Divergent Transport Coefficients and the Binary-Collision Expansion*

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The binary-collision expansion for the viscosity of a two-dimensional gas of hard disks is discussed. A divergence appears in η_1, the first correction to the Boltzmann-equation result. The calculations presented here are exact and explicitly demonstrate the dynamical origin of the divergence indicated by Kawasaki and Oppenheim. The coefficient of the divergence is computed and found to be precisely the same as that found by Sengers by an entirely different method. The origin of the divergence is shown to be exactly the same as that found by Dorfman and Cohen.

INTRODUCTION

RECENTLY, several authors[1-3] have discovered that a divergence exists in the coefficients of the virial expansion for the transport coefficients for moderately dense gases. The divergence exists regardless of whether one starts from the generalized Boltzmann equation and evaluates the transport coefficients via a Chapman–Enskog-like method, or alternatively, one evaluates the time correlation function by means of a virial expansion in powers of the density, n.[1]

One finds by such methods that the $l-1$ coefficient in the virial expansion of the viscosity, say, is determined by dynamical events of l particles.[4] The contribution of genuine l-tuple collisions is well behaved.[1,2] However, one also finds contributions from certain sequences of binary, triple, $\cdots$, $(l-1)$-tuple collisions, the precise characterization of which has been given by Green and Piccirelli.[5] The divergence is caused by the failure of the phase-space volume to remain bounded, for such sequences of events, as the time between successive collisions becomes large. In two dimensions the divergence appears in the three-body term. In three dimensions the divergence appears in the four-body term.[1]

Recently Kawasaki, and Oppenheim[6] have given a discussion of the divergence from the point of view of the binary-collision expansion in the formalism of Zwanzig.[7] They have effected a partial resummation of the most divergent terms and found a term proportional to $\ln n$ in the virial expansion for the self-diffusion coefficient. The discussion of the divergence given by Kawasaki does not consider the dynamical events among three particles in an explicit way. This has lead to some confusion in the literature.[8] Furthermore, since an evaluation of the resummed expression for the viscosity given by Oppenheim and Kawasaki will certainly involve careful discussion of the dynamics of three or more particles, it is valuable to provide an example for a relatively simple case.

The purpose of this paper is to make the connection of the existence of the divergence in the binary-collision expansion with the dynamics of a system of a few particles. We consider a two-dimensional system of hard disks and consider η_1, the first density correction to the kinetic part of the shear viscosity. We show that η_1 diverges; that the origin of this divergence is the same as that found by Dorfman and Cohen from phase-space arguments; and that the coefficient of the divergent part is precisely the same as that found by Sengers from the Bogoliubov theory.

Section 1 is devoted to summarizing the pertinent results from the binary-collision expansion. In Sec. 2 we give an explicit evaluation of a general binary-collision operator (T) matrix element for hard disks. In Sec. 3 we list the sequences of T's which contribute to η_1, and in Sec. 4 we give the explicit integral expressions for these contributions. Section 5 is devoted to a discussion of the general form of the integrals that appear. In Sec. 6 we evaluate the integral expression for one sequence of T's (the recollision term). In Sec. 7 we evaluate completely the numerical coefficient of the (divergent) recollision term for the viscosity. This enables us to compare our result with the work of Sengers[2] who has recently computed the same number by an entirely different method.

1. BINARY-COLLISION EXPANSION

We use the binary-collision expansion (BCE) in the formalism developed by Zwanzig.[7] The reader is referred

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[1] J. R. Dorfman and E. G. D. Cohen, Phys. Letters **16**, 124 (1965); J. R. Dorfman and E. G. D. Cohen (to be published).

[2] J. V. Sengers, Phys. Rev. Letters **15**, 515 (1965); J. V. Sengers (to be published).

[3] E. A. Frieman and R. Goldman, Bull. Am. Phys. Soc. **10**, 531 (1965); J. Weinstock, Phys. Rev. **140**, A461 (1965).

[4] M. H. Ernst, J. R. Dorfman, and E. G. D. Cohen, Physica **31**, 493 (1965); Phys. Letters **12**, 319 (1964).

[5] M. S. Green and R. A. Piccirelli, Phys. Rev. **132**, 1388 (1963).

[6] K. Kawasaki and I. Oppenheim, Phys. Rev. **139**, A1763 (1965).

[7] R. Zwanzig, Phys. Rev. **129**, 486 (1963).

[8] J. Stecki, Phys. Letters **19**, 123 (1965), who erroneously concluded that a divergence does not exist.

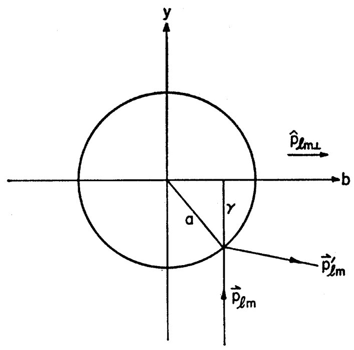

FIG. 1. Coordinate system for evaluation of (2.8).

to his paper for the details of the theory and for the notation, which we adopt in this paper. For formal details concerning the extension of the BCE to computation of the shear viscosity, the reader is referred to the paper of Kawasaki and Oppenheim.[9]

We now summarize a few results from the BCE:

(1) The time-displacement operator $\exp(itL)$ has the property

$$\exp(itL)\alpha(\mathbf{R},\mathbf{p})=\alpha(\mathbf{R}(t),\mathbf{p}(t)) \qquad (1.1)$$

acting on a function α of the initial phase variables $\mathbf{R}$ and $\mathbf{p}$, where

$$iL=iL_0+iL_1 \qquad (1.2)$$

and

$$iL_0=\sum_{j=1}\frac{\mathbf{p}_j}{m}\cdot\frac{\partial}{\partial\mathbf{R}_j};\quad iL_1\equiv\sum_{1\leq i<j}iL_{ij}, \qquad (1.3)$$

with

$$iL_{ij}=-\frac{\partial U(R_{ij})}{\partial\mathbf{R}_i}\cdot\left(\frac{\partial}{\partial\mathbf{p}_i}-\frac{\partial}{\partial\mathbf{p}_j}\right) \qquad (1.4)$$

and where U is the interaction potential.

(2) The Laplace transforms of $\exp(itL)$ and $\exp(itL_0)$ are denoted by

$$\int_0^\infty dt\, e^{-\epsilon t}e^{itL}\equiv G=(\epsilon-iL)^{-1};$$

$$\int_0^\infty dt\, e^{-\epsilon t}e^{itL_0}\equiv G_0=(\epsilon-iL_0)^{-1}. \qquad (1.5)$$

(3) The BCE for G is given by

$$G=G_0-\sum_\alpha G_0T_\alpha G_0+\sum_{\alpha,\beta}{}' G_0T_\alpha G_0T_\beta G_0-\cdots, \qquad (1.6)$$

where the sums are over all distinct pairs (Greek subscripts) and are so restricted that no two consecutive T's refer to the same pair. The operator T_α satisfies

$$T_\alpha=-iL_\alpha+iL_\alpha G_0T_\alpha. \qquad (1.7)$$

(4) The Fourier transform (matrix element) of an operator, say G, in position space is given by

$$\langle\mathbf{k}'|G|\mathbf{k}\rangle\equiv\int d\mathbf{R}e^{-i\mathbf{k}'\cdot\mathbf{R}}Ge^{i\mathbf{k}\cdot\mathbf{R}} \qquad (1.8)$$

and the normalization is chosen so that

$$\langle\mathbf{k}'|G_1G_2|\mathbf{k}\rangle=\int\frac{d\mathbf{k}''}{(2\pi)^d}\langle\mathbf{k}'|G_1|\mathbf{k}''\rangle\langle\mathbf{k}''|G_2|\mathbf{k}\rangle, \qquad (1.9)$$

where d is the number of dimensions. (In the work below we will consider always the limit $N\to\infty$, $V\to\infty$, $N/V\to n$.) We further remark that the matrix element of G_0 is diagonal, while the matrix element of T_{ij} has $\mathbf{k}_i'+\mathbf{k}_j'=\mathbf{k}_i+\mathbf{k}_j$ and is diagonal in all other $\mathbf{k}$'s. We adopt the convention that when the $\mathbf{k}$'s are explicitly enumerated, all those not shown are zero.

2. GENERAL BCE MATRIX ELEMENT FOR HARD DISKS

In this section we compute, for hard disks of diameter a, the general BCE matrix element[10]

$$M_{lm}\equiv\langle\mathbf{k}_1,\cdots,\mathbf{k}_{j-2},\mathbf{k}_l,\mathbf{k}_m|-T_{lm}G_0|\mathbf{k}_1,\cdots,\mathbf{k}_{j-2},\mathbf{k}_l',\mathbf{k}_m'\rangle \qquad (2.1)$$

with $\mathbf{k}_l+\mathbf{k}_m=\mathbf{k}_l'+\mathbf{k}_m'$, acting on a function of the particle momenta, $F(\mathbf{p}_1,\mathbf{p}_2,\cdots)$. We first observe from (1.7) that T_{lm} can be written as

$$T_{lm}=-G_0^{-1}[G(lm)-G_0]G_0^{-1}, \qquad (2.2)$$

where

$$G(lm)=[\epsilon-i(L_0+L_{lm})]^{-1}\equiv[\epsilon-iL(lm)]^{-1}. \qquad (2.3)$$

Since

$$G(lm)-G_0=\int_0^\infty dt\, e^{-\epsilon t}\{\exp[itL(lm)]-\exp(itL_0)\}, \qquad (2.4)$$

we have

$$M_{lm}=\left(\epsilon-\frac{i}{m}\sum_1^j\mathbf{k}_r\cdot\mathbf{p}_r\right)\int d\mathbf{R}_1\cdots d\mathbf{R}_{j-2}d\mathbf{R}_ld\mathbf{R}_m\exp\left[-i\sum_1^{j-2}\mathbf{k}_r\cdot\mathbf{R}_r-i(\mathbf{k}_l\cdot\mathbf{R}_l+\mathbf{k}_m\cdot\mathbf{R}_m)\right]$$
$$\times\int_0^\infty dt\, e^{-\epsilon t}\{\exp[itL(lm)]-\exp(itL_0)\}\exp\left[i\sum_1^{j-2}\mathbf{k}_r\cdot\mathbf{R}_r+i(\mathbf{k}_l'\cdot\mathbf{R}_l+\mathbf{k}_m'\cdot\mathbf{R}_m)\right]. \qquad (2.5)$$

[9] K. Kawasaki and I. Oppenheim, Phys. Rev. **136**, A1519 (1964).

[10] Strictly speaking (2.1) does not exist as written for an infinite system because it contains delta functions of the $\mathbf{k}$'s conserved by T_{lm}. Thus we should compute

$$\langle\mathbf{k}_1,\cdots,\mathbf{k}_{j-2},\mathbf{k}_l,\mathbf{k}_m|-T_{lm}G_0|\mathbf{k}_1',\cdots,\mathbf{k}_{j-2}',\mathbf{k}_l',\mathbf{k}_m'\rangle$$

and keep track of the resulting delta functions. To avoid this insignificant but cumbersome difficulty, we will compute (2.1) for a finite system with unit volume. The results are the same.

In order to evaluate (2.5), we notice that it is sufficient to consider L_0 only for the j particles in question, which we write as $L_0^{(j)}$. Further, we may factor out the motion of all particles except the pair lm so that

$$\exp[itL(lm)]-\exp(itL_0)=\{\exp[itL^{(2)}(lm)]-\exp(itL_0^{(2)})\}\exp(itL_0^{(j-2)}),\tag{2.6}$$

where $L^{(2)}(lm)=L_0^{(2)}+L_{lm}$. Introducing the two-body coordinates

$$\begin{aligned}\mathbf{R}_{lm}&=\tfrac{1}{2}(\mathbf{R}_l+\mathbf{R}_m); & \mathbf{r}_{lm}&=\mathbf{r}_l-\mathbf{r}_m\\ \mathbf{P}_{lm}&=\mathbf{p}_l+\mathbf{p}_m; & \mathbf{p}_{lm}&=\tfrac{1}{2}(\mathbf{p}_l-\mathbf{p}_m)\end{aligned}\tag{2.7}$$

we find

$$\begin{aligned}M_{lm}F(\mathbf{p}_1,\mathbf{p}_2,\cdots)=&\left(\epsilon-\frac{i}{m}\sum_1^j\mathbf{k}_r\cdot\mathbf{p}_r\right)\int d\mathbf{r}_{lm}\exp[-\tfrac{1}{2}i\mathbf{r}_{lm}\cdot(\mathbf{k}_l-\mathbf{k}_m)]\\ &\times\int_0^\infty dt\exp\left\{t\left[-\epsilon+\frac{i}{m}\sum_1^{j-2}\mathbf{k}_r\cdot\mathbf{p}_r+\frac{i}{2m}(\mathbf{k}_l+\mathbf{k}_m)\cdot\mathbf{P}_{lm}\right]\right\}\\ &\times\{\exp[itL^{(2)}(lm)]-\exp(itL_0^{(2)})\}\exp[\tfrac{1}{2}i\mathbf{r}_{lm}\cdot(\mathbf{k}_l'-\mathbf{k}_m')]F(\mathbf{p}_1,\mathbf{p}_2,\cdots).\end{aligned}\tag{2.8}$$

We choose a coordinate system for the initial relative position $\mathbf{r}_{lm}$, as shown in Fig. 1. Then

$$\mathbf{r}_{lm}=b\hat{p}_{lm\perp}+y\hat{p}_{lm}.\tag{2.9}$$

Since the operator $\exp[itL^{(2)}(lm)]-\exp(itL_0^{(2)}$ is zero unless a collision occurs in time t, it is clear that the contribution to M_{lm} comes when

$$-a\leq b\leq a;\quad -\infty\leq y\leq-\gamma;\quad t^*\leq t\leq\infty,\tag{2.10}$$

where

$$\gamma=(a^2-b^2)^{\frac{1}{2}};\quad t^*=-(y+\gamma)m/(2p_{lm}).\tag{2.11}$$

(In making this argument we neglect the contributions from initial positions inside the interaction disk. This contribution leads to a well behaved result as $\epsilon\to0$.) By using the properties of the time-displacement operators and the expressions for the relative position $\mathbf{r}_{lm}'$ and momentum $\mathbf{p}_{lm}'$ after collision

$$\mathbf{r}_{lm}'=b\hat{p}_{lm\perp}-\gamma\hat{p}_{lm}+(2/m)(t-t^*)\mathbf{p}_{lm}',\quad \mathbf{p}_{lm}'=-(1-2b^2/a^2)\mathbf{p}_{lm}+(2b/a)(1-b^2/a^2)^{\frac{1}{2}}\mathbf{p}_{lm\perp},\tag{2.12}$$

one may perform the t and y integrations in (2.8) to find

$$M_{lm}F(\mathbf{p}_1,\mathbf{p}_2,\cdots)=\frac{2p_{lm}}{m}\int_{-a}^{a}db\exp[i(\mathbf{k}_l'-\mathbf{k}_l)\cdot\boldsymbol{\varrho}_{lm}(b)][g(\mathbf{p}_l',\mathbf{p}_m')-g(\mathbf{p}_l,\mathbf{p}_m)],\tag{2.13}$$

where

$$g(\mathbf{p}_l,\mathbf{p}_m)=\left\{\epsilon-\frac{i}{m}\left[\sum_1^{j-2}\mathbf{k}_r\cdot\mathbf{p}_r+\mathbf{k}_l'\cdot\mathbf{p}_l+\mathbf{k}_m'\cdot\mathbf{p}_m\right]\right\}^{-1}F(\mathbf{p}_1,\mathbf{p}_2,\cdots)\tag{2.14}$$

and

$$\boldsymbol{\varrho}_{lm}(b)=b\hat{p}_{lm\perp}-\gamma\hat{p}_{lm};\tag{2.15}$$

and the momenta after collision, $\mathbf{p}_l'$ and $\mathbf{p}_m'$, are given by

$$\mathbf{p}_l'=\mathbf{p}_{lm}'+\tfrac{1}{2}\mathbf{P}_{lm};\quad \mathbf{p}_m'=-\mathbf{p}_{lm}'+\tfrac{1}{2}\mathbf{P}_{lm}.\tag{2.16}$$

3. THREE-COLLISION FIRST DENSITY CORRECTION TO THE VISCOSITY

In two dimensions, the kinetic contribution to the shear viscosity η is given by[11]

$$\eta=\lim_{\epsilon\to0+}\lim_{\substack{N,V\to\infty\\ N/V\to n}}\frac{1}{4kTm^2V}\int_0^\infty dt\,e^{-\epsilon t}\times\left\langle\sum_{i=1}^N\mathbf{p}_i\mathbf{p}_i:\ e^{itL}\sum_{j=1}^N(\mathbf{p}_j\mathbf{p}_j-\tfrac{1}{2}\mathbf{I}p_j^2)\right\rangle,\tag{3.1}$$

where the average is taken over the canonical ensemble.

It has been shown elsewhere[2] that the three-collision contributions to η give rise to a divergence proportional to $\ln T^*$ where T^* is the time between the first and last collisions. Since ϵ^{-1} is the time variable in the Laplace-transformed theory, we expect to find a corresponding divergence proportional to $\ln\epsilon$ by examining the three-collision terms in the BCE. To discuss the divergence it is sufficient to consider the contribution to η when initial particle correlations are neglected ($U=0$ in the canonical distribution).[6] We write

$$\eta=\eta_0+\eta_1+\cdots,\tag{3.2}$$

where we consider only the first Enskog approximation. Here η_0 is the Boltzmann equation viscosity, and the

[11] J. A. McLennan, Phys. Fluids 3, 493 (1960).

three-collision contribution η_1 is given by

$$\eta_1=\frac{n\eta_0^2}{4m^2(kT)^3}\prod_{i=1}^{3}\int d\mathbf{p}_i\Phi(p_i)\mathbf{p}_1\mathbf{p}_1:\quad \epsilon^2 Q\sum_{j=1}^{3}(\mathbf{p}_j\mathbf{p}_j-\tfrac{1}{2}\mathbf{I}p_j^2)\,, \tag{3.3}$$

where

$$\Phi(p)=(\beta/\pi)e^{-\beta p^2};\quad \beta=(2mkT)^{-1} \tag{3.4}$$

and

$$Q=Q_1+Q_2+Q_3+Q_4; \tag{3.5}$$

with

$$\begin{aligned} Q_1&=-\langle 0|G_0T_{12}G_0T_{13}G_0T_{12}G_0|0\rangle, & Q_3&=-\langle 0|G_0T_{12}G_0T_{13}G_0T_{23}G_0|0\rangle,\\ Q_2&=-\langle 0|G_0T_{12}G_0T_{23}G_0T_{12}G_0|0\rangle, & Q_4&=-\langle 0|G_0T_{12}G_0T_{23}G_0T_{13}G_0|0\rangle. \end{aligned} \tag{3.6}$$

Now it follows from (2.13) and (2.14) that, say, Q_1 acting on a function which is conserved during a (1,2) collision gives zero. In particular $\mathbf{P}_{12}$, p_{12} and $\mathbf{p}_3$ are such quantities. Using (2.7), we find

$$Q_1\sum_{i=1}^{3}(\mathbf{p}_i\mathbf{p}_i-\tfrac{1}{2}\mathbf{I}p_i^2)=2Q_1\mathbf{p}_{12}\mathbf{p}_{12}. \tag{3.7}$$

Further, under interchange of particles one and two we see that $Q_2\rightarrow Q_1$ and $Q_4\rightarrow Q_3$. If we write

$$\eta_1/\eta_0\equiv\sigma_R+\sigma_{\mathrm{HC}}, \tag{3.8}$$

we then have

$$\begin{aligned} \sigma_R&=\frac{n\eta_0}{4m^2(kT)^3}\prod_{i=1}^{3}\int d\mathbf{p}_i\Phi(p_i)[4\mathbf{p}_{12}\mathbf{p}_{12}+\mathbf{P}_{12}\mathbf{P}_{12}]:\quad \epsilon^2Q_1\mathbf{p}_{12}\mathbf{p}_{12},\\ \sigma_{HC}&=\frac{n\eta_0}{4m^2(kT)^3}\prod_{i=1}^{3}\int d\mathbf{p}_i\Phi(p_i)[4\mathbf{p}_{12}\mathbf{p}_{12}+\mathbf{P}_{12}\mathbf{P}_{12}]:\quad \epsilon^2Q_3\mathbf{p}_{23}\mathbf{p}_{23}. \end{aligned} \tag{3.9}$$

4. EVALUATION OF Q_1

In this section we discuss Q_1 and Q_3 for hard disks. Writing out Q_1 and Q_3, we have

$$\epsilon^2Q_1J(\mathbf{p}_{12})=\epsilon\int\frac{d\mathbf{k}}{(2\pi)^2}\langle 0|-T_{12}G_0|\mathbf{k},-\mathbf{k}\rangle\langle\mathbf{k},-\mathbf{k}|-T_{13}G_0|\mathbf{k},-\mathbf{k}\rangle\langle\mathbf{k},-\mathbf{k}|-T_{12}G_0|0\rangle J(\mathbf{p}_{12})\,, \tag{4.1}$$

$$\epsilon^2Q_3J(\mathbf{p}_{23})=\epsilon\int\frac{d\mathbf{k}}{(2\pi)^2}\langle 0|-T_{12}G_0|\mathbf{k},-\mathbf{k}\rangle\langle\mathbf{k},-\mathbf{k}|-T_{13}G_0|0,-\mathbf{k},\mathbf{k}\rangle\langle 0,-\mathbf{k},\mathbf{k}|-T_{23}G_0|0\rangle J(\mathbf{p}_{23})\,, \tag{4.2}$$

where $J(\mathbf{p}_{12})$ is some function of the relative momentum, say $\mathbf{p}_{12}\mathbf{p}_{12}$.

We now use (2.13) to evaluate (4.1). Working from left to right, we have first to consider $\langle 0|-T_{12}G_0|\mathbf{k},-\mathbf{k}\rangle$ operating on some function $f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$. For this case, we have

$$l=1,\, m=2;\quad \mathbf{k}_l=\mathbf{k}_m=0;\quad \mathbf{k}_l'=-\mathbf{k}_m'=\mathbf{k};\quad \text{all other } \mathbf{k}\text{'s zero}; \tag{4.3}$$

so that

$$g_1(\mathbf{p}_1,\mathbf{p}_2)=[\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}]^{-1}f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) \tag{4.4}$$

and

$$\langle 0|-T_{12}G_0|\mathbf{k},-\mathbf{k}\rangle f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\frac{2p_{12}}{m}\int_{-a}^{a}db \times\exp[i\mathbf{k}\cdot\boldsymbol{\rho}_{12}(b)][g_1(\mathbf{p}_1',\mathbf{p}_2')-g_1(\mathbf{p}_1,\mathbf{p}_2)]. \tag{4.5}$$

Next we consider the middle matrix element acting on some function $f_2(\mathbf{p}_1,\mathbf{p}_2)$, where

$$f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\langle\mathbf{k},-\mathbf{k}|-T_{13}G_0|\mathbf{k},-\mathbf{k}\rangle f_2(\mathbf{p}_1,\mathbf{p}_2). \tag{4.6}$$

Here we have

$$l=1,\, m=3;\quad \mathbf{k}_l=\mathbf{k}_l'=\mathbf{k};\quad \mathbf{k}_m=\mathbf{k}_m'=0;\quad \mathbf{k}_2=-\mathbf{k};\quad \text{all other } \mathbf{k}\text{'s zero}; \tag{4.7}$$

so that

$$g_2(\mathbf{p}_1,\mathbf{p}_3)=[\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}]^{-1}f_2(\mathbf{p}_1,\mathbf{p}_2) \tag{4.8}$$

and

$$f_1(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\frac{2p_{13}}{m}\int_{-a}^{a}db' \times[g_2(\mathbf{p}_1'',\mathbf{p}_3'')-g_2(\mathbf{p}_1,\mathbf{p}_3)]. \tag{4.9}$$

Finally we compute the final matrix element acting on $J(\mathbf{p}_{12})$:

$$f_2(\mathbf{p}_1,\mathbf{p}_2)=\langle\mathbf{k},-\mathbf{k}|-T_{12}G_0|0\rangle J(\mathbf{p}_{12}). \tag{4.10}$$

We have

$$l=1,\, m=2;\quad \mathbf{k}_l=-\mathbf{k}_m=\mathbf{k};\quad \mathbf{k}_l'=\mathbf{k}_m'=0;\quad \text{all other } \mathbf{k}\text{'s zero}; \tag{4.11}$$

so that

$$g_3(\mathbf{p}_1,\mathbf{p}_2)=\epsilon^{-1}J(\mathbf{p}_{12}) \tag{4.12}$$

and

$$f_2(\mathbf{p}_1,\mathbf{p}_2)=\frac{2p_{12}}{m}\int_{-a}^{a} db''\exp[-i\mathbf{k}\cdot\boldsymbol{\varrho}_{12}(b'')] \times[g_3(\mathbf{p}_1''',\mathbf{p}_2''')-g_3(\mathbf{p}_1,\mathbf{p}_2)]. \tag{4.13}$$

We notice that each of the three contributions to $\epsilon^2Q_1J(\mathbf{p}_{12})$ gives rise to an interacting term (I) and a noninteracting term (N). We therefore define

$$\epsilon^2Q_1J(\mathbf{p}_{12})\equiv\int\frac{d\mathbf{k}}{(2\pi)^2}[NN(IN)+IN(IN)+NI(IN)+II(IN)], \tag{4.14}$$

where, by (IN), we mean that we include both the N and I contributions from the final (1,2) collision. Each of these four terms can be indicated schematically. As an example we indicate the term $NI(IN)$ in Fig. 2. Explicitly, we find for the four contributions to $\epsilon^2Q_1J(\mathbf{p}_{12})$:

$$NN(IN)=\int_{-a}^{a}db\int_{-a}^{a}db'\int_{-a}^{a}db''\left(\frac{2p_{12}}{m}\right)^2\left(\frac{2p_{13}}{m}\right)\exp\{i\mathbf{k}\cdot[\boldsymbol{\varrho}_{12}(b)-\boldsymbol{\varrho}_{12}(b'')]\}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12})^{-2}\Delta J, \tag{4.15}$$

$$IN(IN)=-\int_{-a}^{a}db\int_{-a}^{a}db'\int_{-a}^{a}db''\left(\frac{2p_{12}}{m}\right)^2\left(\frac{2p_{13}'}{m}\right)\exp\{i\mathbf{k}\cdot[\boldsymbol{\varrho}_{12}(b)-\boldsymbol{\varrho}_{12}'(b'')]\}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}')^{-2}\Delta J, \tag{4.16}$$

$$NI(IN)=-\int_{-a}^{a}db\int_{-a}^{a}db'\int_{-a}^{a}db''\left(\frac{2p_{12}}{m}\right)\left(\frac{2p_{13}}{m}\right)\left(\frac{2p_{12}''}{m}\right)\times\exp\{i\mathbf{k}\cdot[\boldsymbol{\varrho}_{12}(b)-\boldsymbol{\varrho}_{12}''(b'')]\}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12})^{-1}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}'')^{-1}\Delta J, \tag{4.17}$$

$$II(IN)=\int_{-a}^{a}db\int_{-a}^{a}db'\int_{-a}^{a}db''\left(\frac{2p_{12}}{m}\right)\left(\frac{2p_{13}'}{m}\right)\left(\frac{2p_{12}''}{m}\right)\times\exp\{i\mathbf{k}\cdot[\boldsymbol{\varrho}_{12}(b)-\boldsymbol{\varrho}_{12}''(b'')]\}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}')^{-1}(\epsilon-(2i/m)\mathbf{k}\cdot\mathbf{p}_{12}'')^{-1}\Delta J, \tag{4.18}$$

where

$$\Delta J=J(\mathbf{p}_{12}''')-J(\mathbf{p}_{12}'') \tag{4.19}$$

and the *primed momenta are always to be computed from the appropriate impact variable and initial momenta* which can be read off from the diagrams.

An exactly similar argument can be made to obtain the contributions to $\epsilon^2Q_3J(\mathbf{p}_{23})$ analogous to (4.15) through (4.18).

5. THE k INTEGRAL

In order to discuss the four terms (4.15) through (4.18) of $\epsilon^2Q_1J(\mathbf{p}_{12})$ and the corresponding four terms of $\epsilon^2Q_3J(\mathbf{p}_{23})$ we must perform the $\mathbf{k}$ integral. We see that the integrals of (4.15) and (4.16) both have the form

$$I_1\equiv\int d\mathbf{k}[\exp(i\mathbf{k}\cdot\mathbf{A})](\epsilon-i\mathbf{k}\cdot\mathbf{B})^{-2} \tag{5.1}$$

while the integrals of (4.17) and (4.18) both have the form

$$I_2\equiv\int d\mathbf{k}[\exp(i\mathbf{k}\cdot\mathbf{A})](\epsilon-i\mathbf{k}\cdot\mathbf{B})^{-1}(\epsilon-i\mathbf{k}\cdot\mathbf{C})^{-1}. \tag{5.2}$$

It can also be shown that all four terms of $\epsilon^2Q_3J(\mathbf{p}_{23})$ have the form (5.2).

Both I_1 and I_2 can be evaluated by writing $d\mathbf{k}=dk_xdk_y$ and computing the integrals over k_x and k_y in the complex k_x and k_y planes. A semicircular contour is used, closed above or below always so that the integral exists. Evaluation of the integrals is simplified by choosing a coordinate system where $B_x=0$. One finds

$$I_1=(2\pi)^2\theta(-A_yB_y)\delta(A_x)|A_y|B_y^{-2}e^{\epsilon A_y/B_y}, \tag{5.3}$$

$$I_2=\frac{(2\pi)^2}{|C_xB_y|}\theta\left(-\frac{A_x}{C_x}\right)\theta\left(-\frac{A_yC_x-A_xC_y}{B_yC_x}\right)\times\exp\left[\epsilon\left(\frac{A_x}{C_x}+\frac{A_yC_x-A_xC_y}{C_xB_y}\right)\right], \tag{5.4}$$

where $\delta(x)$ is the Dirac delta function and $\theta(x)$ is the unit step function which is zero for x negative and one for x positive.

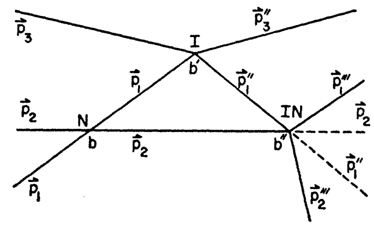

FIG. 2. Schematic illustration of the term $NI(IN)$. At the b'' vertex the solid (dashed) lines correspond to $I(N)$.

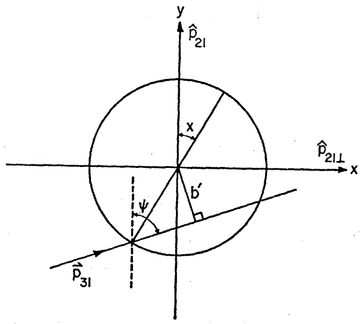

FIG. 3. Geometry of the second collision.

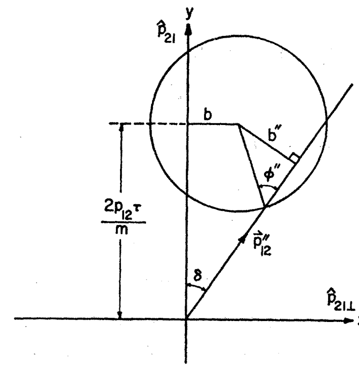

FIG. 4. Geometry of the last collision.

By using the appropriate **A** and **B** from (4.15) and, (4.16) one can see after some calculation that the θ function and δ function in (5.4) give rise to a vanishing region of integration over the impact variables. Thus

$$\int d\mathbf{k}\, NN(IN)=\int d\mathbf{k}\, IN(IN)=0. \tag{5.5}$$

The result (5.5) is not unexpected since it is impossible for the sequence of collisions (1,2) (1,3) (1,2) to occur unless the (1,3) collision physically takes place. Thus (5.5) merely expresses the fact that contributions come only from a physically realizable sequence of collisions. The fact that all four terms contributing to Q_3 are of the form (5.4) is again quite reasonable since the sequence (1,2) (1,3) (2,3) can occur whether or not the (1,3) collision actually takes place. In fact, the sequence (1,2) (1,3) (1,2), where (1,3) takes place, is a "recollision" event, while the sequence (1,2) (1,3) (2,3), where (1,3) does or does not take place, is a "cyclic" or "hypothetical" event, respectively.[1,2]

The following two sections are devoted to the evaluation of the recollision term and its contribution to the viscosity. The evaluation of the cyclic and hypothetical terms is quite similar.

6. THE RECOLLISION TERM

The recollision contribution comes from

$$\epsilon^2 Q_1 J(\mathbf{p}_{12})=\int\frac{d\mathbf{k}}{(2\pi)^2}NI(IN)+\int\frac{d\mathbf{k}}{(2\pi)^2}II(IN). \tag{6.1}$$

We first use (4.17) and (5.4) to evaluate the **k** integral of $NI(IN)$ with

$$\begin{aligned}\mathbf{A}&=b\hat{p}_{12\perp}-\gamma(b)\hat{p}_{12}-b''\hat{p}_{12\perp}''+\gamma(b'')\hat{p}_{12}'';\\ \mathbf{B}&=(2/m)\mathbf{p}_{12};\\ \mathbf{C}&=(2/m)\mathbf{p}_{12}''.\end{aligned} \tag{6.2}$$

We choose the positive y axis in the direction of $\hat{p}_{21}$, define ψ as the angle between $\hat{p}_{21}$ and $\hat{p}_{31}$, and δ as the angle between $\hat{p}_{21}$ and $\hat{p}_{12}''$ (see Figs. 3 and 4). If we change the variables of integration in (4.17) from b to ϕ, b' to χ, and b'' to ϕ'' via

$$b=a\sin\phi;\quad b'=a\sin(\psi-\chi);\quad b''=a\sin\phi'' \tag{6.3}$$

we find

$$\int\frac{d\mathbf{k}}{(2\pi)^2}NI(IN)=\int_{-\pi/2}^{\pi/2}d\phi\,\cos\phi H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi), \tag{6.4}$$

where

$$\begin{aligned}H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi)=&-\frac{4p_{12}p_{13}a^3}{m^2}\int_{\psi-\pi/2}^{\psi+\pi/2}d\chi\cos(\chi-\psi)\int_{-\pi/2}^{\pi/2}d\phi''\cos\phi''\\ &\times\frac{C}{|C_xB_y|}\theta\left(-\frac{A_x}{C_x}\right)\theta\left(-\frac{A_yC_x-A_xC_y}{B_yC_x}\right)\exp\left[\epsilon\left(\frac{A_x}{C_x}+\frac{A_yC_x-C_yA_x}{C_xB_y}\right)\right]\Delta J\end{aligned} \tag{6.5}$$

with

$$\begin{aligned}&A_x=-a[\sin\phi+\sin(\phi''-\delta)]; && A_y=a[\cos\phi+\cos(\phi''-\delta)];\\ &B_x=0; && B_y=-(2/m)p_{12};\\ &C_x=(2p_{13}/m)\cos(\chi-\psi)\sin\chi; && C_y=(2p_{13}/m)\cos(\chi-\psi)\cos\chi-2p_{12}/m,\end{aligned} \tag{6.6}$$

and where δ is given as a function of χ by

$$\cos\delta=C_y/C;\quad \tan\delta=\frac{p_{13}\cos(\chi-\psi)\sin\chi}{p_{13}\cos(\chi-\psi)\cos\chi-p_{12}}. \tag{6.7}$$

The integral (6.5) can be simplified greatly by using a transformation first introduced by Sengers.[2] We change variables from ϕ'' to τ, where physically τ is essentially the time between the first and second collisions (see Fig. 4). Explicitly, we use

$$[2p_{12}\tau/(ma)]\tan\delta=\sin\phi''/\cos\delta+\sin\phi. \tag{6.8}$$

If we rewrite (6.8) as

$$\sin\phi+\sin(\phi''-\delta)=\tan\delta[2p_{12}\tau/(ma)-\cos(\phi''-\delta)] \tag{6.9}$$

and assemble our results, we find

$$\begin{aligned}H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi)=&-4p_{12}p_{13}\left(\frac{a}{m}\right)^2\int_{\psi-\pi/2}^{\psi+\pi/2}d\chi\cos(\chi-\psi)\int_{\tau_-}^{\tau_+}d\tau\\&\times\operatorname{sgn}(\sin\delta)\theta(2p_{12}\tau/(ma)+\cos\phi)\theta([2p_{12}\tau/(ma)-\cos(\phi''-\delta)][p_{13}\cos(\chi-\psi)\cos\chi-p_{12}])\\&\times\exp\left\{-\frac{\epsilon ma}{2p_{12}}\frac{p_{13}\cos(\chi-\psi)\cos\chi[\cos\phi+2p_{12}\tau/(ma)]-p_{12}[\cos\phi+\cos(\phi''-\delta)]}{p_{13}\cos(\chi-\psi)\cos\chi-p_{12}}\right\}\Delta J,\end{aligned} \tag{6.10}$$

where

$$2p_{12}(\tan\delta)\tau_{\pm}/(ma)=\pm1/\cos\delta+\sin\phi \tag{6.11}$$

and $\operatorname{sgn}x$ is the sign of x.

If ϵ is set equal to zero in (6.10), one recovers the divergent phase-space integral of Cohen and Dorfman.[1] However, for $\epsilon>0$ the integral exists, and furthermore it can be verified that its value is independent of the order of the τ and χ integration. Since the evaluation of (6.10) is considerably simpler when the χ integral is performed first, we observe that

$$\begin{aligned}\int_{\psi-\pi/2}^{\psi+\pi/2}d\chi\int_{\tau_-}^{\tau_+}d\tau\operatorname{sgn}(\sin\delta)&=\int d\tau\int d\chi\,\theta(\pi/2-|\chi-\psi|)\\&\times\theta(1-|2p_{12}\tau\sin\delta/(ma)-\cos\delta\sin\phi|).\end{aligned} \tag{6.12}$$

Now the first θ function in (6.10) assures us that τ is bounded from below. Therefore, the τ integral goes from some finite lower limit τ_l to ∞. Furthermore, we see that the coefficient of ϵ in the exponential of (6.10) is bounded provided τ is finite. If we fix T such that

$$\tau_l\ll T<\infty;\quad 2p_{12}T/(ma)\gg1 \tag{6.13}$$

and consider the region

$$\tau_l\le\tau\le T \tag{6.14}$$

then the τ integral in the region (6.14) is bounded by T and exists for $\epsilon=0$. Therefore, any divergence as $\epsilon\to0$ must come from $\tau>T$, and hence we find for the (divergent) part of H_N as $\epsilon\to0$:

$$\begin{aligned}H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi)=&-4p_{12}p_{13}\\&\times\left(\frac{a}{m}\right)^2\int d\tau\int d\chi\cos(\chi-\psi)\,\theta(\tau-T)\\&\times\theta(\pi/2-|\chi-\psi|)\,\theta(p_{13}\cos(\chi-\psi)\cos\chi-p_{12})\\&\times\theta(1-|2p_{12}\tau\sin\delta/(ma)-\cos\delta\sin\phi|)\\&\times\exp\left\{-\epsilon\tau\frac{p_{13}\cos(\chi-\psi)\cos\chi}{p_{13}\cos(\chi-\psi)\cos\chi-p_{12}}\right\}\Delta J.\end{aligned} \tag{6.15}$$

Now the first, third, and fourth θ functions imply that δ is small, while the third θ function and (6.7) imply that χ is small. We then find

$$\begin{aligned}H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi)=&-4p_{12}p_{13}\cos\psi\left(\frac{a}{m}\right)^2\\&\times\theta(p_{13}\cos\psi-p_{12})\int_T^\infty d\tau\,e^{-\epsilon\tau/\nu}\int_{\chi_-}^{\chi_+}d\chi\Delta J,\end{aligned} \tag{6.16}$$

where

$$\nu=(p_{13}\cos\psi-p_{12})/p_{13}\cos\psi;$$
$$\chi_{\pm}=(\nu ma/2p_{12}\tau)(\pm1+\sin\phi). \tag{6.17}$$

Making a final change of variable from χ to α via

$$\chi=(\alpha+\sin\phi)\nu ma/(2p_{12}\tau), \tag{6.18}$$

we find

$$\begin{aligned}H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi)=&(-2a^3/m)(p_{13}\cos\psi-p_{12})\\&\times\theta(p_{13}\cos\psi-p_{12})\int_T^\infty\frac{d\tau}{\tau}e^{-\epsilon\tau/\nu}\int_{-1}^1d\alpha\Delta J.\end{aligned} \tag{6.19}$$

Since

$$\Delta J=J(\mathbf{p}_{12}''')-J(\mathbf{p}_{12}''), \tag{6.20}$$

we must write out $\mathbf{p}_{12}'''$ and $\mathbf{p}_{12}''$ in terms of the variables of integration of (6.19). We find

$$\begin{aligned}
p_{12x}''' &= 2(p_{13}\cos\psi - p_{12})\alpha(1-\alpha^2)^{\frac{1}{2}};\\
p_{12y}''' &= 2(p_{13}\cos\psi - p_{12})(\alpha^2 - \tfrac{1}{2});\\
p_{12x}'' &\sim ma/(p_{12}\tau) \to 0;\\
p_{12y}'' &= p_{13}\cos\psi - p_{12}
\end{aligned} \tag{6.21}$$

where we have computed (6.21) in the approximation needed for (6.19).

We now observe that:

(1) $H_N = 0$ unless $p_{13}\cos\psi > p_{12}$ (or unless the second collision is such that particle one can catch up to particle two to make the last collision).

(2) The variables of integration in (6.19) imply that contributions to H_N come from (a) all initial impact variables $-\pi/2 \leq \phi \leq \pi/2$, (b) long times τ, (c) values of α (or better χ) such that the last collision occurs for large τ.

(3) Equations (6.21) are independent of τ. Thus, we may do the τ integral to find

$$\int_T^\infty \frac{d\tau}{\tau} e^{-\epsilon\tau/\nu} = \int_{\epsilon T/\nu}^\infty \frac{d\zeta}{\zeta} e^{-\zeta} \equiv -\mathrm{Ei}\left(-\frac{\epsilon T}{\nu}\right), \tag{6.22}$$

and as $\epsilon \to 0$,

$$-\mathrm{Ei}(-\epsilon T/\nu) \to \ln(\nu/\epsilon T) \to \ln(1/\epsilon T). \tag{6.23}$$

Summarizing our results, we have

$$H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi) = -\frac{2a^3}{m}[\ln(1/\epsilon T)](\mathbf{p}_{13}\cdot\hat{p}_{12} - p_{12})$$
$$\times \theta(\mathbf{p}_{13}\cdot\hat{p}_{12} - p_{12}) \int_{-1}^{1} d\alpha [J(\mathbf{p}_{12}''') - J(\mathbf{p}_{12}'')], \tag{6.24}$$

where

$$\begin{aligned}
\mathbf{p}_{12}''' &= -2(\mathbf{p}_{13}\cdot\hat{p}_{12} - p_{12}) \times [(\alpha^2 - \tfrac{1}{2})\hat{p}_{12} + \alpha(1-\alpha^2)^{\frac{1}{2}}\hat{p}_{12\perp}],\\
\mathbf{p}_{12}'' &= -(\mathbf{p}_{13}\cdot\hat{p}_{12} - p_{12})\hat{p}_{12}.
\end{aligned} \tag{6.25}$$

Thus we have shown that H_N diverges as $\ln\epsilon$ for any function $J(\mathbf{p}_{12})$ if and only if the collision sequence (1,2) (1,3) (1,2) occurs physically.

A quite similar argument can be made for the $\mathbf{k}$ integral of $II(IN)$. Defining H_I by

$$\int \frac{d\mathbf{k}}{(2\pi)^2} II(IN) = \int_{-\pi/2}^{\pi/2} d\phi \cos\phi H_I(\mathbf{p}_{12}',\mathbf{p}_{13}';\phi), \tag{6.26}$$

one finds for the divergent part of H_I as $\epsilon \to 0$:

$$H_I(\mathbf{p}_{12}',\mathbf{p}_{13}';\phi) = -H_N(\mathbf{p}_{12}',\mathbf{p}_{13}';\phi). \tag{6.27}$$

Finally, we remark that if H_N and H_I are computed in *three* dimensions for a gas of hard spheres, there is no divergence since the ϵ dependence is given by $\epsilon \ln\epsilon$ as $\epsilon \to 0$.

7. RECOLLISION CONTRIBUTION TO THE VISCOSITY

To obtain the recollision contribution to the viscosity we recall the first equation of (3.9). Using the value of η_0 for hard disks as computed by Sengers[2]

$$\eta_0 = (1/2a)(mkT/\pi)^{\frac{1}{2}}, \tag{7.1}$$

and writing

$$\sigma_R \equiv \sigma_R{}^I + \sigma_R{}^N, \tag{7.2}$$

we find

$$\sigma_R{}^N = \frac{n}{(2\pi)^{\frac{1}{2}}}\beta^{5/2}\frac{m}{a}\prod_{i=1}^{3}\int d\mathbf{p}_i\Phi(p_i)\int_{-\pi/2}^{\pi/2} d\phi \cos\phi [4\mathbf{p}_{12}\mathbf{p}_{12} + \mathbf{P}_{12}\mathbf{P}_{12}] : H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi), \tag{7.3}$$

$$\sigma_R{}^I = \frac{n}{(2\pi)^{\frac{1}{2}}}\beta^{5/2}\frac{m}{a}\int d\mathbf{p}_1'\Phi(p_1')\int d\mathbf{p}_2'\Phi(p_2')\int d\mathbf{p}_3\Phi(p_3\int_{-\pi/2}^{\pi/2} d\phi \cos\phi [4\mathbf{p}_{12}\mathbf{p}_{12} + \mathbf{P}_{12}'\mathbf{P}_{12}'] : H_I(\mathbf{p}_{12}',\mathbf{p}_{13}';\phi). \tag{7.4}$$

In writing (7.4) we have transformed to the momentum variables after the first collision and used

$$\partial(\mathbf{p}_1,\mathbf{p}_2)/\partial(\mathbf{p}_1',\mathbf{p}_2') = 1; \quad \Phi(p_1)\Phi(p_2) = \Phi(p_1')\Phi(p_2'); \quad \mathbf{P}_{12} = \mathbf{P}_{12}'. \tag{7.5}$$

We now express $\mathbf{p}_{12}$ in (7.4) as a function of $\mathbf{p}_{12}'$ by

$$\mathbf{p}_{12} = -\cos2\phi\,\mathbf{p}_{12}' - \sin2\phi\,\mathbf{p}_{12\perp}'. \tag{7.6}$$

If we now use (6.27) and relabel the dummy variables $\mathbf{p}_1'$ and $\mathbf{p}_2'$ in (7.4) as $\mathbf{p}_1$ and $\mathbf{p}_2$, we find

$$\sigma_R = \frac{4n}{(2\pi)^{\frac{1}{2}}}\beta^{5/2}\frac{m}{a}\prod_{i=1}^{3}\int d\mathbf{p}_i\Phi(p_i) F(\mathbf{p}_{12},\mathbf{p}_{13}), \tag{7.7}$$

where

$$F(\mathbf{p}_{12},\mathbf{p}_{13}) = \int_{-\pi/2}^{\pi/2} d\phi \cos\phi [\sin^2 2\phi\,(\mathbf{p}_{12}\mathbf{p}_{12} - \mathbf{p}_{12\perp}\mathbf{p}_{12\perp}) - \cos2\phi \sin2\phi\,(\mathbf{p}_{12}\mathbf{p}_{12\perp} + \mathbf{p}_{12\perp}\mathbf{p}_{12})] : H_N(\mathbf{p}_{12},\mathbf{p}_{13};\phi). \tag{7.8}$$

Using (6.24) for H_N with

$$J(\mathbf{p}_{12})=\mathbf{p}_{12}\mathbf{p}_{12}, \tag{7.9}$$

we find

$$F(\mathbf{p}_{12},\mathbf{p}_{13})=-(8a^3/m)p_{12}^2(\mathbf{p}_{13}\cdot\hat{p}_{12}-p_{12})^3\theta(\mathbf{p}_{13}\cdot\hat{p}_{12}-p_{12})$$
$$\times\ln\frac{1}{\epsilon T}\int_{-\pi/2}^{\pi/2}d\phi\,\cos\phi[\sin^2 2\phi(\hat{p}_{12}\hat{p}_{12}-\hat{p}_{12\perp}\hat{p}_{12\perp})-\cos 2\phi\,\sin 2\phi(\hat{p}_{12}\hat{p}_{12\perp}+\hat{p}_{12\perp}\hat{p}_{12})]:$$
$$\int_{-1}^{1}d\alpha[(\alpha^4-\alpha^2)(\hat{p}_{12}\hat{p}_{12}-\hat{p}_{12\perp}\hat{p}_{12\perp})+\alpha(1-\alpha^2)^{\frac{1}{2}}(\alpha^2-\tfrac{1}{2})(\hat{p}_{12}\hat{p}_{12\perp}+\hat{p}_{12\perp}\hat{p}_{12})]. \tag{7.10}$$

The $\hat{p}_{12}\hat{p}_{12\perp}+\hat{p}_{12\perp}\hat{p}_{12}$ terms in (7.10) are odd in α or in ϕ and give no contribution. We find

$$F(\mathbf{p}_{12},\mathbf{p}_{13})=(32/15)^2(a^3/m)p_{12}^2(\mathbf{p}_{13}\cdot\hat{p}_{12}-p_{12})^3\theta(\mathbf{p}_{13}\cdot\hat{p}_{12}-p_{12})\ln(\epsilon T)^{-1}. \tag{7.11}$$

We now change the integration variables in (7.7) to $\mathbf{p}_{12},\mathbf{p}_{13}$, and $\mathbf{P}$, where

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3;\quad \partial(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)/\partial(\mathbf{p}_{12},\mathbf{p}_{13},\mathbf{P})=16/9. \tag{7.12}$$

Then

$$\sigma_R=\frac{64}{9}\frac{n\beta^{5/2}}{(2\pi)^{\frac{1}{2}}}\frac{m}{a}\left(\frac{\beta}{\pi}\right)^3\int d\mathbf{P}\,\exp\left(-\frac{\beta}{3}P^2\right)\int d\mathbf{p}_{12}\int d\mathbf{p}_{13}\,\exp\left[-\frac{8\beta}{3}(p_{12}^2+p_{13}^2-\mathbf{p}_{12}\cdot\mathbf{p}_{13})\right]F(\mathbf{p}_{12},\mathbf{p}_{13}) \tag{7.13}$$

and using (7.11):

$$\sigma_R=\frac{(32)^3}{(15)^2}\frac{na^2\beta^4}{\sqrt{3}\pi}\ln\frac{1}{\epsilon T}\int_0^\infty dp_{12}p_{12}^3\int_{p_{12}}^\infty dp_{13y}(p_{13y}-p_{12})^3\exp[-(8\beta/3)(p_{12}^2+p_{13y}^2-p_{12}p_{13y})], \tag{7.14}$$

where we have chosen $\mathbf{p}_{12}$ as the y axis for the $\mathbf{p}_{13}$ integration and the $\mathbf{p}_{12}$ integral is done in polar coordinates. By a change of variable, we find

$$\sigma_R=[24\sqrt{3}/(25\pi)]Kna^2\ln(\epsilon T)^{-1}, \tag{7.15}$$

where

$$K=\int_0^\infty dx\int_0^\infty dy(xy)^3\exp[-(x^2+y^2+xy)]. \tag{7.16}$$

The integral K can be evaluated in polar coordinates. After the radial integration, K is

$$K=6\int_0^{\pi/2}d\theta\frac{\sin^3\theta}{(2+\sin\theta)^4}. \tag{7.17}$$

The substitution

$$x=(\sin\theta)/(2+\sin\theta) \tag{7.18}$$

brings the integral to a standard form. We find

$$K=1-14\pi/(27\sqrt{3}), \tag{7.19}$$

so that

$$\sigma_R\equiv\left.\frac{\eta_1}{\eta_0}\right|_R=\frac{24\sqrt{3}}{25\pi}\left(1-\frac{14}{27}\frac{\pi}{\sqrt{3}}\right)na^2\ln\frac{1}{\epsilon T} \tag{7.20}$$

in complete agreement with the result obtained by Sengers.[2]

8. SUMMARY

The binary-collision expansion expression for the viscosity of a two-dimensional gas of hard disks has been discussed in detail. We have found a divergence in η_1, the first correction to the Boltzmann-equation result. The physical origin of the divergence is precisely the same as that found by Dorfman and Cohen from phase-space estimates and by Sengers from an exact calculation of η_1 using the Choh-Uhlenbeck formalism. Furthermore, we have demonstrated the precise numerical agreement of the coefficient of the divergence, as obtained by us from the binary-collision expansion, with that obtained by Sengers from the Bogoliubov theory. It must be concluded, therefore, that a divergence does indeed appear in the density expansion of transport coefficients.

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