\Omega_n=1$. This potential is similar to that of the Nilsson model. The $g=0$ limits of the pair-energies for the lowest states of a given isospin are given in Table IV. These

TABLE IV. The $g=0$ limits of the pair energies E_i for the lowest states of a given isospin for 4 pairs in a system with $\epsilon_n=n$, $n=1\cdots\Omega$.

i \ T	0	2	4
1	2	2	2
2	2	2	4
3	4	4	6
4	4	6	8

limits may be deduced by looking at what occupations are consistent with the Pauli principle for the state $T_0=T$. This state has $\frac{1}{2}(N+T)$ proton pairs and $\frac{1}{2}(N-T)$ neutron pairs. The allowable $g=0$ limits of the pair energies then come from the possible ways that these proton and neutron pairs may be placed in the levels of the potential.

VI. OCCUPATION PROBABILITIES

In Ref. 15, algebraic methods were developed for calculating the expectation values of operators in the eigenstates of the pairing Hamiltonian with identical nucleons. We will not develop similar methods for the states of the charge-invariant pairing Hamiltonian here. However, we will give a short derivation of an expression for the occupation probabilities of the single-particle levels.

Since our states have $T_0=0$ and are isospin-symmetric, we have

$$\langle\psi|N_{fp}|\psi\rangle=\langle\psi|N_{fn}|\psi\rangle=\tfrac{1}{2}\langle\psi|N_f|\psi\rangle. \quad (6.1)$$

We also note that we can write symbolically

$$N_f=\tfrac{1}{2}(\partial H/\partial\epsilon_f).$$

We therefore have

$$\langle\psi|N_f|\psi\rangle=\tfrac{1}{2}\langle\psi|\partial H/\partial\epsilon_f|\psi\rangle$$
$$=\tfrac{1}{2}(\partial/\partial\epsilon_f)\langle\psi|H|\psi\rangle$$
$$=\tfrac{1}{2}\sum_{i=1}^{N}(\partial E_i/\partial\epsilon_f), \quad (6.2)$$

where, since $|\psi\rangle$ is an eigenstate of H, we can take the derivative outside the brackets and we have used (5.10) for the energy. We can now differentiate Eqs. (5.12) with respect to ϵ_f and obtain a linear set of equations for $\partial E_i/\partial\epsilon_f$. Solving these equations, substituting the results into (6.2), and rearranging the terms, we obtain

$$\langle\psi|N_f|\psi\rangle=\sum_{i=1}^{N} a_i/(2\epsilon_f-E_i)^2, \quad (6.3)$$

where the a_i satisfy the linear system of equations

$$[C_i-\kappa\sum_{j=1}^{N}{}'(E_j-E_i)^{-2}]a_i+\kappa\sum_{j=1}^{N}{}'(E_j-E_i)^{-2}a_j=1, \quad i=1,\cdots,N, \quad (6.4)$$

with

$$C_i=\textstyle\sum_f 1/(2\epsilon_f-E_i)^2. \quad (6.5)$$

Once Eqs. (5.12) are solved for the pair-energies, then Eqs. (6.4) can be solved for a_i. The occupation probabilities (6.1) and (6.3) can then be easily calculated.

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APPENDIX I: SOLUTION OF EQ. (5.9)

For the sake of completeness, we sketch here the solution of Eq. (5.9) as given in Ref. 14. For $N=2$, Eq. (5.9) reduces to Eq. (4.10) and therefore its solution is included in this Appendix.

If we substitute (5.11) into the three terms of (5.9), then we obtain

$$(2\epsilon_{f_1}+\cdots+2\epsilon_{f_N}-E)\psi(f_1\cdots f_N) = S\sum_{i=1}^{N}\prod_{k\neq i}(2\epsilon_{f_k}-E_k)^{-1}, \quad \text{(I.1)}$$

$$\sum_{i=1}^{N}\sum_{f}\psi(f_1\cdots f_{i-1}ff_{i+1}\cdots f_N) = S\sum_{i=1}^{N}[\sum_{f}(2\epsilon_f-E_i)^{-1}]\prod_{k\neq i}(2\epsilon_{f_k}-E_k)^{-1}, \quad \text{(I.2)}$$

and

$$\sum_{i,j}{}'(\psi(f_1\cdots f_{i-1}f_jf_{i+1}\cdots f_N) = S\sum_{i,j}{}'(2\epsilon_{f_j}-E_i)^{-1}(2\epsilon_{f_j}-E_j)^{-1}\prod_{k\neq i,j}(2\epsilon_{f_k}-E_k)^{-1}$$

$$= 2S\sum_{i,j}{}'\frac{1}{E_j-E_i}\prod_{k\neq i}(2\epsilon_{f_k}-E_k)^{-1}, \quad \text{(I.3)}$$

where we have used a partial-fraction expansion (under the assumption that $E_i\neq E_j$) of the first factor in (I.3). Substituting these expressions into (5.9), we get

$$S\sum_{i=1}^{N}[1-g\sum_{f}1/(2\epsilon_f-E_i)+\kappa g\sum_{j}{}'1/(E_j-E_i)]\times\prod_{k\neq i}(2\epsilon_{f_k}-E_k)^{-1}=0,$$

which then leads to Eqs. (5.12).

APPENDIX II: ISOSPIN EIGENSTATES

In this Appendix we will show that out states are eigenstates of $\mathbf{T}^2$ and give an alternative derivation of the value of κ.

We consider a state $|T\rangle$ which is an isospin-symmetric eigenstate of $\mathbf{T}^2$ and T_0 with eigenvalues $T(T+1)$ and zero respectively. We have shown in Sec. V that this state must have the form

$$|T\rangle=\sum_{t_1\cdots t_N}\delta(\sum t_i,0)\alpha(\sum|t_i|)b_{t_1}{}^{\dagger}\cdots b_{t_N}{}^{\dagger}|0\rangle, \quad \text{(II.1)}$$

where δ is a Kronecker delta, α is to be determined, and we have suppressed the variables $f_1\cdots f_N$. Since $T_0=0$, we have

$$\mathbf{T}^2|T\rangle=T_+T_-|T\rangle$$

and then

$$\mathbf{T}^2|T\rangle=\sum_{t_1\cdots t_N}\delta(\sum t_i,0)\alpha(\sum|t_i|)T_+T_-\prod_{k=1}^{N}b_{t_k}{}^{\dagger}|0\rangle$$

$$=\sum_{t_1\cdots t_N}\delta(\sum t_i,0)\alpha(\sum|t_i|)\{\sum_{i=1}^{N}(\prod_{k\neq 1}b_{t_k}{}^{\dagger})[T_+[T_-,b_{t_i}{}^{\dagger}]]+\sum_{i,j=1}^{N}(\prod_{k\neq i,j}b_{t_k}{}^{\dagger})[T_+,b_{t_i}{}^{\dagger}][T_-,b_{t_j}{}^{\dagger}]\}|0\rangle. \quad \text{(II.2)}$$

In order to write the commutators in (II.2), let us define $C_{\pm}$ by

$$[T_{\pm},b_t{}^{\dagger}]=\sqrt{2}\sum_{t'}C_{\pm}(t,t')b_{t'}{}^{\dagger}, \quad \text{(II.3)}$$

then, using Table II, we have

$$C_+(t,t')=\delta(t,0)\delta(t',+)+\delta(t,-)\delta(t',0) \quad C_-(t,t')=\delta(t,+)\delta(t',0)+\delta(t,0)\delta(t',-). \quad \text{(II.4)}$$

Substituting (II.3) and (II.4) into (II.2) and relabelling some of the summation indices, we have

$$\mathbf{T}^2|T\rangle=2\sum_{t_1\cdots t_N}\delta(\sum t_i,0)\{\sum_{i=1}^{N}(\delta(t_i,0)+\delta(t_i,+))\alpha(\sum|t_k|)+\sum_{i,j=1}^{N}{}'[(\delta(t_i,+)\delta(t_j,0)+\delta(t_i,0)\delta(t_j,-))\alpha(\sum|t_k|)$$

$$+\delta(t_i,+)\delta(t_j,-)\alpha(\sum|t_k|-2)+\delta(t_i,0)\delta(t_j,0)\alpha(\sum|t_i|+2)]\}\prod_{k=1}^{N}b_{t_k}{}^{\dagger}|0\rangle. \quad \text{(II.5)}$$

The sums on i and j may be performed using

$$\sum_i\delta(t_i,0)=N-m \quad \sum_i\delta(t_i,\pm)=m/2,$$

where

$$m=\sum|t_i|.$$

With these results, the isospin eigenvalue problem is

$$\tfrac{1}{2}m^2\alpha(m-2)+[2N+(2N-1)m-2m^2]\alpha(m)+2(N-m)(N-m-1)\alpha(m+2)=T(T+1)\alpha(m). \quad \text{(II.6)}$$

If we now add $N(N-3)\alpha(m)$ to both sides of (II.6), then we obtain Eq. (5.8) with κ given by (5.17). Thus we have proven that the states (5.2) are isospin eigenstates and we have given an alternative derivation of the value of κ (5.17).

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Effective Interaction for Nuclear Matter

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The effective interaction in nuclear matter is defined as the long-range part of the two-body potential, which, in the Born approximation, gives the single-particle potential energy for the average momentum in the Fermi sea. For the Brueckner-Gammel-Thaler potential the effective interaction has been calculated, first for the free-particle propagator and then for the nuclear spectrum. The result shows that in the first case the separation distance ξ_0 is constant over a wide range of densities and does not lead to saturation. The nuclear separation distance $\xi(k_F)$ changes quite rapidly with the Fermi momentum k_F; for low densities it is very close to ξ_0, while for higher densities it becomes very much larger. At the density corresponding to $k_F=1.5$ F^{-1}, the long-range potential starts at $\xi=1.16$ F, and the rate of change of ξ with k_F is $(d\xi/dk_F)=0.8$ F^2. The minimum of the total energy per particle occurs at $k_F=1.35$ F^{-1} and is about -9 MeV. For $k_F=1.5$ F^{-1} the contributions of different partial waves are also calculated by a variational technique, and the results have been compared with previous calculations.

1. INTRODUCTION

NUCLEON-nucleon scattering at high energies suggests that nuclear forces become strongly repulsive at small distances. If these forces bind a system of many nucleons together, then at least for low densities, the interaction on the average must be attractive, and the effect of the repulsive part cancels only part of the effect of the attractive potential. Therefore it is possible to find an effective interaction which depends on the density and represents the remaining part of the attractive force.

Moszkowski and Scott[1] originally introduced the idea of separating the interaction in the two-body t matrix in such a way that the short-range part contributes nothing to the phase shift in each partial wave. Then the effective interaction, to the first order, is the Born approximation of the long-range part. Here we introduce an average separation distance for all of the states in the Fermi sea by the requirement that the expectation value of the short-range part of the many-body Hamiltonian should vanish. The long-range part which is the effective Hamiltonian will depend on the Fermi momentum (or density) of the system of nucleons. Thus the effect of the short-range part of the interaction may be replaced by a separation distance $\xi(k_F)$ and other physical quantities of interest, like the rearrangement energy and the compressibility, can be expressed in terms of the long-range interaction $\xi(k_F)$ and its derivatives with respect to k_F.[2] From the definition of $\xi(k_F)$ it is clear that its functional form depends on the shape and the strength of the short-range interaction and for different potentials it takes different forms. Here it is assumed that the two-nucleon interaction is given by the Brueckner-Gammel-Thaler (BGT) potential.[3] This potential is preferred over the semiphenomenological potentials of Breit[4] and Hamada and Johnson[5] for two reasons: (a) The BGT potential has a simple analytic form and (b) there are at least three other independent calculations of the binding energy of nuclear matter with this potential. It is therefore possible to compare our calculations with the results of other calculations.

Since the exact solution of the reaction matrix is quite complicated we calculate the separation distance by iteration. In Sec. 2 we find the integral equation for the reaction matrix and the condition that the solution of this equation must satisfy in order that the energy shift due to the perturbation be zero. In Sec. 3 we solve the reaction matrix by neglecting the exclusion principle and using the free-particle propagator e^F. We find that

[1] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960), hereafter referred to as MS.

[2] M. Razavy and S. J. Stack, Can. J. Phys. **43**, 605 (1965).

[3] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958).

[4] K. E. Lassila, M. H. Hull, Jr., H. M. Rupple, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).

[5] T. Hamada and I. D. Johnson, Nucl. Phys. **34**, 383 (1962).