Proton-Neutron Mass Difference

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Incorporating the methods of the preceding paper we examine the electromagnetic contributions to the masses of the nucleons in terms of the graviton-nucleon vertex. Using the methods of sidewise dispersion relations, we consider this vertex as a function of the mass of one of the external nucleons, which enables us to express the contributions to $\delta M = M_p - M_n$ in terms of integrals over scattering amplitudes. Applying the hypothesis of threshold dominance, we find that a simple explanation for the observed mass difference $\delta M = -1.3$ MeV emerges. In the low-energy region to which we restrict our attention, the only contributing states are the intermediate $N\gamma$ and $N\pi$ states. For the photon contributions we find the usual unsuccessful result $\delta M^{\gamma} \sim +0.5$ MeV as a consequence of the dominance of the Coulomb over the magnetic energy in the threshold region. However, in calculating the contribution of the nucleon mass shift back on itself from the $N\pi$ states of energy W, there is a term under the dispersion integral proportional to the difference of nucleon pole terms $(W^2-M_n)^{-1}-(W^2-M_p^2)^{-1} \simeq -2M\delta M\,(W^2-M^2)^{-2}$, implying a large contribution due to the enhancement at threshold $W^2 \to M^2$. Including this contribution we have $\delta M = +0.5$ MeV$+1.3\delta M$ or $\delta M \simeq -1.7$ MeV, which suggests how the nucleon mass difference emerges in spite of the sign of the photon contribution.

I. INTRODUCTION

THE purpose of this paper is to suggest a simple physical picture of the origin of the proton-neutron mass difference. Usually the mass difference $\delta M = M_p - M_n = -1.3$ MeV has been thought of as arising purely as a consequence of the difference in the electromagnetic interactions of the proton and neutron. Here one may consider separately the first order contribution to δM arising from the presence of photons δM^{γ}, which is proportional to $\alpha = 1/137$ and the contributions arising from differences in proton-neutron kinematics, electromagnetic shifts in strong interaction couplings, etc., which are denoted by δM^S.

The first attempts to understand the magnitude and sign of the mass shift centered around estimating the photon contributions δM^{γ} while neglecting the effects of higher mass states given by δM^S. The photon contributions to the proton mass arise predominantly from the interaction of the Dirac current $e\gamma_\mu \times e\gamma_\mu$ to produce a Coulomb contribution proportional to e^2 and the Dirac current interacting with the anomalous magnetic moment current $e\gamma_\mu \times e\kappa_p \sigma_{\mu\nu} q^\nu$ giving a contribution proportional to $e^2\kappa_p$, where $\kappa_p = 1.79$ is the anomalous moment of the proton. There is also a contribution to the n-p mass difference proportional to $\kappa_p^2 - \kappa_n^2$ but this is relatively small because of the small isoscalar static anomalous moment $\kappa_S = \frac{1}{2}(\kappa_p + \kappa_n) = -0.06$. Because of the attractive nature of the Coulomb force the e^2 contribution is positive, thus implying a proton heavier than the neutron in contradiction with experiment. But, as was first pointed out by Feynman and Speisman,[1] the magnetic energy $e^2\kappa_p$ results in a contribution of opposite sign. Consequently, if one picks a high enough cutoff the extra factors in the momentum arising from the derivative coupling of the anomalous-moment current to the photon will dominate thus yielding the correct sign for δM from the photon contributions alone.

Subsequent investigations using dispersion-theory techniques related δM^{γ} to integrals over the nucleon electromagnetic form factors.[2–5] The experimentally observed form factor could then be used to provide a strong natural damping of the high-frequency $e^2\kappa_p$ contribution so that the e^2 contribution, which dominates at low energy, gave the major contribution with the result $\delta M^{\gamma} \sim +0.6$ MeV.[2] Only if one assumed a pathological behavior of the charge distribution near the core of the nucleon, corresponding to high-momentum transfers, could the sign of δM^{γ} be reversed. Various attempts to account for the observed mass difference without relying on the high-momentum transfer dependence of the form factors have been proposed.

The tadpole mechanism[6] suggested that the radiative correction to a virtual 0^+ meson disintegrating into the vacuum would provide a dominant contribution to δM and with the correct sign. There is, however, little experimental evidence for the 0^+ octet.

More recently Dashen,[7] using the S-matrix perturbation method,[8,9] was able to obtain the observed δM and in particular found for the photon contribution $\delta M^{\gamma} = -1.6$ MeV. The major hypothesis of this calculation is that the nucleons are bound states of the pion-nucleon system. It would appear that as a consequence of the composite nature of the nucleons the Coulomb term is dominated by the magnetic term, resulting in a negative sign for δM^{γ}. The relation between this approach and the examination of the low-energy contributions to the self-energy diagrams is consequently

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not clear. Moreover it has been shown by Shaw and Wong[10] that the Dashen calculation of δM depends sensitively on the details of the strong interactions and that even the sign of the mass difference cannot be independently established by this method.[11] It is this criticism of the Dashen calculation which has provided the motivation for the present investigation.

In the alternative approach adopted here we will not assume a necessary composite nature for the nucleons. We will instead assume that the major contribution to δM arises from the low-energy region, and thus appeal to the method of threshold dominance already successfully applied to the calculation of the electron, nucleon,[12] and baryon magnetic moments.[13] Incorporating the viewpoint of the preceding paper,[14] we will relate the mass shift δM to dispersion integrals over scattering amplitudes. Convergence properties of these dispersion integrals will also be assumed, as is spelled out in the next section.

The calculation we present considers the contribution to δM arising from virtual nucleon excitations to the $N\gamma$ and $N\pi$ states, the only states which can contribute at low energy. We find for the photon contribution $\delta M \sim +0.5$ MeV as a consequence of the dominance of the Coulomb energy over the magnetic energy at low energies. We accept this result for the sign of the photon contribution. However, in estimating the $N\pi$ contribution to δM^S there is a large effect of the nucleon mass shift δM back on itself, $\delta M^S \sim 1.3\delta M$. In the present calculation this is seen to arise from the virtual transitions

$$n \rightleftarrows p+\pi^-,$$
$$p \rightleftarrows n+\pi^+,$$

which contribute to δM^S a term proportional to a difference in nucleon pole terms

$$\frac{g^2}{W^2-M_n^2}-\frac{g^2}{W^2-M_p^2}\simeq\frac{-2g^2M\delta M}{(W^2-M^2)^2},$$

where W is the total energy and g the pion-nucleon coupling constant. The usual single pole now has an extra factor of W^2-M^2 in the denominator and strongly weights the low-energy region, $W \approx M$, under the dispersion integral, thus yielding a large contribution. The other contributions to δM^S are found to be small so that the total contribution may be written as $\delta M \sim +0.5$ MeV$+1.3\delta M$ or $\delta M \sim -1.7$ MeV in approximate agreement with experiment. Here a simple picture of the neutron-proton mass difference emerges on the basis of the hypothesis of low-energy dominance.

In the next section we present the method of calculation followed by the section in which we calculate δM.

II. METHOD OF CALCULATION

We will make use of the viewpoint presented in the previous paper[14] and examine the matrix elements $\langle \mathbf{p}|\theta(0)|\mathbf{p}+\mathbf{l}\rangle$ of the trace $\theta(x)$ of the total energy-momentum tensor $\theta_{\mu\nu}(x)$ taken between single-nucleon states of momentum p and $p+l$. In the limit of zero momentum transfer $l\rightarrow 0$, we have $\langle \mathbf{p}|\theta(0)|\mathbf{p}\rangle = M\bar{u}(p)u(p)$, where M is the total mass of the nucleon and $u(p)$ is the Dirac four-component spinor satisfying $(p\!\!\!/-m)u(p)=0$. In the previous paper we considered the analytic properties of the matrix element $\langle \mathbf{p}|\theta(0)|\mathbf{p}+\mathbf{l}\rangle$ in the variable l^2. Assuming the convergence properties of a dispersion integral, we were then able to relate the contributions to the mass M to scattering processes. Here, instead of analytically continuing the graviton-nucleon vertex in the variable l^2, we will set $l^2=0$ and examine the analytic properties of this vertex as a function of the mass of one of the nucleons, $W^2=(p+l)^2$. This leads to the sidewise dispersion relations first studied by Bincer in connection with nucleon electromagnetic structure.[15] We are motivated to apply this approach to estimating the low-energy contribution to the mass splitting because of the direct examination of the threshold behavior it offers. The approach here is similar to that already exploited in calculating the static electromagnetic properties of of the electron, nucleon,[12] and baryons[13] but instead of examining the transitions induced by the fermion source operator $\eta(x)=(i\gamma\cdot\partial-M)\psi(x)$ to the one-photon–nucleon state $\langle N\gamma|\bar{\eta}(0)|0\rangle$, we consider the transitions to the one gravitation-nucleon states $\langle Ng|\bar{\eta}(0)|0\rangle$.

Contracting the graviton out of the final state of the matrix element $\langle Ng|\bar{\eta}(0)|0\rangle$ one finds from the Lehmann-Symanzik-Zimmermann reduction formalism that the vertex shown in Fig. 1 may be written[16]

$$\bar{u}(p)\Sigma(W^2)=\left(\frac{p_0}{M}\right)^{1/2} i\int d^4x \times e^{il\cdot x}\hat{\theta}(x_0)\langle \mathbf{p}|[\theta(x),\bar{\eta}(0)]|0\rangle, \quad (1)$$

where $\hat{\theta}(x_0)=\frac{1}{2}(1+x_0/|x_0|)$ and $p^2=M^2$, $l^2=0$ and $W^2=(p+l)^2$. Assuming invariance under the Lorentz group, parity, and time inversions, the most general

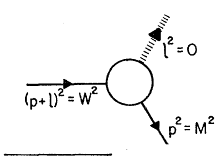

FIG. 1. Graviton-nucleon vertex with virtual nucleon.

[10] G. L. Shaw and D. Y. Wong, Phys. Rev. (to be published).
[11] G. Barton has also shown that in Dashen's calculation the infrared divergence was incorrectly subtracted resulting in the wrong sign for δM. G. Barton, Phys. Rev. (to be published).
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[14] H. Pagels, preceding paper, Phys. Rev. **144**, 1250 (1966).
[15] A. Bincer, Phys. Rev. **118**, 855 (1960).
[16] We refer the reader to Appendix I for details.

form for the vertex is

$$\bar{u}(p)\Sigma(W^2) = \bar{u}(p)\left[G(W^2)\frac{p+l+M}{2M}+G_1(W^2)\frac{p+l-M}{2M}\right] = \bar{u}(p)[G(W^2)+G_2(W^2)l], \quad (2)$$

where from $\bar{u}(p)\Sigma(M^2)u(p)=M\bar{u}(p)u(p)$ we recognize

$$G(M^2)=M \quad (3)$$

as the total mass. We may extract the form factor $G(W^2)$ from the general vertex, Eq. (2), by using the projection operator

$$P=\frac{Ml}{W^2-M^2}u(p) \quad (4)$$

with the property[17]

$$\sum_{\text{spins}} \bar{u}(p)\Sigma(W^2)P=G(W^2). \quad (5)$$

From Eq. (1) we obtain for $G(W^2)$

$$G(W^2)=\left(\frac{p_0}{M}\right)^{1/2} i\int d^4x\, e^{il\cdot x}\hat{\theta}(x_0) \times\sum_{\text{spin}}\langle \mathbf{p}|[\theta(x),\bar{\eta}(0)]|0\rangle P. \quad (6)$$

In order to calculate the nucleon mass difference we will make use of the analytic properties of $G(W^2)$ as a function of the complex variable W^2. Although it was not possible to prove dispersion relations for the function $G(l^2)$ introduction in Ref. 14 as a function of l^2, it is possible to prove dispersion relations rigorously for $G(W^2)$ in the variable W^2 starting from the representation Eq. (6). Here the proof is identical to that given by Bincer[15] for the electromagnetic form factors and we will not repeat it here. Suffice it to say that if we assume $G(W^2)/W^2\rightarrow 0$ as $|W^2|\rightarrow\infty$, then the analytic properties of the proton and neutron form factors $G_{p,n}(W^2)$ are specified by the subtracted dispersion relations

$$G_p(W^2)=M_p+\frac{W^2-M_p^2}{\pi}\times\int_{M_p^2}^{\infty}\frac{\mathrm{Im}G_p(W'^2)dW'^2}{(W'^2-M_p^2)(W'^2-W^2-i\epsilon)}, \quad (7)$$

$$G_n(W^2)=M_n+\frac{W^2-M_n^2}{\pi}\times\int_{M_n^2}^{\infty}\frac{\mathrm{Im}G_n(W'^2)dW'^2}{(W'^2-M_n^2)(W'^2-W^2-i\epsilon)}. \quad (8)$$

Here we have written the dispersion relation for the proton form factor $G_p(W^2)$ subtracted at $W^2=M_p^2$, where $G_p(M_p^2)=M_p$, and a similar relation for the neutron form factor $G_n(W^2)$ subtracted at $W^2=M_n^2$, where $G_n(M_n^2)=M_n$. The thresholds of the dispersion integrals begin at the square of the energy of the lightest state that can be formed from the virtual nucleon, the $N\gamma$ state.

To calculate $\delta M=M_p-M_n$ from a knowledge of the absorptive parts $\mathrm{Im}G_p(W^2)$ and $\mathrm{Im}G_n(W^2)$, one introduces the further assumption that $G_p(W^2)-G_n(W^2)\rightarrow 0$ as $|W^2|\rightarrow\infty$.[18] From this prescribed asymptotic behavior we may subtract Eq. (8) from Eq. (7) and take the limit $W^2\rightarrow\infty$, obtaining

$$\delta M=\frac{1}{\pi}\int_{M_p^2}^{\infty}\frac{\mathrm{Im}G_p(W^2)dW^2}{W^2-M_p^2}-\frac{1}{\pi}\int_{M_n^2}^{\infty}\frac{\mathrm{Im}G_n(W^2)dW^2}{W^2-M_n^2}. \quad (9)$$

This is the basic equation from which we will compute δM from the absorptive parts.

Finally we introduce the assumption of low-energy dominance, namely that it is the difference in the interactions of the lighter states in the cloud surrounding the nucleons that generates the major contribution to to the mass difference. Hence it will be the threshold contributions to $\mathrm{Im}G_{p,n}(W^2)$ that will give us the dominant part of δM. To incorporate this hypothesis into the calculation we include only the low-energy part of the dispersion integrals, Eq. (9),

$$\delta M=\frac{1}{\pi}\int_{M_p^2}^{\lambda^2M_p^2}\frac{\mathrm{Im}G_p(W^2)dW^2}{W^2-M_p^2}-\frac{1}{\pi}\int_{M_n^2}^{\lambda^2M_n^2}\frac{\mathrm{Im}G_n(W^2)dW^2}{W^2-M_n^2}, \quad (10)$$

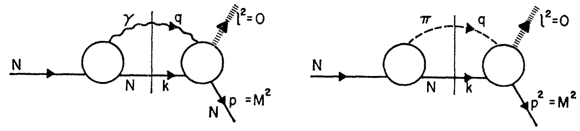
where $\lambda^2>1$ serves to define the low-energy region. Furthermore, to estimate λ^2 we can appeal to a previous calculation of the nucleon anomalous moments[12] which used the same method of sidewise dispersion relations in the variable W^2 and yielded the correct sign and approximate magnitude of the nucleon moments by including intermediate states of energy W with $M\leq W\leq 1.5M$ corresponding to a $\lambda^2=2.3$. Thus we are motivated by this success to pick the same low-energy range for the present calculation and set $\lambda^2\sim 3$. In this way the only contributing states to δM are the $N\gamma$ and $N\pi$ states.

III. CALCULATION OF THE LOW-ENERGY CONTRIBUTION TO δM

Now we turn to the task of calculating the absorptive parts $\mathrm{Im}G_{p,n}(W^2)$ in the neighborhood of threshold. By

[17] Here we have normalized our spinor according to $\sum_{\text{spin}}\bar{u}(p)u(p)=(p+M)/2M$.

[18] This asymptotic behavior is vitally necessary for our calculation. If the difference $G_p(W^2)-G_n(W^2)$ diverged as $W^2\rightarrow\infty$ or approached some arbitrary constant value we could not hope to calculate δM using this method.



(a) (b)

FIG. 2. The $N\gamma$ and $N\pi$ contribution to the absorption $\mathrm{Im}G(W^2)$.

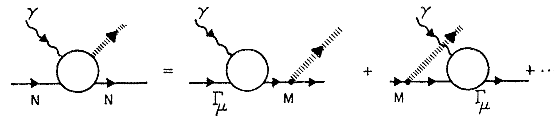


FIG. 3. Pole terms contributing to the scattering amplitude for $N\gamma$ intermediate state.

examining the threshold behavior of these functions we can see precisely how the contributions to δM arise.

The absorptive part may be obtained from Eq. (6) by the replacement $i\theta(x_0)\rightarrow\frac{1}{2}$ as follows from the assumed parity and time reversal invariance of the theory.[19] Furthermore by inserting a complete set of states in the commutator of Eq. (6) the integral over x may be explicitly performed with the result[15]

$$\mathrm{Im}G(W^2)=\left(\frac{p_0}{M}\right)^{1/2}\pi\sum_{\text{spin}}\Big[\sum_n 2p_0{}^n\hat{\theta}(p_0{}^n)\delta(W^2-M_n{}^2)\times\langle\mathbf{p}|\theta(0)|n,\mathbf{p}^n\rangle\langle n,\mathbf{p}^n|\bar{\eta}(0)|0\rangle+\sum_{n'}2l_0\hat{\theta}(-l_0)\delta(l^2+M_{n'}{}^2)\langle\mathbf{p}|\bar{\eta}(0)|n'-\mathbf{l}\rangle\times\langle n'-\mathbf{l}|\theta(0)|0\rangle\Big]P. \quad (11)$$

The sum over the states n' cannot contribute, since $l^2=0$ and thus the delta function vanishes unless $M_{n'}{}^2=0$. The only state n' with $M_{n'}=0$ is the vacuum and it cannot contribute since $\langle\mathbf{p}|\bar{\eta}(0)|0\rangle=0$. We also note that the sum on n does not include the one-nucleon state, again because $\langle\mathbf{p}|\bar{\eta}(0)|0\rangle=0$. However, n can be any multiparticle state with the same quantum numbers as the nucleon. The lightest such states are the $N\gamma$ and $N\pi$ states,[20] and in the low-energy limit these are the only states contributing to $\mathrm{Im}G(W^2)$ (see Fig. 2). We proceed now to calculate the contribution of these intermediate states.

A. Photon Contribution

Including only the state $n=N\gamma$ in the sum, Eq. (11), we have for the contribution of intermediate photons

$$\mathrm{Im}G^\gamma(W^2)=\left(\frac{p_0}{M}\right)^{1/2}\pi\sum_{\text{spins}}\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\delta^4(k+q-p-l)\times\hat{\theta}(k_0)\hat{\theta}(q_0)\langle\mathbf{p}|\theta(0)|\mathbf{k}s;\mathbf{q}\lambda\rangle\langle\mathbf{k}s;\mathbf{q}\lambda|\bar{\eta}(0)|0\rangle P. \quad (12)$$

Here k and s are the momentum and spin of the intermediate nucleon and $\mathbf{q}$ and λ the momentum and spin of the intermediate photon. The sum in Eq. (12) is over all nucleon and photon spin variables. Finally to compute the absorptive part we must estimate the vertex $\langle\mathbf{k}s;\mathbf{q}\lambda|\bar{\eta}(0)|0\rangle$ and the amplitude $\langle\mathbf{p}|\theta(0)|\mathbf{k}s;\mathbf{q}\lambda\rangle$.

The most general form for the transition matrix elements for a virtual nucleon to go into a $J=\frac{1}{2}$, one-nucleon, one-photon state with $k^2=M^2$ and $q^2=0$ is given by[21]

$$\langle\mathbf{k}s;\mathbf{q}\lambda|\bar{\eta}(0)|0\rangle=(M/2q_0k_0)^{1/2}\bar{u}(k,s)\epsilon_\lambda{}^\mu\Big[e\gamma_\mu-F_2{}^+(W^2)(e/2M)i\sigma_{\mu\nu}q^\nu\times\frac{k+q+M}{2M}+F_2{}^-(W^2)i\sigma_{\mu\nu}q^\nu\frac{k+q-M}{2M}\Big], \quad (13)$$

where $W^2=(k+q)^2$ and $\epsilon_\lambda{}^\mu$ is the polarization vector of the photon. Since we are interested only in the threshold contributions to $\mathrm{Im}G^\gamma(W^2)$ we may, to a first approximation, set $F_2{}^+(W^2)=F_2{}^+(M^2)=\kappa$, the static anomalous moment of the nucleon. We also note that as $W^2\rightarrow M^2$ the term in Eq. (13) proportional to $F_2{}^-(W^2)$ contributes to the absorptive part at least one-order higher in the frequency $\omega=W-M$ than that arising from the $F_2{}^+(W^2)$ term since $k+q-M$ vanishes in this limit. Consequently we neglect its contribution in this threshold approximation and write for the vertex

$$\langle\mathbf{k}s;\mathbf{q}\lambda|\eta(0)|0\rangle=(M/2p_0k_0)^{1/2}\times\bar{u}(k,s)\epsilon_\lambda{}^\mu\Big[e\gamma_\mu-i(e\kappa/2M)\sigma_{\mu\nu}q^\nu\frac{k+q+M}{2M}\Big]. \quad (14)$$

We will approximate the scattering amplitude for a nucleon and photon in the initial state to scatter to a nucleon and a graviton in the final state (see Fig. 3) by including just the pole terms. We are assured that in the threshold limit the contribution from just the pole terms survives and hence we write[22]

$$\langle\mathbf{p}|\theta(0)|\mathbf{k}s,\mathbf{q}\lambda\rangle=(M^2/2p_0k_0q_0)^{1/2}\epsilon_\lambda{}^\nu\bar{u}(p)\{M[1/(p+l-M)]\Gamma_\nu(q)+\Gamma_\nu(q)[1/(p-q-M)]M\}u(k,s), \quad (15)$$

where $\Gamma_\nu(q)=e\gamma_\nu-i(e/2M)\kappa\sigma_{\mu\nu}q_\nu$ is the electromagnetic current including the Dirac and anomalous-magnetic-moment terms.

From the expressions for the vertex and scattering amplitude given above, we compute $\mathrm{Im}G^\gamma(W^2)$ in the low-energy region with the assurance that in the threshold limit $W^2\rightarrow M^2$ it becomes exact. Inserting

[19] See Appendix of R. Oehme, Phys. Rev. **100**, 1503 (1955).

[20] Multiphoton processes contribute only higher order corrections to δM and are neglected.

[21] See Ref. 15. Here we have used $\epsilon_\lambda{}^\mu q_\mu=0$, the transversality condition for real photons, so terms proportional to q_μ do not appear and $\sigma_{\mu\nu}=\frac{1}{2}i[\gamma_\mu,\gamma_\nu]$.

[22] We refer the reader to Appendix II where these pole term contributions to the scattering amplitude are established.

Eqs. (14) and (15) into Eq. (12), performing the spin sums using standard trace techniques, and doing the integral over intermediate scattering angles, one obtains the contributions to $\mathrm{Im}G_p{}^\gamma(W^2)$ which are proportional to e^2, $e\kappa_p$, $\kappa_p{}^2$ and contributions to $\mathrm{Im}G_n{}^\gamma(W^2)$ proportional to $\kappa_n{}^2$. Since $\kappa_n{}^2 \simeq \kappa_p{}^2$ these terms will cancel when computing δM^γ. There remain the terms e^2, $e\kappa_p$ which contribute

$$\mathrm{Im}G_p{}^{e^2}(W^2) = -\frac{\alpha M^3}{4W^2}\left[12 + \frac{(W^2-M^2)^2}{W^2M^2} - \frac{2}{W^2}(W^2+M^2) - \frac{8W^2}{W^2-M^2}\ln\left(\frac{W^2}{M^2}\right)\right]$$

$$\xrightarrow[W\to M]{} \frac{\alpha}{2}\frac{W^2-M^2}{M} > 0, \quad \alpha = e^2/4\pi, \qquad (16)$$

$$\mathrm{Im}G_p{}^{e\kappa}(W^2) = -\frac{\alpha\kappa_p M^3}{16W^2}\left\{\frac{W^2-M^2}{M^2}\left[\frac{4W^2}{W^2-M^2}\times\ln\left(\frac{W^2}{M^2}\right) + 6 - \frac{2(2W^2-M^2)}{W^2}\right] - 16\left[\frac{W^2}{W^2-M^2}\ln\left(\frac{W^2}{M^2}\right) - 1\right]\right\}$$

$$\xrightarrow[W\to M]{} -\frac{\alpha\kappa_p}{6M^3}(W^2-M^2)^2 < 0. \qquad (17)$$

In the threshold behavior of the absorptive parts the character of the electromagnetic contributions is clearly revealed. The e^2 term has a positive contribution at threshold due to the attractive nature of the Coulomb force and it vanishes as W^2-M^2, just proportional to to the phase space of the intermediate $N\gamma$ state. There is an additional factor of W^2-M^2 in the threshold behavior of the $e\kappa$ contribution arising from the derivative coupling of the anomalous moment current and this contribution we find is opposite in sign to the Coulomb term. Because of the derivative coupling the magnetic term is small at low energies but at high energies, unless it is damped by including the energy dependence of the form factor $F_2{}^+(W^2)$, it will dominate.

Writing $\delta M^\gamma = \delta M^{e^2} + \delta M^{e\kappa}$ we may obtain the low energy contribution to δM^γ by substituting Eqs. (16) and (17) into the dispersion integral Eq. (10) with the result

$$\delta M^{e^2} = \frac{\alpha}{4\pi}M\left[7\ln\lambda^2 + \left(\frac{1}{\lambda^2}-1\right) - 8\left(\frac{\lambda^2}{\lambda^2-1}\ln\lambda^2 - 1\right)\right],$$

$$= +0.72\ \mathrm{MeV},\ \lambda^2\sim 2,$$

$$= +1.01\ \mathrm{MeV},\ \lambda^2\sim 3, \qquad (18)$$

$$\delta M^{e\kappa} = \frac{\alpha\kappa_p M}{4\pi}\left[\tfrac{7}{2}(\ln\lambda^2+1) + \frac{1}{2\lambda^2} - \frac{4\lambda^2}{\lambda^2-1}\ln\lambda^2 + \Phi(1-\lambda^2)\right],$$

$$= -0.16\ \mathrm{MeV},\ \lambda^2\sim 2,$$

$$= -0.51\ \mathrm{MeV},\ \lambda^2\sim 3, \qquad (19)$$

where

$$\Phi(x) = -\int_0^x \ln(1-y)dy/y$$

is the Spence function.[23] The total photon contribution from the low-energy region yields a proton heavier than the neutron,

$$\delta M^\gamma \sim +0.5\ \mathrm{MeV}. \qquad (20)$$

If we had included the energy dependence of the magnetic form factor $F_2{}^+(W^2)\delta M^{e\kappa}$ would have been smaller than the value quoted above and thus increasing our estimate of δM^γ. This brings it in approximate agreement with other calculations[2] which include the effect of the experimentally observed form factors, and lends additional support to our choice $\lambda^2\sim 3$ and the hypothesis of low-energy dominance.

B. Pion Contribution

In the low-energy region to which we restrict our attention, there also emerges a contribution to δM from pions in the intermediate state which we denote by δM^π. The contributions to δM^π might arise from three distinct sources, the electromagnetic mass splittings among the pions, the electromagnetic differences in the pion-nucleon coupling strengths, or the mass splittings of the nucleons. The mass splittings in the pion isomultiplet transform like T^2, where T is the isospin, and cannot contribute to the nucleon mass difference which transforms like T_3. Moreover, since we are examining only the low-energy region we will neglect recoil and set the pion nucleon mass ratio $\mu/M=0$, which also considerably simplifies the calculation.

First we examine the effect of the electromagnetic differences in the pion-nucleon coupling constants. To this end we must calculate the contribution to $\mathrm{Im}G_{p,n}(W^2)$ emerging from the presence of an intermediate pion. For a single pion in the intermediate state the contribution to the absorptive part is

$$\mathrm{Im}G^\pi(W^2) = \left(\frac{p_0}{M}\right)^{1/2}\pi\sum_{\mathrm{spin}}\int\frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\delta^4(k+q-p-l)$$

$$\times\hat\theta(k_0)\hat\theta(q_0)\langle\mathbf{p}|\theta(0)|\mathbf{k}s,\mathbf{q}\rangle\langle\mathbf{k}s,\mathbf{q}|\bar\eta(0)|0\rangle P, \qquad (21)$$

where k and s are the momentum and spin of the intermediate nucleon and q the momentum of the intermediate pion [see Fig. 2(b)].

We approximate the pion-nucleon vertex appearing

[23] K. Mitchell, Phil. Mag. 40, 351 (1949).

in Eq. (21) by just its threshold value,

$$\langle \mathbf{k}s,\mathbf{q}|\bar{\eta}(0)|0\rangle=(M/2k_0q_0)^{1/2}\bar{u}(k,s)gi\gamma_5\,, \qquad (22)$$

where g is the pion-nucleon coupling constant appropriate to the particular vertex. For example, the vertex for a virtual proton going into a real π^+n pair will have the coupling constant $g_{p\pi^+}$ appear in Eq. (21).

At low energies, the scattering amplitude appearing in Eq. (21) is given by just its pole terms (see Fig. 4).[22] Since we are neglecting recoil the diagram 4(c) which contributes to the absorptive part, a term proportional to $(\mu/M)^2$ will not enter, and we have from 4(a) and 4(b)

$$\langle \mathbf{p}|\theta(0)|\mathbf{k}s;\,\mathbf{q}\rangle = -(M^2/2p_0k_0l_0)^{1/2}\bar{u}(p)\{M[1/(p+l-M)]gi\gamma_5 + gi\gamma_5[1/(p-q-M)]M\}u(k,s)\,. \qquad (23)$$

At threshold $W^2\to M^2$ this amplitude vanishes. This is because the pseudoscalar pion must be absorbed by the nucleon from a relative P state to conserve parity, introducing an over-all factor proportional to the momentum $\mathbf{q}$ of the incoming pion.

Inserting Eqs. (23) and (22) into Eq. (21) and performing the spin sum and angular integration, one obtains for the absorptive part

$$\mathrm{Im}G^\pi(W^2)=\left(\frac{g^2}{4\pi}\right)\frac{M^3}{2W^2}\times\left[\frac{W^4+4W^2M^2-M^4}{4W^2M^2}-\frac{W^2}{W^2-M^2}\ln\left(\frac{W^2}{M^2}\right)\right] \xrightarrow[W\to M]{} -\left(\frac{g^2}{4\pi}\right)\frac{(W^2-M^2)^2}{24M^3}\,. \qquad (24)$$

The threshold dependence of $\mathrm{Im}G^\pi(W^2)$ emerges as a consequence of the intermediate-state phase space for the zero-mass pion and nucleon which contributes a factor W^2-M^2 and the amplitude, Eq. (23), which contributes the additional factor of W^2-M^2.

We can now use Eq. (24) to estimate the effect of the first-order electromagnetic shifts of $g_{\pi N}$ on δM^π by

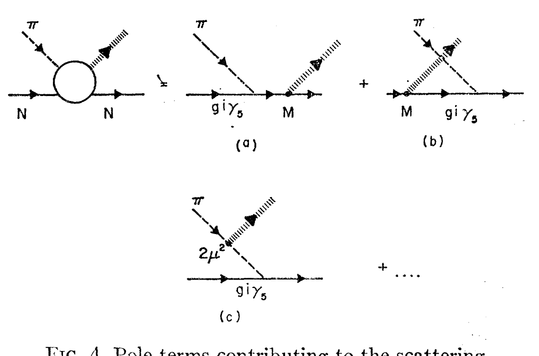


FIG. 4. Pole terms contributing to the scattering amplitude for $N\pi$ intermediate state.

calculating $\mathrm{Im}G_p{}^\pi(W^2)-\mathrm{Im}G_n{}^\pi(W^2)$. The contributions of charged and neutral pions in the intermediate states leads to

$$\mathrm{Im}G_p{}^\pi(W^2)-\mathrm{Im}G_n{}^\pi(W^2)=\left(\frac{g^2}{4\pi}\right)\frac{M^3}{W^2}(\Delta_0+2\Delta_+)\times\left[\frac{W^4+4W^2M^2-M^4}{4W^2M^2}-\frac{W^2}{W^2-M^2}\ln\left(\frac{W^2}{M^2}\right)\right], \qquad (25)$$

where

$$\Delta_0=(g_{p\pi^0}-g_{n\pi^0})/g\,,\quad \Delta_+=(g_{p\pi^+}-g_{n\pi^-})/g\,,\quad g^2/4\pi\sim 15\,, \qquad (26)$$

are the electromagnetic splittings of $g_{\pi N}$. From the dispersion integral one obtains

$$\left.\frac{\delta M^\pi}{M}\right|_{\delta g}=\left(\frac{g^2}{4\pi}\right)\frac{(\Delta_0+2\Delta_+)}{4\pi}\times\left[1-\frac{1}{\lambda^2}+4\left(\frac{\lambda^2}{\lambda^2-1}\ln\lambda^2-1\right)-3\ln\lambda^2\right] = -(\Delta_0+2\Delta_+)4\times10^{-2}\,,\quad \lambda^2\sim3\,. \qquad (27)$$

In Appendix III we estimate the electromagnetic splittings of $g_{\pi N}$; however suffice it here to point out that this calculation indicates that $\delta g/g$ is on the order of $\delta M/M$ with $\Delta_0+2\Delta_+\sim-6.7\delta M/M$. From Eq. (27) we estimate $\delta M^\pi|_{\delta g}\sim+0.27\delta M$.

Finally we turn to estimating the effect of the nucleon mass shift back on itself. First let us examine contributions of intermediate π^0's to $\mathrm{Im}G_{p,n}(W^2)$. Since there are no transitions between the nucleons of different mass in this case, we may easily obtain the absorptive parts from Eq. (24), with the result

$$\mathrm{Im}G_p{}^{\pi^0}(W^2)=M_p\left(\frac{g^2}{4\pi}\right)\frac{1}{2y}\left[\frac{y^2+4y-1}{4y}-\frac{y}{y-1}\ln y\right], \qquad (28)$$

where $y=W^2/M_p{}^2$ and there is a similar equation for $\mathrm{Im}G_n(W^2)$. From the dispersion integral, Eq. (10), we find

$$\delta M^{\pi^0}=\frac{\delta M}{8\pi}\left(\frac{g^2}{4\pi}\right)\left[1-\frac{1}{\lambda^2}+4\left(\frac{\lambda^2}{\lambda^2-1}\ln\lambda^2-1\right)-3\ln\lambda^2\right] = -\delta M\times2\times10^{-2}\,,\quad \lambda^2\sim3 \qquad (29)$$

a very small contribution which may neglected.

Now consider what happens when there is an intermediate π^+ contributing to $\mathrm{Im}G_p{}^{\pi^+}(W^2)$ and an intermediate π^- contributing to $\mathrm{Im}G_n{}^{\pi^-}(W^2)$ (see Fig. 5).

FIG. 5. Contributions of intermediate π^+ to $\mathrm{Im}G_p{}^{\pi^+}(W^2)$.

Then the kinematics and dynamics of the vertex Eq. (22) and the scattering amplitude Eq. (23) will be modified. For example, the scattering amplitude contributing to $\mathrm{Im}G_p^{\pi^+}(W^2)$ will be

$$\langle \mathbf{p}|\theta(0)|\mathbf{k}s,\mathbf{q}\rangle = -\left(\frac{M_pM_n}{2p_0k_0q_0}\right)^{1/2}\bar{u}(p)\left[M_p\frac{1}{p+l-M_p}gi\gamma_5 + gi\gamma_5\frac{1}{p-q-M_n}M_n\right]u(k,s), \quad (30)$$

where $p^2=M_p^2$, $k^2=M_n^2$, $q^2=l^2=0$. The pole terms in Eq. (30) gives rise to singularities of the form $1/(W^2-M_n^2)$. A similar term from the neutron contribution then leads to a term of the form

$$\frac{1}{W^2-M_n^2}-\frac{1}{W^2-M_p^2}=\frac{-2M\delta M}{(W^2-M^2)^2}$$

when calculating δM^π. From the threshold dependence given by Eq. (24) we learn that the difference in pole terms will imply at threshold

$$\mathrm{Im}G_p^{\pi^+}(W^2)-\mathrm{Im}G_n^{\pi^-}(W^2)\rightarrow -(\sqrt{2})^2\left(\frac{g^2}{4\pi}\right)\frac{(W^2-M^2)^2}{24M^3}\times\frac{(-2M)\delta M}{(W^2-M^2)}=\left(\frac{g^2}{4\pi}\right)\frac{W^2-M^2}{6M^2}\delta M. \quad (31)$$

Because of the additional factor of W^2-M^2 in the denominator, we have an improved threshold dependence implying a large contribution. Moreover, the sign of this mass counter term is such as to suggest the possibility that the total mass shift δM will be negative in spite of the positive photon contribution.

It remains only to compute the absorptive part including all the differences in the neutron-proton mass. We find

$$\mathrm{Im}G_p^{\pi^+}(W^2)=-\left(\frac{g^2}{4\pi}\right)\frac{M_n(y-1)}{2y}\left[\frac{-2r(r+1)}{y-r^2}+\frac{r}{2}\left(1+\frac{1}{y}\right)+\frac{r^3(y+1)}{y(y-r^2)}+\frac{2y\ln y}{(y-r^2)(y-1)}-1\right], \quad (32)$$

where $y=W^2/M_n^2$, $r=M_p/M_n$ and a similar equation for $\mathrm{Im}G_n^{\pi^-}(W^2)$ with the roles of M_n and M_p reversed in Eq. (32). Then from the dispersion integral

$$\delta M^{\pi^+}=\frac{1}{\pi}\int_{M_n^2}^{\lambda^2M_n^2}\frac{\mathrm{Im}G_p^{\pi^+}(W^2)dW^2}{W^2-M_p^2}-\frac{1}{\pi}\int_{M_p^2}^{\lambda^2M_p^2}\frac{\mathrm{Im}G_n^{\pi^-}(W^2)dW^2}{W^2-M_n^2} \quad (33)$$

we evaluate δM^{π^+} and then expand in powers of δM keeping only the first-order contribution. The result is[24]

$$\begin{aligned}\delta M^\pi &= -\left(\frac{g^2}{4\pi}\right)\frac{\delta M}{8\pi}\left[5\frac{\lambda^2-1}{\lambda^2}+\ln\lambda^2+16\frac{3-\lambda^2}{\lambda^2-1}-\frac{32\ln\lambda^2}{(\lambda^2-1)^2}+4\left(\frac{\lambda^2}{\lambda^2-1}\ln\lambda^2-1\right)\right]\\ &=+0.86\delta M,\quad \lambda^2\sim 2,\\ &=+1.06\delta M,\quad \lambda^2\sim 3. \end{aligned} \quad (34)$$

This estimate of the pion contribution is then added to the effect due the shifts in the coupling constants, $+0.27\delta M$, to give our total estimate for $\delta M^s=1.06\delta M+0.27\delta M=1.3\delta M$. Along with our estimate of the photon contribution $\delta M^\gamma=0.5$ MeV we may then solve for the mass difference from $\delta M=\delta M^\gamma+\delta M^s$ with the result

$$\delta M\sim -1.7\text{ MeV}\quad(\lambda^2\sim 3) \quad (35)$$

for the low-energy contribution. This is to be compared with $\delta M^{\mathrm{expt}}=-1.3$ MeV. Including a finite pion mass in the calculation increases our estimate by only 5%.

IV. CONCLUSIONS

The purpose of this calculation has not been to obtain precise numerical agreement with the observed mass difference but rather to suggest how at least the sign of this mass difference might emerge in spite of the fact that the photon contributions alone give the wrong sign.[25] The success of this approach depends crucially on the fact that we obtain $\delta M^s\sim+1.3\delta M$; the effect of the nucleon mass shift back on itself is over 100%. Had the coefficient of δM been less than 1.0 the sign of our final result would be reversed, and this would have indeed been the case if we chose a smaller range of intermediate-state energies to integrate over. However, in the threshold approximation and with our choice of λ^2 the coefficient of δM is in fact greater than 1.0, suggesting that a more precise calculation will not change the general conclusions of this work regarding the origin of the mass difference.

APPENDIX I

Here we consider the matrix element $\langle Ng|\bar{\eta}(0)|0\rangle$ for the transition of a virtual fermion of momentum $p+l$ to a real fermion-graviton pair of momentum p and l with $p^2=M^2$, $l^2=0$. We do not require any detailed dynamical theory of gravitation since we consider processes only to first order in the gravitational coupling κ and questions related to the renormalization of gravitational interactions do not concern us. For our purposes it suffices to recognize that there is a field

[24] In Eq. (34) we have also included the small contribution from the π^0's, Eq. (29).

[25] A similar observation has been made using the two point function by H. M. Fried and T. N. Truong, Brown University (unpublished report).

$A_{\mu\nu}(x)$ coupling to the total stress tensor according to

$$\Box A_{\mu\nu}(x)=\kappa\theta_{\mu\nu}(x). \tag{I1}$$

Instead of considering all the polarization states of the final-state graviton[26] we need examine only the sum on these polarization states or equivalently the trace $A(x)=A_{\mu}{}^{\mu}(x)$, where

$$\Box A(x)=\kappa\theta(x). \tag{I2}$$

With this understanding we write for the general form of the matrix element

$$\langle Ng|\bar{\eta}(0)|0\rangle=(M/2p_0l_0)^{1/2}\kappa\bar{u}(p)\Sigma(W^2), \tag{I3}$$

where $\Sigma(W^2)=G(W^2)+G_2(W^2)\boldsymbol{l}$ and $W^2=(p+l)^2$. In the threshold limit $l\to 0$ from $\kappa\bar{u}(p)\Sigma(M^2)u(p)=\kappa M\bar{u}(p)u(p)$ we have that $G(M^2)=M$. We may write the matrix element Eq. (I3) in another way by contracting out the graviton state vector in the final state. The result from the LSZ[27] reduction formalism is

$$\langle Ng|\bar{\eta}(0)|0\rangle=(2l_0)^{-1/2}i\int d^4x\, e^{il\cdot x}\hat{\theta}(x_0) \times\kappa\langle\mathbf{p}|[\theta(x),\bar{\eta}(0)]|0\rangle, \tag{I4}$$

where we have used the equation of motion Eq. (I2) and ignored possible equal-time commutators. Finally we have for the form factors $G(W^2)$, $G_2(W^2)$, combining Eq. (I4) and Eq. (I3),

$$\bar{u}(p)\Sigma(W^2)=\left(\frac{p_0}{M}\right)^{1/2}i\int d^4x\, e^{il\cdot x}\hat{\theta}(x_0) \times\langle\mathbf{p}|[\theta(x),\bar{\eta}(0)]|0\rangle \tag{I5}$$

the same as Eq. (1).

APPENDIX II

Here we establish the expressions Eqs. (15) and (23) for the pole-term contributions to the scattering amplitude. To this end we examine the matrix elements of the fully symmetric stress-energy tensor $\theta_{\mu\nu}$ which satisfies the condition $\partial_\mu\theta_{\mu\nu}=0$, energy-momentum conservation. First we consider the matrix elements of $\theta_{\mu\nu}(0)$ between an initial state consisting of a nucleon of momentum k_α with $k^2=M^2$ and photon of momentum l_α and polarization ϵ_α satisfying $q^2=0$ and $q\cdot\epsilon=0$ and a final single-nucleon state of momentum p_α with $p^2=M^2$. Then we have for the pole terms

$$\begin{aligned}\langle\mathbf{p}|\theta_{\mu\nu}(0)|\mathbf{k},\mathbf{q}\boldsymbol{\epsilon}\rangle &=(M^2/2q_0kp_0)^{1/2}e\bar{u}(p)\{\tfrac{1}{4}\gamma_{(\mu}(2p+l)_{\nu)}\\ &\times[1/(\boldsymbol{p}+\boldsymbol{l}-M)]\gamma\cdot\epsilon+\gamma\cdot\epsilon[1/(\boldsymbol{k}-\boldsymbol{l}-M)]\\ &\times\tfrac{1}{4}\gamma_{(\mu}(2k-l)_{\nu)}+\tfrac{1}{2}[g_{\mu\nu}(\epsilon^\lambda l^2-2\epsilon\cdot l l^\lambda)\\ &-2\epsilon^\lambda(q_{(\mu}l_{\nu)}-2q_\mu q_\nu)+2\epsilon\cdot l(g_\mu{}^\lambda q_\nu+g_\nu{}^\lambda q_\mu)\\ &-2(l-2q)^\lambda\epsilon_{(\mu}(l-q)_{\nu)}][\gamma_\lambda/(q-l)^2]\\ &-\tfrac{1}{2}g_{\mu\nu}\gamma\cdot\epsilon\}u(k),\end{aligned} \tag{II1}$$

where $(\mu\nu)=\mu\nu+\nu\mu$ and we have not included terms from the anomalous-magnetic-moment current for the sake of simplicity. We have introduced $l_\mu=(k+q-p)_\mu$ the momentum of the final graviton which satisfies $l^2=0$. The various terms in Eq. (II1) arise as follows. The first two terms arise from the stress energy in the nucleon field and correspond to inserting a graviton of momentum l_μ in the final and initial nucleons according to the rule given by Weinberg[28] for graviton insertions on spin-$\frac{1}{2}$ systems (see Fig. 3). The third term arises from the graviton interacting with the energy momentum of the incident photon $\frac{1}{2}(F_{(\mu\lambda}F_{\nu)}{}^\lambda-\frac{1}{2}g_{\mu\nu}F_{\sigma\tau}F^{\sigma\tau})$, and corresponds to a graviton insertion on a spin-1 zero-mass system. The final term $-\frac{1}{2}g_{\mu\nu}\gamma\cdot\epsilon$ is the four-point interaction of the two nucleons, photon, and graviton. The presence of this term is required to assure us that the amplitude Eq. (II1) is invariant under the gauge transformation $\epsilon_\mu\to\epsilon_\mu+\lambda q_\mu$ as is imposed by current conservation. Furthermore one may explicitly verify that the amplitude Eq. (II1) is consistent with energy-momentum conservation which implies

$$\begin{aligned}\langle\mathbf{p}|i\partial_\mu\theta_{\mu\nu}(0)|\mathbf{k},\mathbf{q}\boldsymbol{\epsilon}\rangle&=(k+q-p)^\mu\langle\mathbf{p}|\theta_{\mu\nu}(0)|\mathbf{kq},\boldsymbol{\epsilon}\rangle\\ &=l^\mu\langle\mathbf{p}|\theta_{\mu\nu}(0)|\mathbf{k},\mathbf{q}\boldsymbol{\epsilon}\rangle=0.\end{aligned}$$

For our calculation we require only the matrix elements of the trace $\theta(0)$ which is obtained from Eq. (II1) by contraction, with the result

$$\langle\mathbf{p}|\theta(0)|\mathbf{k},\mathbf{q}\epsilon\rangle=(M^2/2q_0k_0p_0)^{1/2} \times e\bar{u}(p)\left[M\frac{1}{\boldsymbol{p}+\boldsymbol{l}-M}\epsilon\cdot\gamma+\epsilon\cdot\gamma\frac{1}{\boldsymbol{k}-\boldsymbol{l}-M}M\right]u(k). \tag{II2}$$

We have not included a possible anomalous moment contribution in Eq. (II2). Including this term according to the substitution $\epsilon\cdot\gamma\to\epsilon_\lambda\Gamma^\lambda(q)$, we have Eq. (15).

In case the initial state consists of a pion of momentum q_α with $q^2=\mu^2$ and nucleon of momentum k_α with $k^2=M^2$ then the pole-term amplitude is given by

$$\begin{aligned}\langle\mathbf{p}|\theta_{\mu\nu}(0)|\mathbf{kq}\rangle &=-(M^2/2q_0k_0p_0)^{1/2}\bar{u}(p)\{\tfrac{1}{4}\gamma_{(\mu}(2p+l)_{\nu)}\\ &\times[1/[\boldsymbol{p}+\boldsymbol{l}-M)]gi\gamma_5+gi\gamma_5[1/(\boldsymbol{k}-\boldsymbol{l}-M)]\\ &\times\tfrac{1}{4}\gamma_{(\mu}(2k-l)_{\nu)}+\tfrac{1}{2}[(2q-l)_\nu(2q-l)_\mu\\ &+l^2g_{\mu\nu}-l_\mu l_\nu][gi\gamma_5/(q-l)^2-\mu^2]\\ &-\tfrac{1}{2}g_{\mu\nu}gi\gamma_5\}u(\kappa).\end{aligned} \tag{II3}$$

Again the first two terms arise from the graviton emissions from the initial and final nucleons [see Fig. 4(a), (b)], while the third term represents the graviton coupling to the energy in the incident pion [see Fig. 4(c)] and the last term is the four-point interaction. The amplitude (II3) satisfies the requirement $l^\mu\langle\mathbf{p}|\theta_{\mu\nu}(0)|\mathbf{kq}\rangle=0$ imposed by energy conservation. Contracting Eq. (II3) to obtain the matrix element of

[26] S. Weinberg, Phys. Rev. **138**, B988 (1965).

[27] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).

[28] S. Weinberg, Phys. Rev. **140**, B516 (1965).

the trace, we have

$$\langle \mathbf{p}|\theta(0)|\mathbf{kq}\rangle = -\left(\frac{M^2}{2q_0p_0k_0}\right)^{1/2} u(p)\left[M\frac{1}{p+l-M}gi\gamma_5 + gi\gamma_5\frac{1}{k-l-M}M + \frac{2\mu^2 gi\gamma_5}{(q-l)^2-\mu^2}\right]u(k). \quad \text{(II4)}$$

In the approximation of neglecting recoil $\mu^2=0$ the last term does not contribute and we have Eq. (23). In case the initial and final nucleons do not have the same mass then we find for the matrix elements Eq. (30).

APPENDIX III

Here we will estimate the electromagnetic splittings of the pion nucleon couplings $g_{\pi N}$. The method we use is the same as that employed in calculating the nucleon mass shift, but instead of examining the nucleon-graviton vertex we examine the nucleon-meson vertex with one of the nucleons off the mass shell. This vertex has the general form[15]

$$\bar{u}(p)\Lambda(W^2) = \bar{u}(p)[i\gamma_5 K(W^2) + i\gamma_5 \bar{K}(W^2) l], \quad \text{(III1)}$$

where $p^2=M^2$, $(p+l)^2=W^2$ and l is the momentum of the outgoing pion. We recognize $K(M^2)=g$, the pion-nucleon coupling constant for this vertex. Our interest is in estimating the quantities

$$\begin{aligned}\Delta_0 &= (g_{p\pi^0} - g_{n\pi^0})/g,\\ \Delta_+ &= (g_{p\pi^+} - g_{n\pi^-})/g,\end{aligned} \quad \text{(III2)}$$

and to do this we make use of the analytic properties of $K(W^2)$ in the complex W^2 plane. Assuming that $K_p^{\pi^0}(W^2)-K_n^{\pi^0}(W^2)$ and $K_p^{\pi^+}(W^2)-K_n^{\pi^-}(W^2)$ vanish as $W^2\to\infty$ we may proceed as we have already shown for $G_p(W^2)$ and $G_n(W^2)$ and write for Δ_0, Δ_+

$$\begin{aligned} g\Delta_0 &= \frac{1}{\pi}\int_{M_p^2}^{\lambda^2 M_p^2} \frac{\mathrm{Im}K_p^{\pi^0}(W^2)dW^2}{W^2-M_p^2} - \frac{1}{\pi}\int_{M_n^2}^{\lambda^2 M_n^2} \frac{\mathrm{Im}K_n^{\pi^0}(W^2)dW^2}{W^2-M_n^2},\\ g\Delta_+ &= \frac{1}{\pi}\int_{M_p^2}^{\lambda^2 M_p^2} \frac{\mathrm{Im}K_p^{\pi^+}(W^2)dW^2}{W^2-M_p^2} - \frac{1}{\pi}\int_{M_n^2}^{\lambda^2 M_n^2} \frac{\mathrm{Im}K_n^{\pi^-}(W^2)dW^2}{W^2-M_n^2},\end{aligned} \quad \text{(III3)}$$

where we again include only the low-energy region with $\lambda^2\sim 3$. This implies that the only intermediate states contributing to $\mathrm{Im}K(W^2)$ are the $N\gamma$ and $N\pi$ states so that the shifts $\Delta_{0,+}$ are related to photopion production and the electromagnetic differences in the pion-nucleon scattering amplitude. Inserting just the pole terms from these amplitudes which are exact in the threshold limit we calculate the low-energy contribution.

The photon diagrams yield a contribution

$$\begin{aligned}\Delta_0^\gamma &= -\left(\frac{\alpha}{4\pi}\right)(1+\tfrac{3}{2}\kappa_p)\left(\ln\lambda^2 + \frac{1}{\lambda^2} - 1\right),\\ \Delta_+^\gamma &= \left(\frac{\alpha}{4\pi}\right)\left\{1 - \frac{1}{\lambda^2} - \frac{\kappa_p}{4}\left[3(\lambda^2-1) - \left(\frac{1}{\lambda^2}-1\right) + 4\Phi(1-\lambda^2)\right]\right\},\end{aligned} \quad \text{(III4)}$$

where we neglect terms proportional to $\kappa_p^2-\kappa_n^2$.

From intermediate pions there are contributions from the shifts $\Delta_{0,+}$ back on themselves and contributions from the nucleon mass shift δM. There are no contributions from the pion mass splittings and again we make the recoil approximation $\mu/M=0$ to simplify the calculation. We only keep $\mu/M\neq 0$, where it is required to prevent a divergence, and these terms will enter in calculating the influence of the nucleon mass shift δM. We find

$$\begin{aligned}\Delta_+^{\delta g} &= -\left(\frac{g^2}{4\pi}\right)\frac{1}{8\pi}\left\{2\Delta_0\left[1-\frac{1}{\lambda^2}+\ln(\lambda^2)\right] + \Delta_+[7(1-1/\lambda^2)+9\ln(\lambda^2)]\right\},\\ \Delta_0^{\delta g} &= -\left(\frac{g^2}{4\pi}\right)\frac{1}{8\pi}\left\{\Delta_0\left[5\left(1-\frac{1}{\lambda^2}\right)+\ln(\lambda^2)\right] + 4\Delta_+[1-1/\lambda^2+3\ln(\lambda^2)]\right\},\\ \Delta_+^{\delta M} &= \left(\frac{g^2}{4\pi}\right)\frac{\delta M}{2\pi M}\left[-12\ln(\lambda^2)+\frac{\ln\lambda^2}{\lambda^2-1}-1-\frac{9}{2}\left(1-\frac{1}{\lambda^2}\right)+12\ln\frac{\lambda^2-1}{2\mu/M}\right],\\ \Delta_0^{\delta M} &= -\left(\frac{g^2}{4\pi}\right)\frac{\delta M}{\pi M}\left[4\ln\frac{\lambda^2-1}{2\mu/M}-4\ln(\lambda^2)+\frac{1}{\lambda^2}-1\right].\end{aligned} \quad \text{(III5)}$$

The equations

$$\begin{aligned}\Delta_0 &= \Delta_0^\gamma + \Delta_0^{\delta g} + \Delta_0^{\delta M},\\ \Delta_+ &= \Delta_+^\gamma + \Delta_+^{\delta g} + \Delta_+^{\delta M},\end{aligned} \quad \text{(III6)}$$

are then an inhomogeneous linear system for Δ_0 and Δ_+. Setting $\lambda^2\sim 3$ and μ/M to its experimental value we solve Eq. (III6) and find

$$\Delta_0 + 2\Delta_+ \sim -6.7\delta M/M. \quad \text{(III7)}$$