Wave Propagation in a Dispersive and Emissive Medium*

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Recently a measurement of the velocity of longitudinal ultrasonic waves propagating through a phonon maser amplifier showed that the pulse traveled with the maximum or cutoff velocity at resonance. With an approximate dispersion rule for a homogeneously broadened spin-resonance transition, the propagation of the wave is analyzed using a method for the asymptotic expansion of Fourier-type integrals. This analysis, contrary to earlier studies, indicates that the pulse observed, or at least the leading edge of that pulse, was an amplified precursor (transient response). For a sinusoidal imput signal, the asymptotic wave form at a given position is given for all possible values of the time.

I. INTRODUCTION

RECENTLY, Shiren[1] has made a measurement of the velocity of longitudinal ultrasonic waves propagating through a phonon maser amplifier. The result shows that the pulse traveled with the maximum or cutoff velocity; the units chosen here are such that this velocity is 1. Shiren's conclusion is that the signal, defined as the steady-state response, travels with the velocity 1 for the case of amplification with the input-signal frequency at resonance. Some theoretical results are presented here, based on a technique of asymptotic expansion of Fourier-type integrals,[2] which differ from the results of the analysis given by Shiren. These results show that the only wave which travels with velocity 1 is an amplified precursor (a transient response) and the signal velocity, Eq. (20a) or (20b), is less than 1.

For a sinusoidal input signal

$$I(0,t)=\sin\omega' t;\quad t>0$$
$$=0;\quad t<0, \tag{1}$$

the solution to the wave equation for a plane wave traveling in the positive x direction, by dispersion theory, is

$$I(x,t)=\lim_{\epsilon\to 0}\operatorname{Re}\left(-\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\exp\{-i[\omega t-kx]\}\times\frac{d\omega}{\omega-\omega'+i\epsilon}. \tag{2}$$

In the following, the approximate dispersion rule for a homogeneously broadened spin resonance transition,

$$k=\omega\mp a/(\omega_0-\omega-i\rho), \tag{3}$$

valid near resonance ($\omega'=\omega_0$) under the conditions $\rho/\omega_0\ll 1$ and $a/\rho\ll\omega_0$, is used.[1,3] Here ρ is the half-width of the resonance and a/ρ with the upper (lower) sign is the amplification (absorption) per unit length at resonance. Thus, for the case of amplification, Eq. (2) is written as

$$I(x,t)=\lim_{\epsilon\to 0}\operatorname{Re}\left(-\frac{1}{2\pi}\right)\int_{-\infty}^{\infty}\exp\left\{-it\left[(1-\xi)\omega+\frac{a\xi}{\omega_0-\omega-i\rho}\right]\right\}\frac{d\omega}{\omega-\omega'+i\epsilon}, \tag{4}$$

where $\xi=x/t$.

Before the asymptotic solution of I is investigated, the special case of $\xi=1$ is considered. For this case, Eq. (4) can be written as

$$I=\lim_{\epsilon\to 0}\frac{(\omega_0-\omega')}{2\pi}\int_{-\infty}^{\infty}\exp\left\{\frac{ita}{z+i\rho}\right\}\times[(z+i\epsilon)^2-(\omega_0-\omega')^2]^{-1}dz. \tag{5}$$

The contour of integration can be completed in either the upper or the lower half complex plane with the integration along the path at ∞ giving zero. This follows from the z^2 dependence in the denominator of the integrand. Since the integrand is analytic in the upper half plane one concludes that $I=0$ for $\xi=1$. In completing the contour in the lower half-plane, since the result must also be zero, the contribution from the poles which gives rise to the steady-state response must exactly cancel the contribution from the essential singularity at $z=-i\rho$. Thus it is concluded that in the neighborhood of $\xi=1$, the signal (steady-state response) cannot dominate and that probably there will be a precursor (a dominant transient response).

II. SIGNAL VELOCITY

For the asymptotic expansion of I given in Eq. (4), the parameter t is considered large. More specific conditions will be given in the sequel. In order to obtain the asymptotic expansion, the integral is converted into a Fourier integral by the transformation

$$\zeta=\omega(1-\xi)+a\xi/(\omega_0-\omega-i\rho), \tag{6}$$

with the inverse transformation

$$\omega=[\zeta+(1-\xi)(\omega_0-i\rho)+(\zeta-\zeta_-)^{1/2}(\zeta-\zeta_+)^{1/2}]/2(1-\xi), \tag{7}$$

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[1] N. S. Shiren, Phys. Rev. Letters **15**, 341 (1965); *ibid.* **15**, 597(E) (1965).
[2] T. A. Weber, D. M. Fradkin, and C. L. Hammer, Ann. Phys. (N. Y.) **27**, 362 (1964).
[3] N. S. Shiren, Phys. Rev. **128**, 2103 (1962).

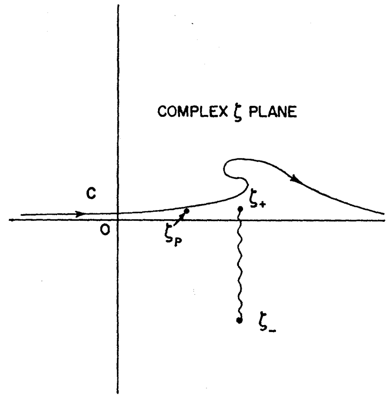

FIG. 1. Schematic drawing of the contour of integration in the complex ζ plane. The phases of the double-valued functions are zero to the right of their respective branch points ζ_+ or ζ_-.

where $\zeta_\pm=(1-\xi)(\omega_0-i\rho)\pm 2i[a\xi(1-\xi)]^{1/2}$. Then I, given by Eq. (4), can be written as

$$I=I_1+I_2, \quad (8)$$

where

$$I_1=-\frac{1}{4\pi}\int_C \frac{e^{-i\zeta t}}{\zeta-\zeta_p}d\zeta, \quad (9)$$

and

$$I_2=-\frac{1}{4\pi B}\int_C e^{-i\zeta t}\frac{[\zeta-(1-\xi)(\omega_0-i\rho)]B-2a\xi}{(\zeta-\zeta_p)(\zeta-\zeta_+)^{1/2}(\zeta-\zeta_-)^{1/2}}. \quad (10)$$

Here $B=\omega_0-\omega'-i\rho$ and the position of the pole is given by $\zeta_p=(1-\xi)\omega'+a\xi B^{-1}$. In all results obtained from Eqs. (9) and (10), the real part must be taken.

A schematic drawing of the contour of integration in the complex ζ plane is given in Fig. 1. The phases of the double-valued functions are zero to the right of their respective branch points at ζ_+ and ζ_- on the sheet shown in the figure. To obtain the asymptotic expansion, the contour is distorted into the lower half complex ζ plane, indenting where necessary around branch points along lines of constant phase of $e^{-i\zeta t}$ as shown in Fig. 2. The integrand has an exponential damping factor along the path at infinity and thus the integration over this path gives zero. Also, along the lines of constant phase (vertical lines) toward infinity, the modulus of $e^{-i\zeta t}$ damps out most rapidly. For each singularity an integral is obtained. For each integral a translation of coordinates is made so that the singularity lies at the origin. This is accomplished by the transformation $\zeta=z+\zeta_s$, where ζ_s is the position of the singularity in the complex ζ plane. Thus, each integral about a branch point will be of the form $e^{-\zeta_s t}$ times a Laplace-type integral. The asymptotic expansion is then obtained by the Laplace method, i.e., by expanding the integrand, except for the exponential and the singular function giving rise to the branch point, about the origin in a power series and integrating term by term. Thus for each branch point one obtains an exponential $e^{-i\zeta_s t}$ times a series of decreasing powers of t. The exact contribution from the pole at ζ_p is of course obtained and has the exponential $e^{-i\zeta_p t}$.

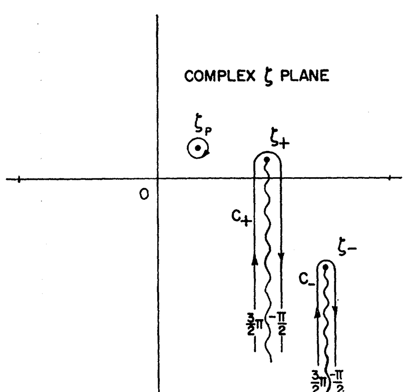

FIG. 2. The distorted contour of integration in the complex ζ plane. The branch point at ζ_- has been displaced to the right.

It then becomes apparent that the dominant contribution to the asymptotic expansion of I will come from the singularity with maximum $\mathrm{Im}\zeta_s$. This presupposes that the given singularity actually exists. For a case given below it is found that the pole dominates in the above sense but that its residue is zero, i.e., the sum of the integrands of Eqs. (9) and (10) is analytic at ζ_p. For the conditions under which the pole contribution will dominate, $\mathrm{Im}\zeta_p$ is set equal to $\mathrm{Im}\zeta_+$. Solving the quadratic equation, one obtains

$$A\equiv\left[\frac{a\xi}{1-\xi}\right]^{1/2}=\frac{1}{\rho}\{D\pm(D^2-\rho^2D)^{1/2}\}\equiv A_\pm, \quad (11)$$

where $D=(\omega_0-\omega')^2+\rho^2$. Thus the pole will dominate for

$$A\geq A_+ \quad (12a)$$

or

$$A\leq A_-. \quad (12b)$$

If both of these actually applied, a peculiar phenomenon would occur. That is, at a given point x the pole contribution or steady state response would first be observed at the cutoff velocity $\xi=1$, according to Eq. (12a). Then as time progressed and neither Eq. (12a) nor (12b) applied, the transient response would dominate and finally, the steady-state response would again dominate according to Eq. (12b) for times to infinity. Evidently Eq. (12a) is not applicable since it contradicts the analysis of Eq. (5). To clear up the contradiction, the contributions from the singularities must be examined in detail.

The contributions from the pole are easily obtained from Eqs. (9) and (10) by the residue theorem, and are

$$I_{1p}=\tfrac{1}{2}ie^{-i\zeta_p t}, \quad (13)$$

and

$$I_{2p}=-\tfrac{1}{2}ie^{-i\zeta_p t}\frac{[B^2+A^2]}{B[(B+A^2/B)^2]^{1/2}}, \quad (14)$$

where

$$B+A^2/B=(\omega_0-\omega')[1+A^2/D]+i\rho[A^2/D-1]. \quad (15)$$

It must now be determined whether a plus or a minus square root should be taken in Eq. (14). The square root arises from $(\zeta_p-\zeta_+)^{1/2}(\zeta_p-\zeta_-)^{1/2}$ and the phase of the square root can be determined by finding the position of ζ_p with respect to $\zeta_\pm$ in the complex ζ plane. Now

$$\zeta_p-\zeta_\pm=(1-\xi)\{[A^2/D-1](\omega_0-\omega')+i\rho[A^2/D+1]\mp 2iA\}. \quad (16)$$

Let $\omega'>\omega_0$. Then for $A^2>D$ it is seen that $\mathrm{Re}(\zeta_p-\zeta_\pm)$ is negative so that the pole lies to the left of $\zeta_\pm$ as shown in Fig. 2. The pole lies in the second or third quadrant as measured from ζ_+ or ζ_- so that the phase of $(\zeta_p-\zeta_\pm)$ is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. Thus the phase of $(\zeta_p-\zeta_\pm)^{1/2}\times(\zeta_p-\zeta_-)^{1/2}$ is between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. From Eq. (15) it is then seen that the positive square root must be taken in Eq. (14) and the sum of the pole contributions I_{p_1} and I_{p_2} is zero. For the case $A^2<D$, the negative square root must be taken and the contribution due to the pole is

$$I_p=i\exp[\rho ax/D]\exp\{-i[(t-x)\omega'+(ax/D)(\omega_0-\omega')]\}. \quad (17)$$

Hence, for the pole to contribute, the necessary condition is[4]

$$A^2\leq D. \quad (18)$$

By using a similar argument to that given above, it is easily shown that Eq. (18) also applies for the case of resonance where $\omega'=\omega_0$.

The condition given in Eq. (18) is compatible only with the condition given in Eq. (12b), since

$$A_+\geq D^{1/2}\geq A_- \quad (19)$$

for all values of ω'. Therefore, it is concluded that the velocity of the signal or steady state response is given by ξ determined from $A=A_-$. Thus,

$$v_s=\frac{D[D^{1/2}-|\omega'-\omega_0|]^2}{\rho^2a+D[D^{1/2}-|\omega'-\omega_0|]^2}<1, \quad (20a)$$

where $D=(\omega'-\omega_0)^2+\rho^2$. This velocity decreases as one moves the input signal away from the resonant frequency ω_0. This comes about since the precursor has the factor

$$\exp\{2[ax(t-x)]^{1/2}-\rho(t-x)-i(t-x)\omega_0\},$$

which is independent of ω'. Thus the amplification due to this factor is independent of the input frequency. On the other hand, the signal has the amplification factor $\exp(a\rho x/D)$ which decreases as ω' moves away from resonance. This seems strange since one would expect not only the signal to decrease but the precursor to disappear as the interaction with the resonance decreases. So far we have assumed the dominance of the exponential factors which is strictly true in the asymptotic limit of large t (large x). For finite times, other factors become important, and it is found in Sec. III that the precursor decreases and approaches zero as $\omega'\rightarrow\infty$. Depending on how much the transient response is decreased due to these other factors, the signal velocity will increase over that given in Eq. (20a). It should be noted that it is not simply a masking effect of the transient over the steady state. If the transient dominates then terms neglected in the expansion about ζ_+ will also dominate the pole contribution which therefore cannot be kept. Thus the theory does not predict an increase in the signal velocity due to tuning the detector so as to discriminate against the transient response. However, if the transient decreases sufficiently—because, for example, $\omega'-\omega_0$ is large,—then the leading term in the asymptotic expansion in the region $D^{1/2}>A>A_-$ comes from the pole and the signal velocity is then given by the ξ determined from $A^2=D$. Thus for this limiting case,

$$v_s'=D/(a+D). \quad (20b)$$

This agrees with the velocity v_s given in Eq. (20a) for the case $\omega'=\omega_0$ but as $\omega'\rightarrow\infty$, $v_s'\rightarrow 1$. The velocity v_s' is the same as the signal velocity for the case of absorption[5] where a is positive but a/ρ has the meaning of absorption per unit length.

In the above analysis the pole enters in a discontinuous way. But for off resonance, the asymptotic solution is continuous since the pole first enters with ζ_+ having the dominant exponential. Therefore, terms neglected in the series generated by the branch point at ζ_+ will dominate the contribution from the pole which should be neglected. The solution for the transition at $A=A_-$ from precursor to signal is continuous since the pole will contribute on both sides of the transition and must be taken into account. Now for the case of resonance, $A_+=A_-=D^{1/2}$ so that the pole first contributes just at the time when it and the singularity at ζ_+ have equal exponentials. The treatment so far suggested would give a discontinuous result. One notes that for resonance as $A\rightarrow D^{1/2}$, $\zeta_p\rightarrow\zeta_+$, so that the power series expansion of the integrand for the contour about ζ_+ will have a radius of convergence which approaches zero. To avoid this difficulty, the singularities at ζ_p and ζ_+ for the contour about ζ_+ are treated jointly, i.e., in the expansion of the integrand both of the functions $(\zeta-\zeta_+)^{-1/2}$ and $(\zeta-\zeta_p)^{-1}$ are left unexpanded. Thus the

[4] In the complex ω plane the pole at $\omega=\omega'-i\epsilon$ gives the steady state response which is present for all values of ξ less than one. Thus it can be said that this response travels with velocity equal to 1; however, the transient response given by the other singularities in the ω plane cancel this for $\xi=1$. Here in the ζ plane there is no pole for $A^2\geq D$, so the cancellation is automatically accomplished in the transformation. What is left after the cancellation is given by the singularities at ζ_+ and ζ_- in the ζ plane, and is also referred to as the transient response.

[5] H. Baerwald, Ann. Physik 7, 731 (1930).

radius of convergence will be the distance to the next nearest singularity at ζ_-.

III. WAVE FORM

The asymptotic wave form is now given for the various cases.

1. First in the region $\xi\simeq 1$, the two branch points must be treated jointly since as $\xi\rightarrow 1$, $\zeta_+\rightarrow\zeta_-$. Here the pole does not contribute. Then for the contribution from ζ_+, the integrand of Eq. (10), except for the two branch points and the exponential, is expanded about ζ_+. The first term gives

$$I_{2+}=\frac{-1}{4\pi B}\left\{\frac{[\zeta_+-(1-\xi)(\omega_0-i\rho)]B-2a\xi}{\zeta_+-\zeta_p}\right\}\times e^{-i\zeta_+t}\int_{-i\infty}^{0-}e^{-izt}z^{-1/2}(z+\zeta_+-\zeta_-)^{-1/2}dz. \quad (21)$$

The contour is C_+ as shown in Fig. 2 with ζ_+ at the origin of the z complex plane. The phase of z at the end of the contour is $\frac{1}{2}\pi$ and the phase of $\zeta_+-\zeta_-$ on the line connecting the points $z=0$ and $\zeta_--\zeta_+$ is $\frac{1}{2}\pi+\delta$. Here δ is a positive parameter introduced so that ζ_+ and ζ_- have the relative positions as shown in Fig. 2. The limit $\delta\rightarrow 0$ is taken at the end of the calculation. The integral which is now well defined can be written in terms of a Hankel function[6,7] to give

$$-\pi i\exp[i(\zeta_+-\zeta_-)t/2]H_0^{(1)}(|(\zeta_+-\zeta_-)t/2|e^{-i\pi/2}).$$

A similar result is obtained from the contour about ζ_- and the sum of the two contributions is

$$I=-\tfrac{1}{2}A\exp[-i(1-\xi)(\omega_0-i\rho)t]\times\{(B-iA)^{-1}H_0^{(1)}(2t[a\xi(1-\xi)]^{1/2}e^{i\pi/2})+(B+iA)^{-1}H_0^{(1)}(2t[a\xi(1-\xi)]^{1/2}e^{-i\pi/2})\}. \quad (22)$$

It is easily found that $\mathrm{Re}I=0$ in the limit $\xi\rightarrow 1$. This is in agreement with the conclusions following Eq. (5). It should be noted that the asymptotic expansion of the Hankel functions cannot be used here since for $\xi=1$ the arguments of these functions are zero.

2. For the case $A>D^{1/2}$ but $\xi\neq 1$, the branch point ζ_+ dominates and the residue of the pole is zero. The first term in the expansion about ζ_+ of Eq. (10) gives

$$I=-\frac{1}{4\pi B}|\zeta_+-\zeta_-|^{-1/2}e^{-i\pi/4}\times\left\{\frac{[\zeta_+-(\omega_0-i\rho)(1-\xi)]B-2a\xi}{\zeta_+-\zeta_p}\right\}\times e^{-i\zeta_+t}\int_{-i\infty}^{0-}e^{-izt}z^{-1/2}dz, \quad (23)$$

where the contour is C_+ of Fig. 2 with ζ_+ at the origin of the z complex plane. The integral[8] is equal to $-2\pi^{1/2}e^{3\pi i/4}t^{-1/2}$ so that

$$I=\tfrac{1}{2}i\pi^{-1/2}[ax(t-x)]^{-1/4}\frac{A}{A-iB}\times\exp\{2[ax(t-x)]^{1/2}-\rho(t-x)-i(t-x)\omega_0\}. \quad (24)$$

This is the same as that obtained from Eq. (22) by taking the asymptotic expansion of the Hankel functions and keeping the leading term.

3. For $A<D^{1/2}$ but $A>A_-$, the pole will have a nonzero residue, but the branch point at ζ_+ will dominate giving the same result as case 2. It should be noted that this transient response is amplified for $A>A_-$, but that for a given position of observation, this transient will eventually become exponentially damped as t increases.

4. For $A<A_-$, the contribution from the pole will dominate giving Eq. (17).

5. For the case of resonance, $\omega'=\omega_0$, the above results apply except in the region of transition from precursor to signal at $A=\rho$. Here $\zeta_p=\zeta_+$ for $A=\rho$ so that the two singularities must be treated jointly. For $A>\rho$, the pole has zero residue so that the only contribution comes from the branch point at ζ_+. The first term in the expansion of Eq. (10) gives

$$I=-(4\pi B)^{-1}e^{-i\pi/4}[a\xi(1-\xi)]^{1/4}(iB-A)e^{-i\zeta_+t}\times\int_{-i\infty}^{0-}e^{-izt}z^{-1/2}[z+\zeta_+-\zeta_p]^{-1}. \quad (25)$$

The integral can be written in terms of regular confluent hypergeometric functions[9] to yield

$$I=(i/2\rho)[a\xi(1-\xi)]^{1/4}[\rho/(1-\xi)]^{1/2}e^{-i\zeta_+t}\times\{\exp[(1-\xi)(A-\rho)^2t/\rho]-2\pi^{-1/2}t^{1/2}[(1-\xi)/\rho]^{1/2}\times|A-\rho|\,{}_1F_1(1,\tfrac{3}{2},(1-\xi)(A-\rho)^2t/\rho)\}. \quad (26)$$

For $A<\rho$, the branch point at ζ_+ gives exactly the same result as that given in Eq. (26) except that the over-all sign is changed. To this must be added the pole contribution which is given by Eq. (17) with $\omega'=\omega_0$. Taking the limit $A\rightarrow\rho$ with either the solution for $A>\rho$ or $A<\rho$ the same result

$$I=\tfrac{1}{2}ie^{ax/\rho}e^{-i(t-x)\omega_0}, \quad (27)$$

which is just half of the pole contribution, is obtained.

[6] See for example Ref. 2, Eqs. (3.6) and (3.9).

[7] The integrations can also be done by using the convention that the branch point at ζ_- approaches the branch line of ζ_+ from the left. A different result from that given in Eq. (22) is obtained since the integrals giving I are not done exactly. However, if B can be neglected compared to A, which is the case for $\xi\rightarrow 1$, then the results are identical. Also, in the limit of large arguments of the Hankel functions, the results will be the same.

[8] See for example Ref. 2, Eq. (2.6).

[9] See for example Ref. 2, Eq. (3.6) and Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi, (McGraw-Hill Book Company, New York, 1953), Vol. 1, Sec. 6.5, Eq. (7).

This solution is therefore continuous in the transition from precursor to signal.

If the incident signal is just off resonance, then in the transition region $A=A_-$ it may be necessary for greater accuracy to treat the singularities jointly because of their proximity. The result for $A^2\leq D$ is

$$I_{2+}=-\frac{1}{2\pi B}[a\xi(1-\xi)]^{1/4}(iB-A)\{\pi g^{-1/2}e^{-i(\zeta_p t+\pi/4)} -2\pi^{1/2}t^{1/2}e^{-i\zeta_+ t}{}_1F_1(1,\tfrac{3}{2},git)\}, \quad (28)$$

where $g=\zeta_+-\zeta_p$ and $-\frac{3}{2}\pi\leq \arg g<-\frac{1}{2}\pi$. To this must be added the contribution from the pole to obtain I. If $A^2\geq D$, the pole contribution is zero and the result for the integral about ζ_+ is the same as that given in Eq. (26) but here $-\frac{1}{2}\pi<\arg g\leq\frac{1}{2}\pi$.

The situation in the region $A^2\approx D$ may best be described as in Fig. 3. Here I is determined using Eq. (10) only. For $A^2>D$, ζ_p gives the location of the pole with no contour about it. As ξ decreases, the pole will move toward the branch line and will be at the branch line for $A^2=D$. Crossing the branch line is done in a continuous manner (the pole is in both Riemann sheets) by continuously distorting the contour and finally pinching off separate contours for the pole as shown in Fig. 3. The positions of the pole in each sheet are the same but are drawn separately in the figure. The dashed contour is on the undersheet. The sum of the contributions from the contours about the pole gives the steady-state response.

The approximations that are made are limited by the fact that the power series expansions have finite radii of convergence. This in turn limits the number of terms one may keep in the expansion (only the first term is kept in the above). The radius of convergence r for the expansion in the neighborhood of a given singularity will be the distance to the next nearest singularity. Then the number of terms $N+1$, if any, that may be taken in the expansion must meet the condition[2]

$$rt>N+\lambda, \quad (29)$$

where λ is the power of the singularity about which the expansion is made ($\lambda=-\frac{1}{2}$ for the expansion about ζ_+). This condition insures that on the path of integration, the terms retained in the series expansion of the integrand have their maximum values within the region of convergence. Therefore, the contribution of these terms to the value of the integral will be significant only in the region of convergence. If more terms are taken than allowed under the above condition, the result will diverge from the best approximate answer. The first neglected term in the series may be used as the error.

It is difficult to set any general condition that would limit the error in the expansion to some tolerable amount. For some cases, however, the expansion is a series in powers of $(rt)^{-1}$ with decreasing coefficients.[8]

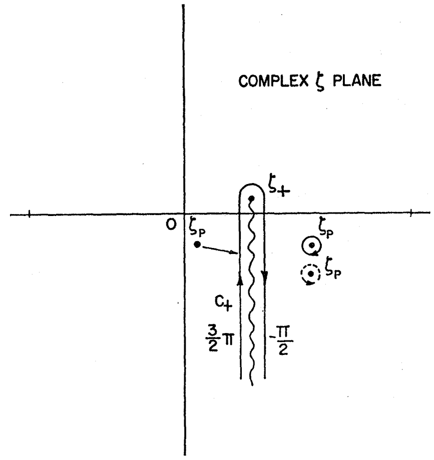

FIG. 3. Motion of the pole with respect to the branch line as ξ increases. As the pole passes continuously across the branch line it picks up a contour on both sheets as shown.

In particular, the second coefficient in the expansion which gives Eq. (24) for the precursor is down at worse by a factor of $\frac{3}{4}$. Therefore, as a rough rule, the minimum allowable value of rt may be taken equal to one. Then, in addition to Eq. (29), one has the condition

$$rt>1. \quad (30)$$

This minimum value of rt will insure that Eq. (29) is fulfilled with $N=0$ in the cases considered here.

The group velocity calculated in the usual way ($v_g=d\omega/dk_R$) actually agrees with Eq. (20a) at resonance but far from resonance becomes greater than one. Here it is assumed that the dispersion rule is not restricted to frequencies near resonance. As noted earlier, in the limit $\omega'\rightarrow\infty$, Eq. (20a) gives a velocity less than 1. However, from Eq. (24) it is seen that the precursor has the factor $[A+\rho-i(\omega'-\omega_0)]^{-1}$. For finite time, one can always find an ω' such that this factor will overcome the amplification of the exponential so that the signal velocity will be increased over that given in Eq. (20a). In particular, as $\omega'\rightarrow\infty$, the signal velocity now given by Eq. (20b) will approach 1.

IV. DISCUSSION AND CONCLUSION

First it must be determined whether the asymptotic conditions are fulfilled in the experiment performed by Shiren. For example, the range of values of $A=[a\xi/(1-\xi)]^{1/2}$ for which Eq. (24) gives the precursor, can be determined. Considering only the case of resonance one sees that as $\xi\rightarrow 1$, ζ_- approaches ζ_+ so that the expansion about ζ_+ will have decreasing radius of convergence. This puts an upper limit on A (for A's larger than this, Eq. (22) gives the precursor). As $A\rightarrow\rho$, ζ_p approaches ζ_+ so this sets a lower limit on A. Letting $A=\alpha\rho$ and applying Eq. (30) for the two cases, one finds that α can range between 3.2 and 12 for $ax/\rho=3$. This value of ax/ρ is the largest used in the experiment and is the best as far as the application of the asymptotic expansion. If ax/ρ is such that no α can be found, then there will be values of A for which none of the solutions will apply.

The smallest value of ax/ρ, for which $A=\rho$ gives the signal velocity at resonance, can be estimated using Eq. (30). For $A=\rho$, the wave is one half of the final steady state as given in Eq. (27). This is considered the beginning of the signal. In obtaining Eq. (27), the singularities at ζ_p and ζ_+ are treated jointly so that the radius of convergence for the expansion is the distance from $\zeta_p=\zeta_+$ to ζ_-. Applying Eq. (30), one finds the minimum value of ax/ρ equal to 0.25. While the above limits are only rough estimates, they do indicate that the asymptotic expansions can be validly applied to the experiment.

Furthermore, if the precursor was observed, it appears that the signal should also have been seen. The signal velocity at resonance is the same for both amplification and absorption so that, in Fig. 1(a) of Ref. 1, one should expect to see the signal delayed by the amount shown in Fig. 1(b). If the signal is a sinusoidal pulse of finite duration τ (an integral number of periods) then the observed pulse is just the solution given above minus the same solution with the time changed to $t-\tau$. Thus the finite pulse can be thought of as two semi-infinite wave trains, one incident at $t=0$ and the other incident at $t=\tau$. Now if τ is of the order of the delay time, then just as the steady-state response is reached for the first wave train, the transient response begins to grow for the second wave train. Taking the difference one would expect to obtain a pulse with roughly a duration of 2τ, the leading half composed of the transient response and the trailing half composed mainly of the steady state response. In the experimental situation τ was equal to 0.5 μsec and it appears that the delay time was almost equal to this. From the wave form given in Sec. III under the conditions of the experiment, there seems to be no reason to expect a sharp division between the transient and the steady-state response, and they may appear as one pulse. Unlike the case of amplification where the main frequency component of the precursor is ω_0, the precursor for the case of absorption has frequency components split above and below the resonant frequency ω_0 and may be discriminated against in the detection. Then, for absorption, the pulse will be composed of the steady state response and have duration τ.

It is dangerous to assume, without making the detailed subtraction, that the signal in the case of amplification is just an amplification factor times the signal in the case of absorption. If such an assumption were true, then the shape of the trailing edge of the amplified pulse would be quite different from that which is observed. It was seen in Sec. III that for resonance in the case of amplification, the signal and the precursor have the same frequency and also the pole at ζ_p and the branch point ζ_+ must be treated jointly in the transition from precursor to signal, since $\zeta_p \rightarrow \zeta_+$. This is not true for the case of absorption. This distinction will surely make a difference in the shape of the pulses for the two cases, particularly in their trailing edges.

It should also be pointed out that no firm conclusion can be drawn from the time duration of the pulses in the study of the figures of Ref. 1 unless one knows the shape of the pulse (i.e., whether there are long tails of small amplitude), the effect of the detection apparatus on the pulse, and the masking effect of the noise on the small amplitude components of the pulse.

It appears that if the approximate dispersion rule applies, the leading edge of the pulse with velocity approximately equal to 1 was actually the amplified precursor. However, Shiren[10] has proposed a further experiment to test whether the pulse was the precursor or the signal by making use of the fact that the main frequency component of the precursor is ω_0.

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[10] N. S. Shiren (private communication).