Spontaneous Symmetry Breakdown without Mass Splitting

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The question whether spontaneous symmetry breakdown can cause n-p and μ-e mass differences is examined, employing a nonperturbative approximation scheme to solve the Dyson-Schwinger equation for the one-particle Green's function. In both cases, mass degeneracy persists in the presence of a nontrivial symmetry-breaking solution. This result is discussed with special reference to a recent theorem of Streater.

1. INTRODUCTION

IN recent years, the idea of spontaneous breakdown of symmetry has been exploited to account for mass differences between elementary particles. On the basis of perturbation theory, Baker and Glashow[1] attempted to explain the muon-electron mass difference, starting from zero bare masses for these particles and Coleman and Glashow[2] succeeded in obtaining the correct sign of K^+-K^0 and n-p mass difference by a phenomenological inclusion of tadpole diagrams in addition to the usual nontadpole contributions. Arnowitt and Deser[3] carried out a nonperturbative analysis of the muon-electron problem, once again starting from a theory admitting an asymmetric vacuum and employing the high-energy approximation due to Baker, Johnson, and Willey[4] and arrived at the conclusion that the muon-electron mass difference could develop spontaneously, while maintaining the electromagnetic stability of the muon.

Recently, Streater has proved[5] a generalization of the Goldstone theorem which appears to shatter the hopes of all such attempts to explain the mass splitting between elementary particles. Streater's result is the following: Any mass difference between particles, one of which is stable, must be caused by a symmetry breaking which is not of the spontaneous type, lest there should occur in the theory bosons of arbitrarily small mass. This result leans heavily on a formal proof which is based on a study of the matrix elements of the product of field operators. We have, therefore, thought it worthwhile to make a detailed examination of the question of mass splitting due to spontaneous breakdown, starting from the Dyson-Schwinger equation for the one-particle Green's function and employing a nontrivial, nonperturbative approximation method due to Falk.[6]

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[1] M. Baker and S. L. Glashow, Phys. Rev. **128**, 2462 (1962); see also S. L. Glashow, *ibid.* **130**, 2132 (1963).

[2] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964); S. Coleman and H. J. Schnitzer, *ibid.* **136**, B223 (1964).

[3] R. Arnowitt and S. Deser, Phys. Rev. **138**, B712 (1965); Th. Maris, V. Herscovitz, and G. Jacob, Nuovo Cimento **34**, 946 (1964).

[4] K. Johnson, M. Baker, and R. Wiley, Phys. Rev. **136**, B1111 (1964). Th. Maris, V. Herscovitz, and G. Jacob, Phys. Rev. Letters **12**, 313 (1964).

[5] R. F. Streater, Phys. Rev. Letters **15**, 475 (1965).

[6] D. S. Falk, Phys. Rev. **115**, 1069 (1959).

Falk's method involves the symmetrical treatment of the Green's function and ultimately, the assumption that certain particles behave in an uncorrelated manner. This approximation method has the merit that the approximate Green's function automatically has the same analytical properties as the exact one and results in a linear integral equation for the renormalized Green's function, which can be solved exactly through the use of a spectral representation.

Both the neutron-proton and the muon-electron problems have been examined in this context. Within the framework of the aforementioned approximation, we have arrived at the following conclusion: No mass difference can develop in either case although a broken symmetry solution exists in both cases. Our conclusion reinforces the result of Streater as far as the neutron-proton problem is concerned. For the muon-electron problem, it is in apparent contradiction with the conclusion of Arnowitt and Deser.

2. FALK'S APPROXIMATION TO THE DYSON-SCHWINGER EQUATION

Falk's approximation to the Dyson-Schwinger equation for the renormalized one-nucleon Green's function is

$$G(p)=G_0(p)+G_0(p)F_r'(p)G_0(p);\quad F_r'=F_r/Z_2,\qquad(1)$$

$$F_r(p)=F(p)-[F(p)]_{\gamma p+m=0}-(\gamma p+m)\left[\frac{\partial}{\partial(\gamma p)}F(p)\right]_{\gamma p+m=0},\qquad(2)$$

where

$$F(p)=-\frac{3ig^2}{(2\pi)^4}\int d^4k\,\gamma_5G(p-k)\gamma_5\frac{1}{k^2+\mu^2},\qquad(3)$$

$$G_0(p)=1/(\gamma p+m),\qquad(4)$$

and $g^2/4\pi$ is the renormalized pion-nucleon coupling constant. Using the spectral representation

$$G(p)=\frac{1}{\gamma p+m}+\int_{m+\mu}^{\infty}\frac{\alpha(\kappa)}{\gamma p+\kappa}d\kappa+\int_{-\infty}^{-m-\mu}\frac{\alpha(\kappa)}{\gamma p+\kappa}d\kappa,\qquad(5)$$

$$\alpha(\kappa)=(\kappa/|\kappa|)[1/\pi(\kappa-m)^2][\mathrm{Im}F_r']_{\gamma p=-\kappa},\qquad(5\mathrm{a})$$

Falk obtains the following integral equation for the spectral function $\alpha(\kappa)$:

$$\alpha(\kappa)=f(\kappa,m)+\int_{-|\kappa|+\mu}^{|\kappa|-\mu} d\kappa' f(\kappa,\kappa')\alpha(\kappa'), \tag{6}$$

where

$$f(\kappa,\kappa')=-\frac{\kappa}{|\kappa|}\frac{3g^2}{16\pi^2}\frac{1}{(\kappa-m)^2}\left[\kappa'-\frac{\kappa}{2}\left(1+\frac{\kappa'^2-\mu^2}{\kappa^2}\right)\right]$$
$$\times[1-2(\kappa'^2+\mu^2)/\kappa^2+(\kappa'^2-\mu^2)^2/\kappa^4] \tag{7}$$

and $f(\kappa,\kappa')\geq 0$ for all κ and κ' with $|\kappa|>|\kappa'|+\mu$. The basic approximation employed in deriving Eq. (6) is the so-called "noncorrelation" assumption. To wit, three- and four-point correlations are neglected; for instance,

$$\begin{aligned}&i\langle T(\phi^i(x)\psi(x)\bar{\psi}(x')\phi^j(x'))\rangle\\ &\quad\simeq -i[i\langle T(\psi(x)\bar{\psi}(x'))\rangle][i\langle T(\phi^i(x)\phi^j(x'))\rangle]\\ &\quad= -iG(x,x')\mathcal{G}_{ij}(x,x'),\end{aligned} \tag{8}$$

where G and $\mathcal{G}$ are the nucleon and meson Green's functions.

In the asymptotic region $|\kappa|>m$, $\alpha(\kappa)$ is an even function of κ and satisfies the differential equation

$$\kappa^2\frac{d^2\alpha}{d\kappa^2}+7\kappa\frac{d\alpha}{d\kappa}+(5-4\beta)\alpha=0, \tag{9}$$

whose solution has the following leading term in κ:

$$\alpha(\kappa)\sim\frac{\beta[1+(1+\beta)^{1/2}]}{[4(1+\beta)^{1/2}]}\frac{1}{m}\left(\frac{|\kappa|}{m}\right)^{-3+2(1+\beta)^{1/2}}, \tag{10}$$

where

$$\beta=3g^2/16\pi^2.$$

3. n-p PROBLEM

We will now employ these results to investigate whether or not the n-p mass difference can develop from a spontaneous breakdown of isotopic-spin symmetry. Let

$$\psi=\begin{pmatrix}\psi_p\\ \psi_n\end{pmatrix},\quad G=\begin{pmatrix}G_p\\ G_n\end{pmatrix} \tag{11}$$

and let us break the symmetry along the "3" direction in isotopic-spin space by introducing a vacuum which does not share in the isotopic-spin invariance of the Lagrangian. Thus,

$$G_{ij}=G_1\delta_{ij}+(\tau_3)_{ij}G_2, \tag{12}$$

where τ_3 is the Pauli matrix

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

so that[7]

$$G_p=G_1+G_2,\quad G_n=G_1-G_2. \tag{13}$$

Falk's procedure can now be carried through on the 2×2 diagonal matrix G. This results in the two equations

$$\begin{aligned}G_1(p)&=(\gamma p+m)^{-1}+(\gamma p+m)^{-1}F_r^{(1)}(p)(\gamma p+m)^{-1},\\ G_2(p)&=(\gamma p+m)^{-1}F_r^{(2)}(p)(\gamma p+m)^{-1},\end{aligned} \tag{14}$$

where

$$m_p=m_n=m,$$

$$F^{(1)}(p)=-\frac{3ig^2}{(2\pi)^4}\int d^4k\,\gamma_5 G_1(p-k)\gamma_5\frac{1}{k^2+\mu^2}, \tag{15}$$

$$F^{(2)}(p)=+\frac{ig^2}{(2\pi)^4}\int d^4k\gamma_5 G_2(p-k)\gamma_5\frac{1}{k^2+\mu^2}, \tag{15a}$$

and $F_r^{(i)}(p)$ is the convergent part of $F^i(p)$ as given by Eq. (2).

One notices immediately that the pole occurs only in $G_1(p)$ by virtue of the linearity of the integral equation for $G(p)$, and, $G_2(p)$ thus contributes only to the continuum. Following Falk, the spectral function corresponding to $G_2(p)$ can be obtained *for all* κ.[8] Hence we have a symmetry breaking solution, $G_2(p)\neq 0$, but without generating any n-p mass difference.

Within the framework of Falk's approximation to the Dyson-Schwinger equation, we are thus led to conclude that spontaneous breakdown of isotopic spin symmetry does not cause n-p mass difference, although a nontrivial symmetry-breaking solution exists.

4. μ-e PROBLEM

In this case, the procedure is similar to that of the n-p problem. Here, one works in the Coulomb gauge[9] so that the spectral function of the electron (muon) propagator is positive definite and further assumes that the μ-e mass difference is *not* of electromagnetic origin,[10] so that the renormalized electron and muon masses are identical in the absence of a symmetry breakdown. Here, the symmetry is that of a formal $SU(2)$ invariance and it operates in the two-dimensional μ-e vector space.[11] The spontaneous breakdown of this formal $SU(2)$ symmetry is achieved by requiring only τ_3 to commute with the

[7] We have chosen a representation in which G_{ij} is diagonal. This is consistent with the absence of the decay $n\to p+\pi^-$ via strong interactions, as will be borne out by the conclusion that the mass degeneracy is not lifted.

[8] One could demonstrate this by first computing $\alpha_2(\kappa)$ in the strip $m+(n-1)\mu\leq|\kappa|\leq m+n\mu$ ($n=2,3,4,\cdots$) and then performing an analytical continuation in κ.

[9] D. S. Falk, thesis, Harvard University, 1959 (unpublished), Chap. IV.

[10] R. Haag and Th. Maris, Phys. Rev. **132**, 2325 (1963); see discussion in Sec. 5 of Johnson, Baker, and Wiley (Ref. 4).

[11] See Arnowitt and Deser, Ref. 3.

μ-e Green's function G, which is now a 2×2 matrix, thus implying

$$G_{ij}=G_1\delta_{ij}+(\tau_3)_{ij}G_2. \tag{16}$$

Following the procedure employed earlier, we find that the spectral functions $\alpha^e(\kappa)$, $\alpha^\mu(\kappa)$ can be computed in the asymptotic region, $m/|\kappa|\ll 1$ and have the form[12]

$$\alpha^e(\kappa)=\alpha^\mu(\kappa)\sim\frac{\beta}{2m}\frac{[1+(1+2\beta)^{1/2}]}{1(+2\beta)^{1/2}}\left(\frac{|\kappa|}{m}\right)^{-3+2(1+2\beta)^{1/2}}, \tag{17}$$

where

$$\beta=e^2/16\pi^2,\quad m_e=m_\mu=m.$$

Once again, we arrive at the conclusion that a nontrivial broken-symmetry solution exists with $G_2\neq 0$ without, however, causing any mass difference between the muon and the electron. The apparent discrepancy between our conclusion and that of Arnowitt and Deser is discussed in Sec. 5.

5. DISCUSSION AND CONCLUSION

Within the framework of the nonperturbative approximation due to Falk, we have demonstrated that spontaneous breakdown of $SU(2)$ symmetry (isospin) does not lead to a mass difference between proton and neutron. Our result is in agreement with the theorem proved by Streater and thus it would appear that the experimentally observed n-p mass difference should arise from a symmetry violation of a type which is different from that of spontaneous breakdown. Since it is known that the contributions of purely electromagnetic origin cannot account for the mass difference,[13] the n-p problem is still "there."

Coming to the problem of the muon-electron mass difference, once again, we find that spontaneous breakdown cannot account for the observed mass difference within the framework of the same approximation[14] (assuming, of course, that the mass difference is not of electromagnetic origin). Our result, then, apparently contradicts the conclusion of Arnowitt and Deser. In the notation of these authors, the inverse Green's function has the form

$$G_{ij}^{-1}(p)=\gamma p h_{ij}(x)+k_{ij}(x);\quad x\equiv p^2, \tag{18}$$

with $h(x)=1$. Further, $k(x)$ satisfies the differential equation

$$(xk)''+3\lambda k(x+k^2)^{-1}=0;\quad \lambda=e_0^2/16\pi^2 \tag{19}$$

which has the following general solution near the origin:

$$k(x)=a_{-1}x^{-1}+a_0+a_1x+\cdots, \tag{20}$$

where all a_m, $m\geqslant 1$ can be determined by recursion relations in terms of a_{-1} and a_0. The condition that $k(x)$ be regular at the origin, forces them to choose $a_{-1}=0$. By virtue of the Hermiticity of k_{ij}, they are able to diagonalize a_0 and thus k_{ij} for all momenta. When a_0 is put in diagonal form, $G_{ij}(p)$ depends upon two arbitrary constants which are the diagonal elements of a_0. They conclude that these may be determined by requiring the two poles of $G_{ij}(p)$ to occur at the experimental masses of the muon and electron.

Crucial to their argument is the vanishing of a_{-1} which is dictated by the boundary condition at the origin. One notices, however, that a_{-1} must, in general, depend upon e_0^2 in order that the solution $k(x)$ be nontrivial. *One, therefore, does not have the freedom of setting $a_{-1}=0$ for arbitrary values of e_0^2.* One would rather expect a_{-1} to vanish only for a specific value of e_0^2 and only for this value can one hope to predict the observed μ-e mass difference by spontaneous breakdown. By the very nature of the approximation method, however, this value of e_0^2 cannot be computed and, therefore, one does not know whether it will lie between 0 and $4\pi^2/3$. The latter restriction is dictated by the requirement that the asymptotic solution for $G(p)$ be a physical solution for time-like p^2.[15] Now, Eq. (19) has been obtained in the high-energy approximation. In order that $G(p)$ may admit of poles at the electron and muon masses, one is led to consider the solution of Eq. (19) near the origin, assuming of course the validity of this procedure.[16] It is therefore necessary to show that the restriction on e_0^2 arising from the condition of regularity of the solution near the origin is consistent with the range of e_0^2 allowed by the condition on the asymptotic solution before one can conclude that the μ-e mass difference can develop spontaneously. In our opinion, therefore, the μ-e problem remains "open."

Finally a word about the possible dependence of the result obtained in this papaer on the approximation method employed. One might, perhaps, argue that were it not for the linearity of the integral equation satisfied by G, and the consequent separation of the pole term so that the latter is exclusively carried by G_1, one might not have arrived at the result of mass degeneracy in the presence of a nontrivial symmetry-breaking solution. In view of Streater's theorem, however, we believe that this possibility is, in fact, ruled out.

[12] See Ref. 9 for details.

[13] M. Cini, E. Ferrari, and R. Gatto, Phys. Rev. Letters **2**, 7 (1959); see, however, R. Dashen, Phys. Rev. **135**, B1196 (1964).

[14] This result for the μ-e problem is not to be taken as a confirmation of Streater's result, for the following reason. Streater's proof depends on having a positive definite metric in the theory and local commutation relations, and, in the Coulomb gauge, we have a positive definite metric but causality is violated. See, R. F. Streater, in *Seminar on High Energy Physics and Elementary Particles, Trieste, 1965* (International Atomic Energy Agency, Trieste, 1965).

[15] See discussion following Eq. (5.22) in Y. Frishman, Phys. Rev. **138**, B1450 (1965).

[16] See, however, Y. Frishman and A. Katz, Phys. Letters **11**, 172 (1964).

Note added in proof. After the completion of our work, we have been informed by Streater that rigorous proofs of his theorem may be found in the literature [R. F. Streater, J. Math. Phys. **5**, 581 (1964); Proc. Roy. Soc. (London) **A287**, 510 (1965)]. Dr. Streater also informs us that his theorem has been generalized to apply to any continuous groups G of automorphisms with a local generator commuting with space-time translations [Robinson *et al.*, Comm. Math. Phys. (to be published)]. We are indebted to Dr. Streater and also to Dr. Arnowitt and Dr. Deser for extremely valuable communications.

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High-Energy Limit of $\pi^+\pi^0$ Total Cross Section*

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The high-energy limit of the $\pi^+\pi^0$ total cross section $\sigma(\infty)$ is estimated upon the assumption that the $\pi^+\pi^0$ forward scattering amplitude is dominated by the S wave below the ρ resonance and becomes purely imaginary fairly rapidly above the ρ resonance. The underlying assumption is that the $\pi^+\pi^0$ forward amplitude satisfies the usual analyticity assumption. Using the experimental values of the mass and the width of the ρ resonance and also assuming an effective-range expansion for the S wave, we estimate $\sigma(\infty)$ as a function of the scattering length and the effective range in the channel of total isospin 2. It is found that, for negative scattering length (corresponding to a repulsive interaction), $\sigma(\infty)$ rises fairly steeply as the scattering length increases its magnitude for all the effective ranges. In the case of positive scattering length (corresponding to an attractive interaction), however, $\sigma(\infty)$ is much less dependent on the scattering length up to $0.8\mu^{-1}$ especially when the effective range lies between μ^{-1} and $2\mu^{-1}$ (μ^{-1} being the pion Compton wavelength). In particular, $\sigma(\infty)$ hardly decreases below 20 mb as long as the effective range is not too much greater than $2\mu^{-1}$. It is argued that various corrections to be considered can quite likely increase the above estimate of $\sigma(\infty)$ 10% to 20% in the case of a weakly attractive S-wave interaction, but hardly can decrease it. It is therefore felt that the present analysis excludes the possibility of a repulsive S-wave interaction and also implies that the high-energy limit of the $\pi^+\pi^0$ total cross section is about the same as that for the pion-nucleon total cross section.

I. INTRODUCTION

IN the dispersion theory there exists a correlation between the low- and high-energy behavior of the scattering amplitude, since analytic continuation correlates uniquely the behavior of the scattering amplitude in various energy regions. This correlation can be expressed in a simple manner in the case of the forward scattering amplitude, mainly because of the positive definiteness of its imaginary part. If there is a low-energy resonance which appears as a pronounced peak in the imaginary part of the forward amplitude and moreover if the forward amplitude becomes pure imaginary sufficiently rapidly in the high-energy limit, the above correlation can be expressed as follows[1]: The high-energy limit of the total cross section for scattering in question can only be a nonzero, finite cross section as long as the low-energy resonance is pronounced, say, as much as the **33** resonance is in the pion-nucleon system.

If one assumes furthermore that the forward amplitude becomes pure imaginary fairly rapidly above the low-energy resonance, one can estimate the high-energy limit of the total cross section in terms of the low-energy parameters including the mass and the width of the resonance. The estimate of this type was carried out[2] in the case of the pion-nucleon system. The estimated value[2] of the high-energy limit of the total pion-nucleon cross section is in good agreement with what the available data of the high-energy total cross section indicate.[3]

The main purpose of the present work is to carry out a similar estimate of the high-energy limit of the pion-pion total cross section by assuming that the ρ resonance ($M_\rho=760$ MeV, $\Gamma_\rho=106$ MeV, $I=1$, $J=1$) is the pronounced low-energy resonance in the above sense. Since it is most convenient[1,2] to work with a crossing-symmetric forward amplitude, the present work deals with the $\pi^+\pi^0$ forward amplitude, thus estimating the high-energy limit of the $\pi^+\pi^0$ total cross section. The $\pi^0\pi^0$ forward amplitude is another crossing symmetric amplitude. This amplitude, however, contains pion-

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[1] M. Sugawara, Phys. Rev. Letters **14**, 336 (1965). It is shown in this reference that this correlation is of considerable physical interest even without the assumption that the forward amplitude becomes pure imaginary in the high-energy limit.

[2] M. Sugawara and A. Tubis, Phys. Rev. **138**, B242 (1965).

[3] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and Matts Roos, Rev. Mod. Phys. **36**, 977 (1964).