Representation of the S Matrix by Regge Parameters

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We express the S matrix of complex angular momentum and positive energy by Regge poles, proving that a knowledge of the Regge poles enables one to determine the S matrix uniquely. The background integral in the Mandelstam-Sommerfeld-Watson transform cannot be made to vanish by closing the contour to the left. Furthermore, one cannot express the scattering amplitude by an infinite-series sum of Regge-pole terms where each term is given by the Khuri representation.

I. REPRESENTATION OF $S(\lambda,s)$ BY REGGE POLES

ONE of the most important questions in the theory of complex angular momentum is whether the scattering amplitude $A(s,t)$ is determined if the location of all Regge poles and the residue functions are given. This may point to a new way of constructing the S matrix.

We give here a representation of the S matrix of complex angular momentum by the Regge poles, proving that the scattering amplitude is uniquely determined once the Regge poles are given.

The unitary condition is

$$S(\lambda,s)S^*(\lambda^*,s)=1\,,\quad s\geq 0\,,$$

where $S(\lambda,s)$ is the S matrix of complex angular momentum $l=\lambda-\frac{1}{2}$. This implies that if λ_n is a pole of $S(\lambda,s)$, then λ_n^* is a zero of $S(\lambda,s)$. Furthermore, for positive energy, $S(\lambda,s)$ takes the asymptotic form[1]

$$S(\lambda,s)\xrightarrow[|\lambda|\to\infty]{}\begin{cases}1\,, & -\pi/2\leq\arg\lambda\leq\pi/2\,,\\ e^{2i\lambda\pi}\,, & \pi/2<\arg\lambda<3\pi/2\,.\end{cases}\tag{1}$$

Defining

$$T(\lambda,s)=[\partial S(\lambda,s)/\partial\lambda]S(\lambda,s)\,,\tag{2}$$

then

$$T(\lambda,s)\xrightarrow[|\lambda|\to\infty]{}\begin{cases}0\,, & -\pi/2\leq\arg\lambda\leq\pi/2\,,\\ 2i\pi\,, & \pi/2<\arg\lambda<3\pi/2\,.\end{cases}\tag{3}$$

From (3), we obtain

$$\frac{1}{2i\pi}\int_c\frac{T(\lambda',s)\,d\lambda'}{\lambda'-\lambda}=i\pi\,,\tag{4}$$

where c is a circle with its center at the origin and with infinite radius. If $S(\lambda,s)$, considered as a function of λ, has poles at $\lambda_n(s)$ and zeros at $\lambda_n^*(s)$, then (2) shows that $T(\lambda,s)$ has poles at $\lambda_n(s)$ with residue -1 and poles at $\lambda_n^*(s)$ with residue 1. Computing the left side of (4) by the Cauchy residue theorem, we get

$$T(\lambda,s)=i\pi+\sum_n\left[\frac{1}{\lambda-\lambda_n^*(s)}-\frac{1}{\lambda-\lambda_n(s)}\right].\tag{5}$$

We may integrate (5) from an arbitrary point λ_a to λ, obtaining

$$\ln\frac{S(\lambda,s)}{S(\lambda_a,s)}=i\pi(\lambda-\lambda_a)+\sum_n\ln\left(\frac{\lambda-\lambda_n^*(s)}{\lambda_a-\lambda_n^*(s)}\,\frac{\lambda_a-\lambda_n(s)}{\lambda-\lambda_n(s)}\right),$$

or

$$S(\lambda,s)=S(\lambda_a,s)e^{i\pi(\lambda-\lambda_a)}\prod_n\left(\frac{\lambda-\lambda_n^*(s)}{\lambda_a-\lambda_n^*(s)}\,\frac{\lambda_a-\lambda_n(s)}{\lambda-\lambda_n(s)}\right).\tag{6}$$

Equation (6) requires, in addition to $\lambda_n(s)$, one subtraction constant $S(\lambda_a,s)$, for the representation of the S matrix. We may take λ_a to be a large positive real number, and make use of (1) to obtain

$$S(\lambda,s)=\lim_{R\to\infty}e^{i\pi(\lambda-R)}\prod_n\left(\frac{\lambda-\lambda_n^*(s)}{R-\lambda_n^*(s)}\,\frac{R-\lambda_n(s)}{\lambda-\lambda_n(s)}\right).\tag{7}$$

Equation (7) shows that $S(\lambda,s)$ can be constructed once all $\lambda_n(s)$ are known.[2]

We may pause to consider how (5) can be consistent with (3). When $|\lambda|\to\infty$, we may neglect any finite number of terms in the summation of (5), since they contribute only to the order $1/\lambda$. Therefore, we may start with n large enough for the asymptotic form of $\lambda_n(s)$ to be valid.

Take the potential

$$V(r)=V_0\int_\mu^\infty\frac{e^{-\mu' r}}{r}e^{-\mu'}\,d\mu'\,;$$

then for the poles in the upper half-plane their asymptotic forms are given by[1]

$$\lambda_n(s)\ln(2\lambda_n(s)e^{-\pi i}/ke)=n\pi i\,,\tag{8}$$

and for those in the lower half-plane, they are given by

$$\lambda_{-n}(s)\ln[2\lambda_{-n}(s)e^{2\pi i}/ke]=-n\pi i\,.\tag{9}$$

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[1] H. Cheng and T. T. Wu, preceding paper, Phys. 144, 1232 (1966).

[2] It is trivial to generalize (7) for the case when s is complex. Instead of (1), we make use of

$$S(\lambda,s)\xrightarrow[|\lambda|\to\infty]{}\begin{cases}1+S_B(\lambda,s), & -\pi/2\leqslant\arg\lambda\leqslant\pi/2,\\ e^{2i\lambda\pi}, & \pi/2<\arg\lambda<3\pi/2,\end{cases}$$

with $S_B(\lambda,s)$, the Born approximation, which is not small.

The sum in (5), for large $|\lambda|$, can be approximated by

$$\sum_n \left[\frac{1}{\lambda-\lambda_n^*(s)}-\frac{1}{\lambda-\lambda_n(s)}\right] \underset{|\lambda|\to\infty}{\longrightarrow} \int_0^\infty dy \left[\frac{1}{\lambda-\lambda^*(y,s)}+\frac{1}{\lambda-\lambda^*(-y,s)}-\frac{1}{\lambda-\lambda(y,s)}-\frac{1}{\lambda-\lambda(-y,s)}\right], \quad (10)$$

where $\lambda(y,s)$ and $\lambda(-y,s)$ satisfy (8) and (9), respectively, with n replaced by y, a continuous variable. For the first term in (10), we change the independent variable to $\lambda^*(y,s)=ix$, and similarly for the other terms, obtaining, after neglecting terms of order $1/\lambda$,

$$\sum_n \left[\frac{1}{\lambda-\lambda_n^*(s)}-\frac{1}{\lambda-\lambda_n(s)}\right] \underset{|\lambda|\to\infty}{\longrightarrow} -i\int_0^\infty dx \left(\frac{1}{\lambda-ix}+\frac{1}{\lambda+ix}\right)=\begin{cases}-i\pi, & \mathrm{Re}\lambda>0,\\ i\pi, & \mathrm{Re}\lambda<0,\end{cases} \quad (11)$$

which together with (5), given (3).

Equation (7) may be a little clumsy to use. We may obtain another representation for $S(\lambda,s)$ by noticing that

$$S(\lambda,s)-1=O(e^{-\lambda\xi(s)}/\sqrt{\lambda}), \quad \mathrm{Re}\lambda\to\infty,$$

with

$$\cosh\xi(s)=1+\mu^2/2s,$$

μ being the lowest mass in the Yukawa potentials. Thus

$$\ln S(\lambda,s)=O(e^{-\lambda\xi(s)}/\sqrt{\lambda}), \quad \mathrm{Re}\lambda\to\infty.$$

Consequently, we have

$$\frac{1}{2\pi i}\int_c d\lambda' \frac{e^{\lambda'\xi(s)}\ln S(\lambda',s)}{\lambda'-\lambda}=0, \quad (12)$$

where c is a circle with infinite radius. Now $\ln S(\lambda,s)$ is an analytic function of λ, with branch points at the zeros $\lambda_n^*(s)$ and the poles $\lambda_n(s)$ of $S(\lambda,s)$. The branch cuts are chosen to lie from $-\infty$ to those branch points. Then we may evaluate the left side of (12) to obtain

$$\ln S(\lambda,s)=e^{-\lambda\xi(s)}\sum_n \int_{\lambda_n(s)}^{\lambda_n^*(s)} d\lambda' \frac{e^{\lambda'\xi(s)}}{\lambda'-\lambda}. \quad (13)$$

In (13), if λ lies on any of the contours, we may add or subtract an infinitesimal quantity to it to take it off the path. This is because adding $2\pi i$ to $\ln S(\lambda,s)$ does not change the value for $S(\lambda,s)$. Equation (13) again enables one to construct $S(\lambda,s)$ once all $\lambda_n(s)$ are given. Now, the contribution of a Regge pole $\lambda_n(s)$ to $S(\lambda,s)$ can be crudely estimated to be proportional to $e^{\xi(s)\,\mathrm{Re}\lambda_n(s)}$, thus the contributions of Regge poles in the left-hand plane are cut off rapidly. Furthermore, (13) automatically incorporates the asymptotic behavior for $\mathrm{Re}\lambda\to\infty$ and the threshold behavior for $s\to 0$. Therefore, (13) may be convenient to use for practical purposes.

A representation for $S(\lambda,s)$ in infinite product form was obtained by Desai and Newton.[3] They did not have the asymptotic form (1) and had to make a guess. As a result, their expression is similar to (6), but requires two subtractions.

Equation (13) may be written in more familiar form. The incomplete gamma function is defined as

$$\Gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt.$$

Then (13) can be expressed by $\Gamma(0,x)$:

$$\ln S(\lambda,s)=\sum_n [\Gamma(0,\,\xi(s)\lambda-\xi(s)\lambda_n(s)) -\Gamma(0,\,\xi(s)\lambda-\xi(s)\lambda_n^*(s))]. \quad (14)$$

II. A FORMULA SATISFIED BY $S(\lambda,s)$

The representations for $S(\lambda,s)$ given previously were obtained with the aid of the unitarity condition. It is not known if one can obtain a representation of the Mittag-Leffler type for $S(\lambda,s)$, based on the meromorphy of $S(\lambda,s)$ and the asymptotic form (1). Nevertheless, we may make use of the mirror property

$$S(n,s)=S(-n,\,s), \quad n=0,1,2,\cdots \quad (15)$$

to obtain a formula satisfied by $S(\lambda,s)$. We have

$$\frac{1}{2\pi i}\int_c \frac{\lambda' S(\lambda',s)e^{-i\lambda'\pi}}{\sin\lambda'\pi(\lambda'^2-\lambda^2)}d\lambda'=0, \quad (16)$$

where c is an infinite circle as before. Applying the Cauchy residue theorem to evaluate the left side of (16), and making use of (15), we have

$$S(\lambda,s)e^{-i\lambda\pi}-S(-\lambda,\,s)e^{i\lambda\pi} =-2\sum_n \frac{\lambda_n(s)\,\sin\lambda\pi}{\sin\lambda_n(s)\pi}\,\frac{r_n(s)e^{-i\lambda_n(r)\pi}}{\lambda_n^2(s)-\lambda^2}, \quad (17)$$

where $r_n(s)=\mathrm{Res}S(\lambda,s)|_{\lambda=\lambda_n(s)}$. Equation (17) relates $S(\lambda,s)$ to $S(-\lambda,\,s)$ by a sum of Regge pole terms.

[3] B. P. Desai and R. G. Newton, Phys. Rev. **129**, 1445 (1963).

[4] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento **23**, 954 (1962).

We also know that[4]

$$S(\lambda,s)e^{-i\lambda\pi}-S(-\lambda,s)e^{i\lambda\pi}=-\frac{4k\lambda}{f(\lambda,ke^{-i\pi})f(-\lambda,ke^{-i\pi})}. \quad (18)$$

Combining (17) and (18), we get

$$\frac{1}{f(\lambda,ke^{-i\pi})f(-\lambda,ke^{-i\pi})}=\frac{1}{2k}\sum_n \frac{\sin\lambda\pi}{\sin\lambda_n(s)\pi}\frac{r_n(s)e^{-i\lambda_n(s)\pi}}{\lambda_n^2(s)-\lambda^2}\frac{\lambda_n(s)}{\lambda}. \quad (19)$$

III. CONSEQUENCES ON THE SCATTERING AMPLITUDE $A(s,t)$

With the asymptotic form (1), it is trivial to show that the background term in the Mandelstam-Watson-Sommerfeld transform equation[5] cannot be made to vanish by closing the contour to the left. The infinite-series sum of Regge-pole terms, with each term given by the Khuri representation,[6] is not convergent.

[5] S. Mandelstam, Ann. Phys. (N. Y.) **19**, 254 (1962).
[6] N. N. Khuri, Phys. Rev. **130**, 429 (1963).

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Some Consequences of Off-Shell Unitarity*

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The implications of off-shell unitarity on the structure of the off-shell, nonrelativistic, two-particle partial-wave amplitudes are investigated. It is found that the unitarity conditions along with time-reversal invariance imply certain useful factorization properties of the off-shell amplitudes.

It has been established in two quite distinct ways that in potential scattering the off-shell, two-particle partial-wave amplitudes exhibit certain factorization properties with respect to the off-shell momenta.[1,2] The possible usefulness of these features in constructing approximations in three-body problems has also been suggested.[1,3–5]

Both existing treatments[1,2] of this problem have a common defect in that the factorization appears to emerge in a somewhat accidental manner. Moreover, both derivations are couched firmly in the language and formalism of potential scattering theory and thus the possible generality of these results is obscured.[6,7]

In the present note we will show that the factorization properties of the complete off-shell partial-wave amplitudes follow from off-shell unitarity[8,9] and time-reversal invariance, while the factorization of the half-off-shell amplitude follows from slightly weaker conditions. This analysis constitutes something analogous to the phase-shift parametrization which follows from on-shell unitarity and has the mutual advantage that no reference to potentials, wave functions, or (dynamical) integral equations is necessary.

Let us introduce a partial-wave decomposition of the matrix elements of the transition operator in the c.m. system,[10]

$$\langle \mathbf{p}'|t_k|\mathbf{p}\rangle=(1/4\pi)\sum_l (2l+1)t_k{}^l(p',p)P_l(\cos\theta)\,,$$

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[1] H. P. Noyes, Phys. Rev. Letters **15**, 538 (1965).
[2] K. L. Kowalski and D. Feldman, J. Math. Phys. **4**, 507 (1963); K. L. Kowalski, Phys. Rev. Letters **15**, 798, 908 (1965).
[3] It has been pointed out (Ref. 4) that great care must be exercised when employing approximate off-shell amplitudes in Faddeev-type three-body calculations. The crucial requirement is that these amplitudes satisfy the unitary condition in the unphysical region as well as in the physical region. The (separable) approximations considered in Refs. 1 and 2 satisfy only physical unitarity; the further application of unphysical unitarity leads to restrictions on the analytic properties of the (approximate) quantities (in Refs. 1 and 2) which correspond to the functions F and R in the present article.
[4] J. L. Basdevant (private communication). See also J. L. Basdevant and R. E. Kreps, Phys. Rev. (to be published).
[5] The factorization properties were exploited in an impulse approximation calculation of nucleon-deuteron scattering. See K. L. Kowalski and D. Feldman, Phys. Rev. **130**, 276 (1963).
[6] We have in mind here the extension to the relativistic problem. See Ref. 7 and the works cited therein.
[7] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B167 (1965).
[8] The first clear formulation of off-shell unitarity appears to have been given by Lovelace (Ref. 9). See also Ref. 7.
[9] C. Lovelace, in *Lectures at the 1963 Edinburgh Summer School*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964); Phys. Rev. **135**, B1225 (1964).
[10] In the entirety of this paper we will be concerned only with the scattering of two massive, spinless, nonrelativistic particles. The extension to more complicated nonrelativistic two-particle scattering problems, for example particles with spin, is, for the most part, primarily a matter of introducing an appropriate matrix notation. We presuppose translational, Galilean, and rotational invariance and we use units $(2\mu/\hbar^2)=1$, where μ is the reduced two-particle mass.