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Surface-Wave Instability in Helicon-Wave Propagation

G. A. BARAFF AND S. J. BUCHSBAUM
Bell Telephone Laboratories, Murray Hill, New Jersey
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A novel instability is described which can be utilized to amplify helicon waves in a single-component solid-state plasma. The instability is created by carrier drift at threshold velocities which can be smaller than the phase velocity of the wave which is being amplified. The instability can be excited in composite structures made up of two or more layers of solid-state plasma of different carrier densities with the interface parallel to an applied static magnetic field. It is intrinsically connected with the presence of a surface wave at the interface between the two media.

I. INTRODUCTION

It is now well established that transverse electromagnetic waves can propagate with little attenuation through solid-state plasmas in metals and semiconductors.[1] Because the phase velocity of these waves (helicon or Alfven waves) can be made very much smaller than the speed of light, the idea of amplifying them has intrigued many workers. The most notable of the various amplification schemes is that of Bok and Nozières.[2] These authors pointed out that a helicon wave in a two-component plasma (of unequal electron and hole concentration) can be unstable when the electrons and holes are made to drift relative to each other. A necessary, though not sufficient, condition for the instability to occur is that the drift velocity exceed the phase velocity of the wave. Although the Bok-Nozières scheme has been criticized by some authors,[3] there is no question that various instabilities can be induced by drifts in excess of the phase velocity of the wave. In gaseous plasmas, a myriad of such instabilities is well known. In solids, the establishment of large drift is not easy since it is then accompanied by large heat dissipation.

In the present paper, we discuss a novel instability associated with propagation of helicon waves in bounded composite plasmas (waveguides) which can be excited with threshold drift velocities *smaller* than the phase velocity of the wave. The structure which we consider is shown in Fig. 1. Media I and II are dissimilar solid-state plasmas in the form of thin, infinite slabs. A static magnetic field B_0 is oriented parallel to the interfaces between the two media. As pointed out by Legendy and by Klozenberg, McNamara, and Thonemann,[4] in a finite medium (such as a single slab or a cylinder oriented parallel to a static magnetic field), the propagation of a helicon wave is accompanied by a surface wave, which is required to match boundary conditions at the plasma-vacuum interface. Such a surface wave in a passive medium contributes to loss. In a preliminary publication[5] we pointed out that in a composite structure of the sort shown in Fig. 1 an instability, which is associated with the "surface" wave at the interface between media I and II, can be excited. The instability is produced by carrier drift at a threshold drift velocity which can be made much smaller than the helicon-wave velocity in the guiding structure. At threshold, the loss associated with the wave at the interface vanishes, and at velocities exceeding threshold, the loss turns into gain.

The purpose of this paper is to derive the conditions for the instability and to describe its properties in a detail not possible in Ref. 5. Since the mathematical complexities associated with the derivation are considerable, we first summarize in words the physics of the instability.

Let us assume that the two media in Fig. 1 can be characterized, so far as helicon wave propagation is considered, by dielectric constants K_1 and K_2, where

$$K_i = \omega_{pi}^2/\omega\omega_{ci}. \tag{1.1}$$

Here $\omega_{pi} = (N_i g^2/m_i\epsilon_0)^{1/2}$ is the plasma frequency in medium i, ω_{ci} is the cyclotron frequency in medium i, and ω the frequency of the wave. K_i plays the role of the dielectric constant in the sense that the speed of helicon waves propagating along a static magnetic field in medium i, *if it were infinite in extent*, would be $c/\sqrt{K_i}$. As we mentioned earlier, in a structure of the sort depicted in Fig. 1, a "surface" wave must exist at the interface between the two media in order to match the boundary conditions there. It is a surface wave in that, for conditions of interest in the present paper, the

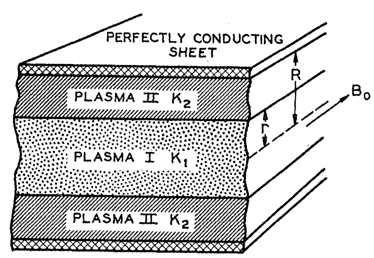

FIG. 1. Schematic of the sandwich structure analyzed in the text. Media I and II are assumed to contain different concentrations of free carriers leading to different effective dielectric constants K_1 and K_2.

[1] See, for example, S. J. Buchsbaum, and R. Bowers, in *Proceedings of the Symposium on Plasma Effects in Solids, Paris, 1964* (Dunod Cie., Paris, 1965), pp. 3–18, 19–35.
[2] J. Bok and P. Nozières, J. Phys. Chem. Solids **24**, 709 (1963).
[3] T. Misawa, Japan J. Appl. Phys. **2**, 500 (1963); A. Bers and A. L. McWhorter, Phys. Rev. Letters **15**, 755 (1965); A. Hasegawa, J. Phys. Soc. Japan **20**, 1072 (1965).
[4] C. R. Legendy, Phys. Rev. **135**, A1713 (1964); J. P. Klozenberg, B. McNamara, and P. Thonemann, J. Fluid Mech. **21**, 545 (1965).
[5] G. A. Baraff and S. J. Buchsbaum, Appl. Phys. Letters **6**, 219 (1965).

strength of the fields associated with the wave decays exponentially in the direction transverse to the interface. With such a wave is associated loss. The origin of that loss is similar to that of the plasma-vacuum surface wave of Legendy. Here, however, the strength of the surface wave at the interface depends essentially on the difference in dielectric constant across the interface; it can be made arbitrarily small by making $|K_1-K_2|$ arbitrarily small.

Now assume that the carriers in one of the two media comprising the structure possess a steady drift in a direction parallel to the interface. As we shall show in Sec. II, the presence of the drift effectively modifies the dielectric constant in that medium from K_i to $K_i(1-V_d/V_\Phi)$, where V_d is the drift velocity of the carriers and V_Φ is the (as yet undetermined) phase velocity of the wave propagating in the guide. For the sake of an example, assume that the carriers in medium I drift. Then by making

$$K_1(1-V_d/V_\Phi)=K_2, \tag{1.2}$$

the surface wave at the interface between the two media vanishes, because the two media now have the same effective dielectric constant; and to the wave, the two media appear to be identical. The drift velocity given by Eq. (1.2) is the threshold drift velocity. For drift velocities larger than the threshold, the surface wave reappears. But now its phase is reversed with respect to the phase below threshold. We shall show that it is this reversal in phase and the interaction between the surface wave and helicon-like wave in the bulk (the bulk wave) which leads to gain. It is clear that by making K_1 and K_2 nearly equal, the threshold drift velocity can be made arbitrarily small, albeit at the price of low gain. In practice the over-all losses in the system will, by setting a minimum value of amplification needed for net gain, establish the minimum discontinuity in dielectric constant, and will determine, thereby, the threshold drift velocity.

The structure of this paper is as follows: in Sec. II, we study the effect of drift on propagation in infinite media. Most of the physics of the model and most of the approximations are contained in this section. As a result, it contains most of the steps needed in the derivations. In Sec. III, we derive and discuss the dispersion relation for the sandwich structure shown in Fig. 1, and in particular, the damping or growth constant of the waves near threshold. The derivation and discussion here allows us to infer what the damping or growth constant is likely to be for the multilayered structure shown in Fig. 2. In that section, as in the fourth section, most of the algebra is relegated to the appendices. Finally, in Sec. IV, we analyze the instability on the basis of power-flow considerations.

II. INFINITE-MEDIUM SOLUTIONS

The sandwich structure depicted in Fig. 1 is to be considered as being infinite in the x and z directions.

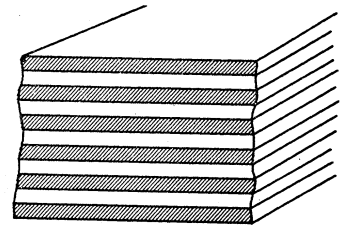

FIG. 2. Multilayer generalization of the sandwich structure depicted in Fig. 1.

The drift current and magnetic field are parallel, and will be taken in the z direction. Although our interest is in waves which propagate along the z direction, the boundary conditions will force the electromagnetic fields to depend on the y coordinate as well. There is no essential reason for any of the fields to depend on x, and so, for simplicity, we shall be concerned only with x independent fields and currents.

In this section, we shall calculate the various infinite medium solutions possible in the helicon regime. By helicon regime, we mean the conditions

$$\omega_c\tau\gg 1, \tag{2.1a}$$

$$\omega/\omega_c\ll 1, \tag{2.1b}$$

$$K\equiv\omega_p^2/\omega\omega_c\gg 1. \tag{2.1c}$$

We assume that the motion of an *average* carrier in the plasma is governed by the transport equation[6]:

$$m[\dot{\mathbf{V}}+(\mathbf{V}\cdot\nabla)\mathbf{V}]=q[\mathbf{E}+\mathbf{V}\times\mathbf{B}]-m\nu\mathbf{V}. \tag{2.2}$$

Here $\mathbf{V}$ and $\nu=1/\tau$ are the velocity and collision frequency of the carrier; $\mathbf{E}$ and $\mathbf{B}$ are the total electric and magnetic fields. The equation is linearized by writing

$$\begin{aligned}\mathbf{V}&=\mathbf{V}_0+\mathbf{v},\\ \mathbf{B}&=\mathbf{B}_0+\mathbf{b},\\ \mathbf{E}&=\mathbf{E}_0+\mathbf{e}.\end{aligned} \tag{2.3}$$

We drop products of first-order (lower-case) quantities, and regard the zeroth-order quantities as uniform, time-independent, externally determined quantities. The resulting equation is treated by postulating an exponential variation $\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$ for all first-order quantities. Maxwell's third equation,

$$\nabla\times\mathbf{e}=-\dot{\mathbf{b}}, \tag{2.4}$$

is then used to eliminate $\mathbf{b}$, leaving the following vector

[6] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1962). Deleting the pressure tensor ψ and the gravitational potential φ from Eq. (6.16) of this reference leads essentially to (2.2). The neglect of the pressure tensor means in effect that we are neglecting the random or thermal motion of the carriers, an approximation which is justified when $kv_F\tau<1$.

equation relating the velocity $\mathbf{v}$ to the electric field $\mathbf{e}$:

$$(1-\mathbf{n}\cdot\mathbf{U}+i\gamma)\mathbf{v}+i\boldsymbol{\beta}\times\mathbf{v} = -a^2(\omega\epsilon_0/iN_0q)[\mathbf{e}+\mathbf{U}\times(\mathbf{n}\times\mathbf{e})], \quad (2.5a)$$

$$a^2=\omega_p^2/\omega^2=N_0q^2/(m\epsilon_0\omega^2), \quad (2.5b)$$

$$\boldsymbol{\beta}=\boldsymbol{\omega}_c/\omega=q\mathbf{B}_0/(m\omega), \quad (2.5c)$$

$$\gamma=\nu/\omega, \quad (2.5d)$$

$$\mathbf{U}=\mathbf{V}_0/c, \quad (2.5e)$$

$$\mathbf{n}=c\mathbf{k}/\omega. \quad (2.5f)$$

The index-of-refraction vector $\mathbf{n}$ is not to be confused with the scalar n which emerges when we linearize the density by writing

$$N=N_0+n.$$

The solution to (2.5) may be expressed in matrix form as

$$(iN_0q/\omega\epsilon_0)\mathbf{v}=\mathsf{M}\cdot\mathbf{e}, \quad (2.6a)$$

where

$$\mathsf{M}=\frac{a^2/\zeta}{(\beta^2-\zeta^2)}\begin{bmatrix}\zeta^2 & i\beta\zeta & 0\\ -i\beta\zeta & \zeta^2 & 0\\ 0 & 0 & \zeta^2-\beta^2\end{bmatrix}\begin{bmatrix}1-Un_z & 0 & 0\\ 0 & 1-Un_z & Un_y\\ 0 & 0 & 1\end{bmatrix}, \quad (2.6b)$$

$$\zeta=1-Un_z+i\gamma, \quad (2.6c)$$

$$\beta=|\boldsymbol{\beta}|\times\mathrm{sig}q, \quad (2.6d)$$

$$U=|\mathbf{U}|. \quad (2.6e)$$

In obtaining Eqs. (2.6), we have used the fact that $\mathbf{U}$ and $\boldsymbol{\beta}$ are parallel to z and that $\mathbf{n}$ has no x component. Thus, we are here restricting our solutions to be independent of x. Having obtained the velocity, we calculate the particle current $\mathbf{G}=N\mathbf{V}$ and concentrate our attention on the first-order part

$$\mathbf{g}=N_0\mathbf{v}+n\mathbf{V}_0. \quad (2.7)$$

From the continuity equation

$$\nabla\cdot\mathbf{g}+\dot{n}=0,$$

or

$$\mathbf{k}\cdot(N_0\mathbf{v}+n\mathbf{V}_0)-\omega n=0,$$

we can solve for n and, with that solution, eliminate n from (2.7):

$$\mathbf{g}=N_0[\mathbf{v}+\mathbf{U}(\mathbf{n}\cdot\mathbf{v})/(1-\mathbf{n}\cdot\mathbf{U})] \quad (2.8)$$

The dependence of the electric current $\mathbf{J}=q\mathbf{g}$ on the electric field $\mathbf{e}$ now follows by combining (2.6) and (2.8):

$$(i/\omega\epsilon_0)\mathbf{j}=\mathsf{S}\cdot\mathbf{e}, \quad (2.9a)$$

$$\mathsf{S}=\frac{a^2\beta}{(\beta^2-\zeta^2)}\begin{bmatrix}A\zeta/\beta & iA & iB\\ -iA & A\zeta/\beta & \zeta B/\beta\\ -iB & \zeta B/\beta & (\zeta^2-\beta^2+\zeta^2B^2)/\zeta\beta A\end{bmatrix}, \quad (2.9b)$$

$$A=1-Un_z, \quad (2.9c)$$

$$B=Un_y. \quad (2.9d)$$

This current acts as a source of the magnetic field $\mathbf{b}$ via the fourth Maxwell equation

$$\nabla\times\mathbf{b}=\mu_0\mathbf{j}+\mu_0\epsilon_0\dot{\mathbf{e}}.$$

Using (2.4) to eliminate $\mathbf{b}$ and (2.9a) to eliminate $\mathbf{j}$ converts this to an equation for $\mathbf{e}$ alone:

$$\mathsf{K}\cdot\mathbf{e}+\mathbf{n}\times(\mathbf{n}\times\mathbf{e})=0, \quad (2.10a)$$

$$\mathsf{K}=1+\mathsf{S}. \quad (2.10b)$$

It is now appropriate to make the helicon approximation in (2.10). For simplicity, we also assume that the frequency is well below the scattering frequency so that $\gamma=1/(\omega\tau)$ is much larger than unity. Thus

$$\zeta\approx i\gamma, \quad (2.11)$$

because even at the largest drift currents, Un_z will be comparable to unity. Now the ratio ζ/β becomes

$$\zeta/\beta=i\gamma/\beta=i/\omega_c\tau. \quad (2.12)$$

The first approximation is that $\omega_c\tau$ is large enough that $(1/\omega_c\tau)^2$ is negligible compared to unity. (It will soon be evident why we cannot drop terms of order $1/\omega_c\tau$.) This approximation gives

$$a^2\beta/(\beta^2-\zeta^2)\approx a^2/\beta=\omega_p^2/\omega\omega_c\equiv K. \quad (2.13)$$

The second helicon approximation (2.1c) allows us to neglect the unit tensor in (2.10b). With these approximations, Eq. (2.10) determining the electric field is

$$\begin{bmatrix}S-n_y^2-n_z^2 & iX & iYn_y\\ -iX & S-n_z^2 & (iZ+n_z)n_y\\ -iYn_y & (iZ+n_z)n_y & P+Qn_y^2\end{bmatrix}\begin{bmatrix}e_x\\ e_y\\ e_z\end{bmatrix}=0, \quad (2.14a)$$

$$S=iK(1-Un_z)/\omega_c\tau, \quad (2.14b)$$

$$X=K(1-Un_z), \quad (2.14c)$$

$$Y=KU, \quad (2.14d)$$

$$Z=KU/\omega_c\tau, \quad (2.14e)$$

$$P=iK\omega_c\tau/(1-Un_z), \quad (2.14f)$$

$$Q=\frac{iKU^2/(1-Un_z)}{\omega_c\tau}-1. \quad (2.14g)$$

The vanishing of the determinant in (2.14a) gives a quadratic equation for n_y^2:

$$an_y^4+bn_y^2+c=0, \tag{2.15a}$$

$$a=-SQ+2in_zZ-Z^2+(1+Q)n_z^2, \tag{2.15b}$$

$$b=(X+n_zY)^2+(n_z^2-S)(P+S-Z^2+2in_zZ) -SY^2+2iXYZ+(Q+1)[-X^2+(n_z^2-S)^2], \tag{2.15c}$$

$$c=P[S^2+n_z^2(n_z^2-2S)-X^2], \tag{2.15d}$$

and a relation for the components transverse to the magnetic field, e_x and e_y, in terms of e_z, the component along the magnetic field:

$$De_x=-in_y[Y(S-n_z^2)-iX(Z-in_z)]e_z, \tag{2.16a}$$

$$De_y=-in_y[(S-n_y^2-n_z^2)(Z-in_z)+iXY]e_z, \tag{2.16b}$$

$$D=S^2-S(2n_z^2+n_y^2)+n_z^2(n_z^2+n_y^2)-X^2. \tag{2.16c}$$

Here, some simplification of the coefficients is possible because of the definitions (2.14b)–(2.14f) which give us

$$(X+n_zY)^2=K^2=-PS, \tag{2.17a}$$

$$SY=iXZ. \tag{2.17b}$$

We now regard $\omega_c\tau$ as large and expand a, b, and c in terms of inverse powers of this parameter. To do this, however, we must first assign an $(\omega_c\tau)$ dependence to n_z. Since our idea is that the whole structure should propagate helicon-like (i.e., infinite-medium–like) waves in the z direction, it is easily shown that we should regard n_z as being of order $(\omega_c\tau)^0$.

We shall now consistently retain only the lowest order terms in $(1/\omega_c\tau)$. This approximation, passage to the high $\omega_c\tau$ limit, discards all dissipative effects, except those which result from the presence of a surface wave. The lowest order terms in (2.15) are

$$a=S+2iZn_z+(1+Q)n_z^2 =iK/[\omega_c\tau(1-Un_z)], \tag{2.18a}$$

$$b=Pn_z^2, \tag{2.18b}$$

$$c=P(n_z^4-X^2). \tag{2.18c}$$

In the formal solution to (2.15a),

$$n_y^2=[-b\pm(b^2-4ac)^{1/2}]/2a,$$

we find from (2.18) and (2.14) that $4ac/b^2$ is of order $(\omega_c\tau)^{-2}$. Thus the leading term here may be obtained by expanding the square root to lowest order, so that the two roots for n_y^2 are $n_y^2=-c/b$ and $n_y^2=-b/a$. In the second solution, we shall always write

$$N_y=in_y, \tag{2.19}$$

so that the two roots are

$$n_y^2=-c/b=X^2/n_z^2-n_z^2, \tag{2.20a}$$

$$N_y^2=+b/a=n_z^2(\omega_c\tau)^2. \tag{2.20b}$$

We shall designate the two forms of electric field which result when Eqs. (2.16) are evaluated using Eqs. (2.20) by superscripts b (for the bulk wave) and s (for the surface wave), referring to (2.20a) and (2.20b), respectively. Making use of (2.17) to simplify forms, and retaining only the lowest terms in $(\omega_c\tau)^{-1}$, the results of substituting Eqs. (2.20) into Eq. (2.15) are[7]

$$D^{\mathrm{b}}e_x^{\mathrm{b}}=in_yn_zKe_z^{\mathrm{b}}, \tag{2.21a}$$

$$D^{\mathrm{b}}e_y^{\mathrm{b}}=(n_yXK/n_z)e_z^{\mathrm{b}}, \tag{2.21b}$$

$$D^{\mathrm{b}}=+Sn_y^2, \tag{2.21c}$$

and

$$D^{\mathrm{s}}e_x^{\mathrm{s}}=N_yn_zKe_z^{\mathrm{s}}, \tag{2.22a}$$

$$D^{\mathrm{s}}e_y^{\mathrm{s}}=iN_y^3n_ze_z^{\mathrm{s}}, \tag{2.22b}$$

$$D^{\mathrm{s}}=-n_z^2N_y^2. \tag{2.22c}$$

Each type of electric field generates a magnetic field $\mathbf{b}=\mathbf{k}\times\mathbf{e}/\omega=\mathbf{n}\times\mathbf{e}/c$. Hence using (2.19), (2.20), (2.21), and (2.22), we have[8]

$$b_x^{\mathrm{b}}=-(XK/cSn_y)e_z^{\mathrm{b}}, \tag{2.23a}$$

$$b_y^{\mathrm{b}}=+(in_z^2K/n_ycS)e_z^{\mathrm{b}}, \tag{2.23b}$$

$$b_z^{\mathrm{b}}=-(in_zK/cS)e_z^{\mathrm{b}}, \tag{2.23c}$$

$$b_x^{\mathrm{s}}=[iN_y(S+iZn_z)/(cn_z^2)]e_z^{\mathrm{s}}, \tag{2.24a}$$

$$b_y^{\mathrm{s}}=(N_yK/cn_z^2)e_z^{\mathrm{s}}, \tag{2.24b}$$

$$b_z^{\mathrm{s}}=(-iK/cn_z)e_z^{\mathrm{s}}. \tag{2.24c}$$

It is clear from (2.20a), (2.21) and (2.23) that if the type-b solution is normalized so that Ke_z^{b} is some constant, then for a given n_z, the field components e_x^{b}, e_y^{b}, $\mathbf{b}^{\mathrm{b}}$, and the transverse propagation constant n_y are functions only of $X=K(1-Un_z)$ and $S=iX/\omega_c\tau$. Thus, except for e_z, which is one order of the $\omega_c\tau$ smaller than the other $\mathbf{e}$ components and therefore negligible, the effect of a drift current on all aspects of the bulk solution is completely described by the replacement of K by X. For this reason, the quantity X can be called the effective dielectric constant.

[7] Some care is required in obtaining (2.21c) because straightforward substitution of (2.20a) into (2.16c) leads to a complete cancellation of terms of order $(\omega_c\tau)^0$. It is therefore necessary that the root $n_y^2=-c/b$ be correct to order $(\omega_c\tau)^{-1}$ before substituting into (2.16c). Hence the proper procedure is to retain the two lowest powers of $(\omega_c\tau)^{-1}$ in (2.15c), (2.16c), and (2.15d). This leads to

$$b=(X+n_zY)^2+P(n_z^2-S)=P(n_z^2-2S),$$
$$c=P[n_z^2(n_z^2-2S)-X^2],$$
$$n_y^2=X^2/(n_z^2-2S)-n_z^2,$$

and thence to (2.21c).

[8] Again some care is required to obtain (2.24a) because the lowest order terms in $(\omega_c\tau)^{-1}$ cancel. This time, it is necessary to retain an extra order of $(\omega_c\tau)^{-1}$ in passing from (2.16b) and (2.16c) to (2.22). These give

$$D^{\mathrm{s}}e_y^{\mathrm{s}}=iN_y^3(n_z+iZ)e_z^{\mathrm{s}}$$
$$D^{\mathrm{s}}=N_y^2(S-n_z^2)$$

and now, (2.24a) follows immediately.

TABLE I. Solution types. A zero indicates that the entry is one order of $\omega_c\tau$ smaller than the other entries in the same row.

Type j	$e_x^j=$	$e_z^j=$	$b_x^j=$	$b_z^j=$
$j=1$	$E_x^1\sin(ky)$	0	$B_x^1\sin(ky)$	$B_z^1\cos(ky)$
2	0	$E_z^2\cosh(\kappa y)$	$B_x^2\sinh(\kappa y)$	$B_z^2\cosh(\kappa y)$
3	$E_x^3\cos(ky)$	0	$B_x^3\cos(ky)$	$B_z^3\sin(ky)$
4	0	$E_z^4\sinh(\kappa y)$	$B_x^4\cosh(\kappa y)$	$B_z^4\sinh(\kappa y)$
5	0	$E_z^5\exp(\kappa y)$	$B_x^5\exp(\kappa y)$	$B_z^5\exp(\kappa y)$
6	0	$E_z^6\exp(-\kappa y)$	$B_x^6\exp(-\kappa y)$	$B_z^6\exp(-\kappa y)$

There are four tangential field components, e_x, e_z, b_x, b_z, in each solution. In type-b, the component e_z^{b} is one order of $\omega_c\tau$ smaller than the other three tangential fields, while in type s, it is e_x^{s} that is one order of $\omega_c\tau$ smaller than the other three tangential fields. Therefore, when boundary conditions are set up, these two small components can be taken equal to zero and the results will still be correct to lowest order in $(\omega_c\tau)^{-1}$. This same conclusion can be obtained, more convincingly but also more laboriously, be retaining one extra power of $(\omega_c\tau)^{-1}$ throughout the entire calculation.

Equations (2.20) for n_y^2 and N_y^2 allow both signs for n_y and N_y. It is convenient to combine the two solutions corresponding to the two possible signs so as to obtain solutions with definite y symmetry. Specifically, we take half the sum of the two type-b solutions and designate that as type 1, half the sum of the two type-s solutions and designate that as type 2; $(i/2)$ times the difference of the two type-b solutions we designate as type 3; half the difference of the two type-s solutions will be type 4, and finally the two type-s solutions themselves we now denote as types 5 and 6. The six solutions are exhibited in tabular form as Tables I and II, in which we have written

$$k\equiv\omega n_y/c\,, \tag{2.25a}$$

$$\kappa\equiv\omega N_y/c\,. \tag{2.25b}$$

III. THE DISPERSION RELATION AND PROPAGATION CONSTANT

The actual fields in regions I and II of Fig. 1, will be sums of solutions of the types tabulated in Tables I and II, each solution in the sum being multiplied by an arbitrary constant. The arbitrary constant must be so adjusted that boundary conditions are satisfied. We take for boundary conditions the continuity of the tangential electric and magnetic fields at $y=\pm r$ and the vanishing of the tangential electric field at $y=\pm R$. (We assume that the slab is coated by a perfect conductor at $y=\pm R$.) It turns out that there is one more boundary condition than there are arbitrary constants with which to satisfy them[9] and that consequently, a condition on the solutions tabulated in Tables I and II must also be satisfied. This condition provides the dispersion relation for the structure.

TABLE II. Solution constants.

Type j	E_x^j	E_z^j	cB_x^j/i	cB_z^j/i
$j=1$	$-n_zK/Sn_y$		$-XK/Sn_y$	$-n_zK/S$
2		1	$N_y(S+iZn_z)/n_z^2$	$-K/n_z$
3	$E_x^3=E_x^1=-n_zK/Sn_y$		$B_x^3=B_x^1$	$=-B_z^1$
4		1	$=B_x^2$	$=B_z^2$
5		1	$=B_x^2$	$=B_z^2$
6		1	$=-B_x^2$	$=B_z^2$

Note added in proof. The results and conclusions of this paper rest heavily on the assumption that the tangential component of the total magnetic field of the wave is continuous across the interface, i.e., that there is no surface current localized strictly to the interface driven by the drifting carriers. If such a current is introduced into the model, many of the present results will have to be modified. The consequence of such a current under various conditions of surface recombination and mobility will be discussed elsewhere.

The derivation of the dispersion relation is a straightforward but tedious task. We relegate it to Appendix A where we show that the dispersion relation is

$$n_2\cot k_2(r-R)=n_1\cot(k_1r+\eta\pi/2)-\frac{i(K_1-K_2)(X_1-X_2)}{(K_1+K_2)n_z}\,, \tag{3.1a}$$

where

$$X_i=K_i(1-U_in_z)\,, \tag{3.1b}$$

$$k_i=\omega n_i/c\,, \tag{3.1c}$$

$$n_i^2=X_i^2/n_z^2-n_z^2\,. \tag{3.1d}$$

In Eqs. (3.1), $\eta=0$ for modes in which the fields e_z and b_z are symmetric about $y=0$, and $\eta=1$ for modes in which e_z and b_z are antisymmetric about $y=0$. This form is valid provided the surface wave excited at any boundary damps out before reaching the next boundary.

We now study the dispersion relation for those values of (normalized) drift velocity U near the threshold value U^0. At threshold, the wave changes from attenuating to growing and, hence, n_z must be purely real. Finding that value of U^0 which leads to a real n_z is therefore the first problem.

If n_z is real, then Eqs. (3.1b) and (3.1d) yield a real n_i^2. It follows that the first two terms in (3.1a) are also real. Therefore, at threshold, the last term in (3.1a) must vanish. That is,

$$M=\frac{(K_1-K_2)(X_1-X_2)}{(K_1+K_2)n_z}$$

[9] We regard one of the arbitrary constants as being fixed by the over-all normalization of the fields. Otherwise, there are as many arbitrary constants as there are boundary conditions and we are led to a set of homogeneous equations whose compatibility demands that the determinant of coefficients be annulled. The equation resulting is the same.

must vanish at threshold. This can happen only because $K_1=K_2$ or because $X_1=X_2$. The first possibility gives n_z real at all values of U. The second possibility is the interesting one. From it, we have the threshold condition $X_1=X_2$ or

$$K_1(1-U_1^0n_z^0)=K_2(1-U_2^0n_z^0). \tag{3.2}$$

One of U_1^0 and U_2^0 can, of course, be equal to zero provided that the other satisfies (3.2). It is clear from (3.1b)–(3.1d) that the threshold condition $X_1=X_2$ yields $k_1=k_2$, which common value we designate by k_0. At threshold, then, Eq. (3.1a) reduces to

$$\cot k_0(r-R)=\cot(k_0r+\eta\pi/2) \tag{3.3a}$$

whose solution is

$$k_0=m\pi/2R \quad m=1,2,3,\cdots. \tag{3.3b}$$

The integer m is even for $\eta=0$ (symmetric z fields) and odd for $\eta=1$ (antisymmetric z fields). The condition (3.3b) says that the transverse wavelength at threshold is determined geometrically, with an even or odd number of half-wavelengths spanning the full thickness $2R$ of the sandwich. The reason this occurs is that at $X_1=X_2$, the two media have the same effective dielectric constants and thus the bulk wave is unaware of any interface or difference between the media. One would suspect (and indeed we shall show in the next section) that, at threshold, the surface wave is not needed to satisfy boundary conditions at the interface.

Having found k_0, one can solve (3.1d) for n_z. The solution is easiest if one of the currents, U_2, say, is zero, for then the solution is

$$(n_z^0)^2=(K_2/2)[(4+T^4)^{1/2}-T^2], \tag{3.4a}$$

$$T=ck_0/\omega\sqrt{K_2}. \tag{3.4b}$$

With n_z determined and $U_2^0=0$, the threshold drift velocity $U_1^0\equiv u_t$ follows simply from (3.2).

We now solve for n_z when U_1 is in the neighborhood of u_t. To do this, we expand all quantities to first order in U_1-u_t. The details of the expansion and the resulting solution are presented in Appendix B. Of particular interest is the imaginary part of n_z, since this determines growth or attenuation. Writing

$$n_z=n_z^0[1+(U_1-u_t)\delta], \tag{3.5a}$$

we find that for K_1 and K_2 nearly equal, the imaginary part of n_z follows from

$$\mathrm{Im}\delta=\frac{(1-\xi)\varphi^2(r)}{4(\omega R/c)(1+T^4/4)^{1/2}}, \tag{3.5b}$$

where

$$\xi=K_1/K_2, \tag{3.5c}$$

$$\varphi(r)=\sin[(m\pi/2)(1-r/R)]. \tag{3.5d}$$

We observe that the y dependence of any of the bulk-wave transverse field components, e_x^{b}, e_y^{b}, b_x^{b}, b_y^{b}, is $\varphi(y)$. Therefore, the imaginary part of n_z is proportional to the square of the transverse bulk-wave field evaluated at the interface position. This dependence is a consequence of the physical mechanism for the instability, as will be verified in the next section. The mechanism involves the interaction between the surface wave and the bulk wave. The surface wave exists only at the interface, and only for the purpose of satisfying boundary conditions which the bulk wave cannot quite do. Clearly then, the amplitude of the surface wave will be proportional to the amplitude of the bulk wave at the interface.[10] Therefore *each* of the interacting fields is proportional to $\varphi(r)$. Since the power gain or loss is quadratic in the fields, the imaginary part of the propagation constant, which is proportional to the power gain or loss, exhibits a $\varphi^2(r)$ dependence.

The implication of this mechanism is that $\mathrm{Im}n_z$ will be half as large for a one-interface structure as for the corresponding two-interface structure. Since the even m modes of the sandwich have all tangential $\mathbf{e}$ fields vanishing at $y=0$, the $y>0$ half of the sandwich structure reproduces, for even m, all possible modes of the single interface structure of thickness R. The dispersion relation for the one-interface structure of width $2R$, the width of the sandwich, then follows by replacing R by $2R$ in Eqs. (3.3), (3.4), and (3.5). It is evident that the one-interface structure indeed supports a wave which grows or attenuates half as fast as does the wave in the corresponding two-interface sandwich structure. These observations suggest that it should be possible to obtain $\mathrm{Im}n_z$ for a multilayered structure such as shown in Fig. 2 by summing an expression of the form (3.5) over all values of r_i, the position of the ith interface. Although this is a reasonable conjecture, its proof will have to await the completion of our studies of the multilayered structure. [*Note added in proof*. It has been shown recently that the gain in multilayered structures, indeed, is additive. See L. M. Saunders and G. A. Baraff (to be published).]

IV. CURRENTS, FIELDS, AND POWER LOSS

In order to bring out the physical nature of the instability, consider the power loss per unit length of slab:

$$W=\tfrac{1}{2}\,\mathrm{Re}\int \mathbf{e}\cdot\mathbf{j}^*da. \tag{4.1}$$

The integration is over the cross-sectional area of the sandwich structure. Since the electric field ($\mathbf{e}=\mathbf{e}^{\mathrm{b}}+\mathbf{e}^{\mathrm{s}}$) and the current ($\mathbf{j}=\mathbf{j}^{\mathrm{b}}+\mathbf{j}^{\mathrm{s}}$) are each composed of the fields and currents of the bulk and surface waves, the

[10] It turns out that the longitudinal components of the bulk wave can be matched without the surface wave and therefore that the surface wave is proportional to the *transverse* bulk wave components. One expects in general that factors proportional to the amplitude of the longitudinal components could enter $\mathrm{Im}n_z$.

power loss W is composed of three terms:

$$W = W_{\mathrm{bb}} + W_{\mathrm{bs}} + W_{\mathrm{ss}}, \tag{4.2}$$

where

$$W_{\mathrm{bb}} = \tfrac{1}{2}\,\mathrm{Re}\int (\mathbf{e}^{\mathrm{b}}\cdot\mathbf{j}^{\mathrm{b}*})da, \tag{4.2a}$$

$$W_{\mathrm{bs}} = \tfrac{1}{2}\,\mathrm{Re}\int (\mathbf{e}^{\mathrm{b}}\cdot\mathbf{j}^{\mathrm{s}*}+\mathbf{e}^{\mathrm{s}}\cdot\mathbf{j}^{\mathrm{b}*})da, \tag{4.2b}$$

$$W_{\mathrm{ss}} = \tfrac{1}{2}\,\mathrm{Re}\int (\mathbf{e}^{\mathrm{s}}\cdot\mathbf{j}^{\mathrm{s}*})da. \tag{4.2c}$$

The term W_{bb}, resulting from bulk-bulk interaction, vanishes in the high-$\omega_c\tau$ limit. (See Appendix C.) The other two loss terms, W_{bs} and W_{ss}, are of interest to us here. We show in Appendix C that only the terms $(e_x{}^{\mathrm{b}}j_x{}^{\mathrm{s}*})$ and $(e_y{}^{\mathrm{s}}j_y{}^{\mathrm{b}*})$ contribute to W_{bs} in the high $\omega_c\tau$ limit and that only $(e_y{}^{\mathrm{s}}j_y{}^{\mathrm{s}*}+e_z{}^{\mathrm{s}}j_z{}^{\mathrm{s}*})$ contributes to W_{ss}. Each of these terms has the $\varphi^2(r)$ dependence mentioned earlier.

It is not difficult to show (See Appendix C) that the amplitude of the surface wave is proportional to (X_1-X_2) so that the surface wave vanishes at threshold. At drift velocities greater than threshold, where (X_1-X_2) has the opposite sign from (X_1-X_2) below threshold, the surface wave reappears with its phase reversed relative to the phase it had below threshold. This behavior is reflected in the vanishing of W_{bs} and W_{ss} at threshold, and in a reversal of the sign of W_{bs} (but not of W_{ss}) as the drift velocity crosses threshold. The calculation in Appendix C shows that for $(\omega_c\tau)\rightarrow\infty$

$$W_{\mathrm{bs}} = \frac{QQ^*(X_1-X_2)[(K_1-X_1)-(K_2-X_2)]}{K_1+K_2}, \tag{4.3}$$

and

$$W_{\mathrm{ss}} = \frac{QQ^*(X_1-X_2)^2}{K_1+K_2}. \tag{4.4}$$

Here, Q is the arbitrary constant multiplying the bulk-wave solution so that QQ^* is a constant proportional to the energy contained in a unit length of the sandwich. We see that W_{ss} is positive, but that it vanishes at threshold. That is, the surface wave interacting with itself always gives rise to a power loss except when that wave disappears. The gain can arise only from W_{bs}, the surface-bulk interaction term which changes sign at threshold, when the surface field reverses phase.[11] Thus, the mechanism of the instability is that the surface wave, which is needed to satisfy boundary conditions at an interface across which the effective dielectric constant changes, can be made to reverse phase by using carrier drift to reverse the sense of the relative effective dielectric mismatch. The reversal causes the bulk-wave–surface-wave interaction to change loss to gain sufficient to overcome the surface-surface losses. That the loss is overcome may be seen by noting that

$$\frac{(W_{\mathrm{bs}}+W_{\mathrm{ss}})}{QQ^*} = \frac{(X_1-X_2)(K_1-K_2)}{K_1+K_2},$$

which changes sign at threshold. Note that the net power loss divided by the energy density is essentially the imaginary term M in the dispersion relation (3.1a) whose presence led to an imaginary part of n_z. Our analysis suggests that if $(W_{\mathrm{bs}}+W_{\mathrm{bb}})$ can be made sufficiently negative to overcome collisional loss (which has here been neglected by passage to the infinite $\omega_c\tau$ limit) net gain can be achieved. The problem of extending these results to the next order in $(\omega_c\tau)^{-1}$ is being pursued.

ACKNOWLEDGMENT

We should like to express our deep appreciation to Dr. G. A. Pearson whose careful reading of the manuscript uncovered an error in our work. As a result of this error, the factor

$$(1-\beta^2u^2)^{-1/2}$$

in Eq. (1) of Ref. 5 is spurious, and it should be deleted. The other equations and conclusions of that reference are correct as they stand.

APPENDIX A

As stated in Sec. III, the actual fields in region I and region II are sums of solutions of the types tabulated in Tables I and II, each solution in the sum being multiplied by an arbitrary constant. These arbitrary constants are adjusted so that boundary conditions are satisfied. It is useful to designate each solution by an alphabetic superscript which will denote both the type of solution and the region in which it is to be used. We shall use the capital letter corresponding to the superscript to represent the arbitrary constant which multiplies that solution. Using the nomenclature of Table

TABLE III. Solution nomenclature.

Superscript	Region	Type	To be multiplied by
n	I	3	$-N$
p	I	4	$-P$
q	I	1	$-Q$
r	I	2	$-R$
s	II	1	S
t	II	3	T
u	II	5	U
v	II	6	V

[11] Note that this term vanishes when there is no drift, $(X_i=K_i)$, as it must because the surface wave and bulk wave are orthogonal to each other. The presence of drift in either medium removes the orthogonality and renders W_{bs} finite.

III, the x component of the magnetic field in region II is

$$b_x = Sb_x^s + Tb_x^t + Ub_x^u + Vb_x^v$$

and the x component of electric field in region I is

$$e_x = -Qe_x^q - Ne_x^n.$$

The minus signs appear here because the multiplier for the q and n solutions are, by our choice, $-Q$ and $-N$; the constants P and R fail to appear because the x component of electric field in these solutions is of order $(\omega_c\tau)^{-1}$ smaller than the other field components.

Because of the $y \leftrightarrow -y$ symmetry of the sandwich, the complete solutions can be classified as being of one or the other of two types, which we designate as even or odd depending on whether $b_z(y) = \pm b_z(-y)$. In the even mode $b_z(y) = +b_z(-y)$ and therefore, solutions n and p do not appear because they have the wrong symmetry. (There is no corresponding restriction on the solutions used in the outer region II, because here the symmetry may be maintained by having the arbitrary constants in one of the two outer regions either equal or opposite to the arbitrary constants in the other.) Similarly, solutions q and r will fail to appear in the mode for which $b_z(y) = -b_z(-y)$. We first show how the boundary conditions lead to the dispersion equation, k_z versus ω, for the even mode.

It is convenient to let a field component evaluated at $y = r$ temporarily be denoted by a lower case letter and temporarily, to use a capital letter to denote a field component evaluated at $y = R$. Having chosen the symmetry, we need no longer be concerned with what happens at $y = -r$ and at $y = -R$. Thus, there are six boundary conditions, four expressing the continuity of tangential electric and magnetic field components at $y = r$ and two expressing the vanishing of the tangential electric-field components at $y = R$.

$$Qe_x^q + Se_x^s + Te_x^t = 0, \tag{A1a}$$

$$Re_z^r + Ue_z^u + Ve_z^v = 0, \tag{A1b}$$

$$Qb_x^q + Rb_v^r + Sb_x^s + Tb_x^t + Ub_x^u + Vb_x^v = 0, \tag{A1c}$$

$$Qb_z^q + Rb_z^r + Sb_z^s + Tb_z^t + Ub_z^u + Vb_z^v = 0, \tag{A1d}$$

$$SE_x^s + TE_x^t = 0, \tag{A1e}$$

$$UE_z^u + VE_z^v = 0. \tag{A1f}$$

The condition that this set of six homogeneous equations for the six arbitrary constants $Q, R, \cdots, V$ has a nontrivial solution is that the determinant of the coefficients vanish. Annulling the six-by-six determinant leads, in fact, to the dispersion equation. It is somewhat easier, however, to reduce the size of the determinant by solving some of the equations first. This task is greatly simplified when we note that if solution v plays a reasonable role at $y = r$, then it will be completely negligible at $y = R$, while, if solution u plays a reasonable role at $y = R$, it will be completely negligible at $y = r$. This comes about because each of these solutions is reduced by a factor $\exp(\kappa|R - r|)$ in crossing region II. Since κ is, by (2.20b) and (2.25b), proportional to $\omega_c\tau$, this reduction is severe. Hence, we should set

$$e_z^u = b_x^u = b_z^u = 0, \tag{A2a}$$

$$E_z^v = 0. \tag{A2b}$$

Now, Eq. (A1f) is satisfied by taking $U = 0$, and independently of this, (A1e) is satisfied by taking

$$T = -SE_x^s / E_x^t. \tag{A3}$$

Consider now the combinations of S and T solutions which appear in (A1a), (A1c), and (A1d). Using (A3), we may express these combinations as

$$Se_x^s + Te_x^t = (S/E_x^t)(e_x^s E_x^t - e_x^t E_x^s) \equiv We_x^w, \tag{A4a}$$

$$Sb_x^s + Tb_x^t = (S/E_x^t)(b_x^s E_x^t - b_x^t E_x^s) \equiv Wb_x^w, \tag{A4b}$$

$$Sb_z^s + Tb_z^t = (S/E_x^t)(b_z^s E_x^t - b_z^t E_x^s) \equiv Wb_z^w. \tag{A4c}$$

Thus, the four equations (A1a)–(A1d), which are still to be solved, have, by the definition (A4), been reduced to the form

$$Qe_x^q + We_x^w = 0, \tag{A5a}$$

$$Re_z^r + Ve_z^v = 0, \tag{A5b}$$

$$Qb_x^q + Rb_x^r + Wb_x^w + Vb_x^v = 0, \tag{A5c}$$

$$Qb_z^q + Rb_z^r + Wb_z^w + Vb_z^v = 0; \tag{A5d}$$

i.e., four homogeneous equations for the four arbitrary constants Q, R, W, V.

The functional form of the type-w solution follows from the defining Eq. (A4) and the various definitions and nomenclatures presented in Tables I and III. For example,

$$\begin{aligned} e_x^w = e_x^s E_x^t - e_x^t E_x^s &= e_x^s(y=r) e_x^t(y=R) - e_x^t(y=r) e_x^t(y=R) \\ &= [E_x^1 \sin(kr) E_x^3 \cos(kR) - E_x^3 \cos(kr) E_x^1 \sin(kR)]_{\rm II} \\ &= [E_x^1 E_x^3 \sin(k(r-R))]_{\rm II}. \end{aligned}$$

The bracket and subscript II denote that the quantities bracketed are to be computed using medium-II parameters. The same information can be conveyed by putting a subscript 2 on the k and changing the superscripts 1 and 3 back to s and t. The other forms needed follow in much the same way. The only difference is that we use the relation $E_x^1 = E_x^3$, $B_x^1 = B_x^3$, and $B_z^1 = -B_z^3$, evident from Table II. We have then

$$e_x^w = E_x^t E_x^s \sin[k_2(r-R)], \tag{A6a}$$

$$b_x^w = E_x^t B_x^s \sin[k_2(r-R)], \tag{A6b}$$

$$b_z^w = E_x^t B_z^s \cos[k_2(r-R)]. \tag{A6c}$$

There is no further need for functions evaluated at

$y=R$. Hence, from now on, the capital-letter field component will always be used to designate the y-independent constant, as in Eqs. (A6). With this understood, and with the further observation that the combination WE_x^t is just the arbitrary constant S, we write Eqs. (A5) as

$$QE_x^q \sin(k_1r)+ \qquad SE_x^s \sin[k_2(r-R)] \qquad =0, \tag{A7a}$$

$$RE_z^r \cosh(\kappa_1 r) \qquad VE_z^v\psi=0, \tag{A7b}$$

$$QB_x^q \sin(k_1r)+RB_x^r \sinh(\kappa_1r)+SB_x^s \sin[k_2(r-R)]+VB_x^v\psi=0, \tag{A7c}$$

$$QB_z^q \cos(k_1r)+RB_z^r \cosh(\kappa_1r)+SB_z^s \cos[k_2(r-R)]+VB_z^v\psi=0, \tag{A7d}$$

$$\psi\equiv\exp(-\kappa_2 r).$$

It is evident that none of the manipulations to this point has involved the Q and R solutions at all. Hence, in calculating the dispersion equation for the odd modes, one can start at Eqs. (A7), merely replacing the Q and R solutions and constants by P and N solutions and constants.

In order that (A7) have a nontrivial solution, the determinant of coefficients must vanish. Factoring $\sin(k_1r)$ out of the first column of that determinant, $\cosh(\kappa_1r)$ out of the second, etc., and using the fact that $E_z^r=E_z^v=1$ leaves us with

$$\begin{vmatrix} E_x^q & 0 & E_x^s & 0 \\ 0 & 1 & 0 & 1 \\ B_x^q & \mathcal{B}_x^r & B_x^s & B_x^v \\ B_z^q\cot(k_1r) & B_z^r & B_z^s\cot[k_2(r-R)] & B_z^v \end{vmatrix}=0, \tag{A8a}$$

where

$$\mathcal{B}_x^r=B_x^r\tanh(\kappa_1r). \tag{A8b}$$

Expanding,

$$A_1\cot(k_1r)+A_2\cot[k_2(r-R)]+A_3=0, \tag{A9a}$$

$$A_1=(B_x^v-\mathcal{B}_x^r)B_z^qE_x^s, \tag{A9b}$$

$$A_2=-(B_x^v-\mathcal{B}_x^r)B_z^sE_x^q, \tag{A9c}$$

$$A_3=(B_z^r-B_z^v)(B_x^qE_x^s-B_x^sE_x^q). \tag{A9d}$$

It is useful to rewrite this equation as

$$\cot[k_2(r-R)]=C_1\cot(k_1r)+C_2, \tag{A10a}$$

$$C_1=B_z^qE_x^s/B_z^sE_x^q, \tag{A10b}$$

$$C_2=\left(\frac{B_z^r-B_z^v}{\mathcal{B}_x^r-B_x^v}\right)\left(\frac{B_x^sE_x^q-B_x^qE_x^s}{B_z^sE_x^q}\right). \tag{A10c}$$

To evaluate C_1 and C_2, we refer to information in Tables II and III. For instance, Table III indicates that both q and s are type-1 solutions, while Table II shows that in a type-1 solution, the ratio B_z/E_x is essentially the n_y of the medium. Hence, we have

$$C_1=n_1/n_2, \tag{A11a}$$

$$n_1\equiv(n_y)_{\text{I}}, \tag{A11b}$$

$$n_2=(n_y)_{\text{II}}. \tag{A11c}$$

In a similar fashion, we can express C_2 in the form

$$C_2=\frac{(K_2-K_1)(X_2-X_1)/n_2}{(\mathcal{B}_x^r-B_x^v)n_z^2}.$$

In evaluating the denominator of C_2, it is useful to combine (2.20b) with (2.14b)-(2.14f) to obtain

$$N_y^2=n_z^2(\omega_c\tau)^2,\quad N_y(S+iZn_z)=iKn_z.$$

When the denominator of C_2 is evaluated using Tables II and III, we have

$$n_z^2(\mathcal{B}_x^r-\mathcal{B}_x^v)=N_1(S_1+iZ_1n_z)\tanh(\kappa_1r)+N_2(S_2+iZ_2n_z),$$

$$N_1\equiv(N_y)_{\text{I}},$$

$$N_2\equiv(N_y)_{\text{II}}.$$

Putting these last few results together, and, for convenience, rewriting Eqs. (2.13), (2.14), (2.20), and (2.25), gives us the system of equations which serves as the dispersion equation for the even modes:

$$n_2\cot k_2(r-R)=n_1\cot k_1r-\frac{i(K_1-K_2)(X_1-X_2)}{n_z(K_1\tanh\kappa_1r+K_2)}, \tag{A12a}$$

$$X_i=K_i(1-U_in_z), \tag{A12b}$$

$$n_i^2=X_i^2/n_z^2-n_z^2, \tag{A12c}$$

$$k_i=\omega n_i/c. \tag{A12d}$$

The dispersion equation for the odd modes arises, as we mentioned, by replacing solution type q by solution type n and solution type r by type p, in (A7). This will, in Eqs. (A8), replace $\cot(k_1r)$ by $\tan(k_1r)$ and $\tanh(\kappa_1r)$ by $\coth\kappa_1r$, and, in Eqs. (A9), replace the superscripts q and r by n and p. However, this replacement will, as reference to Tables II and III, and Eqs. (A10) shows, replace C_1 by its negative and (neglecting the coth $\rightarrow$ tanh replacement) leave C_2 unchanged. Thus, the essential change is to replace

$$\cot(k_1r)\rightarrow-\tan(k_1r)=\cot(k_1r+\pi/2).$$

For the large κ_1 (proportional to $\omega_c\tau$), the tanh is equal to the coth, both being equal to ±1, depending on the sign of the real part of κ_1. Hence, the dispersion equation for the odd modes is

$$n_2 \cot[k_2(r-R)] = n_1 \cot(k_1 r + \pi/2) - \frac{i(K_1-K_2)(X_1-X_2)}{n_z(K_1 \coth(\kappa_1 r)+K_2)}. \quad \text{(A13)}$$

We suppress the tanh and coth in (A12) and (A13), since they are effectively unity and the result is Eq. (3.1) of the text.

APPENDIX B

Here we wish to study how n_z depends on U near threshold. For simplicity, we put $U_2=0$ and from (3.2) and (3.5c) obtain

$$U_t \equiv U_1^0 = (\xi-1)/\xi n_z^0. \quad \text{(B1)}$$

We expand all quantities in (3.1) to first order in $U-U_t$ about their threshold values:

$$U = U_t + v, \quad \text{(B2a)}$$

$$n_z = n_z^0 + \gamma v, \quad \text{(B2b)}$$

$$n_1 = n_0 + \gamma_1 v, \quad \text{(B2c)}$$

$$n_2 = n_0 + \gamma_2 v, \quad \text{(B2d)}$$

where

$$n_0 = ck_0/\omega.$$

Since the last term in (3.1a) vanishes at threshold, its expansion must start with a linear term in v which can be evaluated, using (B1), (B2), and (3.1):

$$\frac{(K_1-K_2)(X_1-X_2)/n_z}{K_1+K_2} \approx \frac{K_2(1-\xi)\xi(n_z^0+\gamma U_t)v}{n_z^0(1+\xi)} \quad \text{(B3a)}$$

$$\equiv Kv. \quad \text{(B3b)}$$

Insert (B2) and (B3) into (3.1) and differentiate with respect to v at $v=0$. The resulting equations are

$$\gamma_2[\cot(k_0(r-R)) - k_0(r-R)\csc^2(k_0(r-R))] = \gamma_1[\cot(k_0 r + \eta\pi/2) - k_0 r \csc^2(k_0 r + \eta\pi/2)] - iK, \quad \text{(B4a)}$$

$$2n_0\gamma_2 = -\frac{2\gamma}{(n_z^0)}\left[\frac{K_2^2}{(n_z^0)^2} + (n_z^0)^2\right], \quad \text{(B4b)}$$

$$2n_0\gamma_1 = \frac{-2\gamma}{(n_z^0)}\left[\frac{\xi K_2^2}{(n_z^0)^2} + (n_z^0)^2\right] - \frac{2\xi K_2^2}{n_z^0}. \quad \text{(B4c)}$$

Using (3.3b) in (B4a) gives

$$(\gamma_2-\gamma_1)\cot Z - (\csc^2 Z)[(\gamma_2-\gamma_1)k_0 r - \gamma_2 k_0 R] = -iK, \quad \text{(B5a)}$$

$$Z = k_0(r-R). \quad \text{(B5b)}$$

Using (B3), (B4b), and (B4c) in (B5a) gives a linear equation for γ whose solution is

$$\gamma = \frac{\xi K_2(\cot Z - k_0 r \csc^2 Z) + i\xi(1-\xi)n_z^0 n_0/(1+\xi)}{\alpha k_0 R \csc^2 Z + (1-\xi)J},$$
$$J = \epsilon(\cot Z - k_0 r \csc^2 Z) - i\xi U_t n_0/(1+\xi), \quad \text{(B6)}$$
$$\alpha = \epsilon + 1/\epsilon, \quad \epsilon = K_2/(n_z^0)^2.$$

Finally, letting ξ approach 1, we find that the imaginary part of (B6) becomes

$$\mathrm{Im}\gamma = \frac{n_z^0(1-\xi)\sin^2 Z}{2\alpha\omega R/c}. \quad \text{(B7)}$$

Using (3.4a) and (B5b) here reduces (B7) to Eq. (3.5) of the text.

APPENDIX C

The currents associated with the magnetic field are given by

$$\mathbf{j} = (1/\mu_0)i\mathbf{k}\times\mathbf{b} = (i\omega/\mu_0)\mathbf{n}\times\mathbf{b},$$

in this low-frequency regime where neglect of displacement current is justified. It is easiest for our purposes to evaluate $\mathbf{j}$ for the type-b and type-s solutions (2.21), (2.24) and then later to combine these into type 1–6 solutions. Before doing this, however, we will change the normalization of these solutions, using a type-b solution Sn_y/K times larger than that of (2.21) and (2.23), and using a type-s solution $-n_z$ times larger than (2.2) and (2.24). The coefficients for the type-b and type-s electric fields, magnetic fields, and electric currents are given in Table IV. Also indicated in Table IV is the power of $(\omega_c\tau)$ to which the entry is proportional in the high $-\omega_c\tau$ limit. Constant factors of c, ω, and μ_0 have been deleted.

The boundary conditions which brought the type-s fields into the problem were essentially continuity of $\mathbf{b}$ tangential. The normalization of the solutions in Table

TABLE IV. $(\omega_c\tau)$ Dependence of solution constants.

	Type b Coefficient	$(\omega_c\tau)^m$ $m=$	Type s Coefficient	$(\omega_c\tau)^m$ $m=$
e_x	in_z	0	K/N_y	−1
e_y	X/n_z	0	iN_y	+1
e_z	Sn_y/K	−1	$-n_z$	0
b_x	$-X$	0	$\pm K$[a]	0
b_y	in_2^2	0	Kn_z/n_y	−1
b_z	$-in_zn_y$	0	iK	0
j_x	$(n_y^2+n_z^2)n_z$	0	iN_yK[b]	+1
j_y	$-in_zx$	0	$\pm iKn_z$[a]	0
j_z	in_yx	0	$\mp N_yK$[a]	+1

[a] The term $\pm K$ arises as it did at the end of Appendix A, namely, multiplied by a tanh or a coth of large argument. As a result, the $\pm$ sign must be taken to agree with the sign of N_y.
[b] We have written $(N_y^2-n_z^2)/N_y=N_y$.

TABLE V. Effective power of $\omega_c\tau$ in $\int e_i^\alpha \cdot j_i^\beta da$.

$\mathbf{e}^\alpha\cdot\mathbf{j}^\beta$	(α,β) (b,b)	(b,s)	(s,b)	(s,s)
$e_x^\alpha j_x^\beta$	0	0	-2	-1
$e_y^\alpha j_y^\beta$	0	-1	0	0
$e_z^\alpha j_z^\beta$	-1	-1	-1	0

IV has been so chosen that all tangential **b** fields in both b and s solutions have coefficients with the same power of $(\omega_c\tau)$. This choice will insure that the arbitrary constants corresponding to $(Q,R,\cdots,V)$ of Appendix A will be of order $(\omega_c\tau)^0$.

The evaluation of W [Eq. (4.1)] involves evaluating an integral of products of solutions, electric fields and currents, of the various types. In carrying out this integral, one should recall that the s-type solutions fall off rapidly with distance from the interface, essentially as $\exp(-\kappa|y-r|)$, with κ being proportional to $\omega_c\tau$. Thus, the distance over which the solution is important is proportional to $1/\omega_c\tau$ and we may treat the b-type solution as being constant over this range. The integral of a b$\times$s product is then proportional to $1/\kappa\sim(1/\omega_c\tau)$ and the integral of the s$\times$s product is proportional to $1/2\kappa\sim(1/\omega_c\tau)$. This means that integration has the effect of decreasing by one the $(\omega_c\tau)$ power of all $\mathbf{e}\cdot\mathbf{j}$ products except $\mathbf{e}^{\mathrm{b}}\cdot\mathbf{j}^{\mathrm{b}}$. The effective, or lowered, order of the various $e_i^\alpha j_i^\beta$ integrals is as given in Table V.

Table V shows that in the high $\omega_c\tau$ limit, the only terms which contribute to W are $(e_x^{\mathrm{b}}j_x^{\mathrm{b}*}+e_y^{\mathrm{b}}j_y^{\mathrm{b}*})$, $e_x^{\mathrm{b}}j_x^{\mathrm{s}*}$, $e_y^{\mathrm{s}}j_y^{\mathrm{b}*}$, and $(e_y^{\mathrm{s}}j_y^{\mathrm{s}*}+e_z^{\mathrm{s}}j_z^{\mathrm{s}*})$. Since bulk helicons are known to propagate without loss in an infinite medium (in this limit), the (b,b) term vanishes. The reader may verify directly that the real part of this term is proportional to $\mathrm{Im}Un_z$, and thus, assuming n_z is real, this term contributes nothing. The terms (b,s) and (s,s) are not difficult to evaluate in the limit in which n_z is taken real, and they do contribute.

We take the b fields to be constant near the interface and write the actual fields and currents in the two media as follows: In medium I,

$$\begin{aligned} e_x&=-Qe_x^q, & j_x&=-Qj_x^q-Rj_x^r\psi;\\ e_y&=-Qe_y^q-Re_y^r\psi, & j_y&=-Qj_y^q-Rj_y^r\psi;\\ e_z&=\quad -Re_z^r\psi, & j_z&=-Qj_z^q-Rj_z^r\psi;\\ && \psi&=\exp[\kappa_1(y-r)]. \end{aligned}\tag{C1}$$

In medium II,

$$\begin{aligned} e_x&=We_x^w, & j_x&=Wj_x^w+Vj_x^v\varphi;\\ e_y&=We_y^w+Ve_y^v\varphi, & j_y&=Wj_y^w+Vj_y^v\varphi;\\ e_z&=\quad Ve_z^v\varphi, & j_z&=Wj_z^w+Vj_z^w\varphi;\\ && \varphi&=\exp[-\kappa_2(y-r)]. \end{aligned}\tag{C2}$$

[The coefficients Q and W here would be equal to $Q\sin(k_1r)$ and $W\sin[k_2(r-R)]$ of Appendix A; there are also constant factors relating R and V of this section to those of Appendix A, and j_z^q, j_z^w and differ from corresponding entries in Table V by factors of $\tan(k_1r)$ and $\tan(k_2(r-R))$.] The power loss from the two (b,s) terms is then

$$\begin{aligned} P_{\mathrm{bs}}&=\int_{-\infty}^{r}dy[Qe_z^q(Rj_x^r\psi)^*+Re_y^*\psi(Qj_y^q)^*]\\ &\quad+\int_r^\infty dy[We_x^w(Vj_x^v\varphi)^*+Ve_y^{\mathrm{b}}\varphi(Wj_y^w)^*]\\ &=(QR^*/\kappa_1^*)e_x^qj_x^{r*}+(RQ^*/\kappa_1)e_y^rj_y^{q*}\\ &\qquad+(Q\to W,R\to V,1\to 2). \end{aligned}$$

Since we shall be taking the real part of P_{bs}, we substitute for the second term of P_{bs} its complex conjugate, and write

$$\begin{aligned} P_{\mathrm{bs}}=(QR^*/\kappa_1^*)e_x^qj_x^{r*}+(RQ^*/\kappa_1)e_y^rj_y^{q*}\\ +(Q\to W,R\to V,1\to 2). \end{aligned}\tag{C3}$$

In a similar way, the power loss from the (s,s) term is

$$\begin{aligned} P_{\mathrm{ss}}=[RR^*/(\kappa_1+\kappa_1^*)][e_y^rj_y^{r*}+e_z^rj_z^{r*}]\\ +(R\to V,1\to 2). \end{aligned}\tag{C4}$$

The field and current components appearing in (C3) and (C4) are easily evaluated using Table IV and recalling that q and w are type-b solutions, r and v are the $+N_y$ and $-N_y$ forms of type-s solutions. Finally, using $(N_y/\kappa)=c/\omega$, we obtain

$$P_{\mathrm{bs}}=(cn_z/\omega)[QR^*(K_1-X_1)-WV^*(K_2-X_2)],\tag{C5a}$$

$$P_{\mathrm{ss}}=(cn_z/\omega)[RR^*K_1+VV^*K_2].\tag{C5b}$$

In arriving at (C5) we have taken n_z as real.

The coefficients R, W, and V may be expressed in terms of Q by using three interface continuity conditions. For instance,

$$\begin{aligned} Qe_x^q+We_x^w&=0,\\ Re_z^r+Ve_z^v&=0,\\ Qb_x^q+Rb_x^r+Wb_x^w+Ve_z^v&=0. \end{aligned}\tag{C6}$$

(The fourth condition, on continuity of b_z, is redundant, being satisfied automatically as a consequence of the dispersion relationships.) Solving (C6) and evaluating using Table IV gives

$$\begin{aligned} W&=-Q,\\ V&=-R,\\ R&=Q(X_1-X_2)/(K_1+K_2), \end{aligned}\tag{C7}$$

which, inserted in (C5) and ignoring the constant cn_z/ω, gives Eqs. (4.3) and (4.4).