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Asymptotic Form of the S Matrix for Large Angular Momentum in the Left Half-Plane

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Starting with the Schrödinger equation, we prove that for all energies, $S(\lambda,s)$ approaches $e^{2i\lambda\pi}$ as $|\lambda|$ becomes large in the direction $\frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi$, for a class of potentials. These include the square-well potential, the cut-off Coulomb potential, a single Yukawa potential, and a superposition of Yukawa potentials of the form $\int_\mu^\infty (e^{-\mu' r}/r)\, e^{-\mu'}\, d\mu'$. The asymptotic forms of the Regge-pole parameters α_n and β_n are derived. We found that $\arg\lambda_n$ approaches $\frac{1}{2}\pi$ or $\frac{3}{2}\pi$ as $n\to\infty$, and β_n is proportional to $1+e^{2\pi i\alpha_n}$, which grows exponentially for the Regge poles in the lower half plane. The asymptotic forms for the Jost functions and the V function are also given. A general proof for the asymptotic formula $S(\lambda,s)\to e^{2\pi i\lambda}$ as $|\lambda|\to\infty$, $\frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi$, is also outlined.

I. INTRODUCTION

To answer several crucial questions in the theory of complex angular momentum, it is necessary to know the asymptotic form of $S(\lambda,s)$, the S matrix of complex angular momentum $l=\lambda-\frac{1}{2}$, as $|\lambda|\to\infty$ in the direction $\frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi$. These questions include the distribution of Regge poles in the left hand λ plane, the representation of the S matrix by the Regge-pole parameters, and the possibility of eliminating the contour integral in the Sommerfeld-Watson transform. The first question is studied in this paper, while the answer to the latter two questions is presented elsewhere.[1]

II. CUTOFF POTENTIALS

We start by considering a simple example, the square-well-potential case, which offers the suggestion that

$$S(\lambda,s)\to e^{2i\lambda\pi},\quad |\lambda|\to\infty\,,\quad \tfrac{1}{2}\pi<\arg\lambda<\tfrac{3}{2}\pi\,, \tag{1}$$

where $S(\lambda,s)$ is the S matrix of complex angular momentum $l=\lambda-\frac{1}{2}$.

For the square-well potential

$$V(r)=V_0\,,\quad r<a$$
$$\quad\;\;=0\,,\quad r>a$$

we have

$$S(\lambda,s)=-\frac{\eta H_\lambda^{(2)}(ka)J_\lambda'(\eta a)-kJ_\lambda(\eta a)H_\lambda^{(2)\prime}(ka)}{\eta H_\lambda^{(1)}(ka)J_\lambda'(\eta a)-kJ_\lambda(\eta a)H_\lambda^{(1)\prime}(ka)}\,, \tag{2}$$

where $\eta=(k^2-V_0)^{1/2}$ and $J_\lambda'(x_0)=(d/dx)J_\lambda(x)|_{x=x_0}$, etc. As $|\lambda|\to\infty$ with Z fixed, we have

$$J_\lambda(Z)\to(\tfrac{1}{2}Z)^\lambda\left[1-\frac{(\frac{1}{2}Z)^2}{1+\lambda}+\cdots\right]\Big/\Gamma(1+\lambda)\,. \tag{3}$$

Since we have

$$H_\lambda^{(1)}(Z)=[J_{-\lambda}(Z)-e^{-\lambda\pi i}J_\lambda(Z)]/(i\sin\lambda\pi)\,, \tag{4}$$

$$H_\lambda^{(2)}(Z)=[e^{\lambda\pi i}J_\lambda(Z)-J_{-\lambda}(Z)]/(i\sin\lambda\pi)\,, \tag{5}$$

we get, as $|\lambda|\to\infty$,

$$H_\lambda^{(1)}(Z)\to\begin{cases}J_{-\lambda}(Z)/(i\sin\lambda\pi)\,, & -\frac{1}{2}\pi<\arg\lambda<\frac{1}{2}\pi\\ -e^{-\lambda\pi i}J_\lambda(Z)/(i\sin\lambda\pi)\,, & \frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi\end{cases}$$

and

$$H_\lambda^{(2)}(Z)\to\begin{cases}-J_{-\lambda}(Z)/(i\sin\lambda\pi)\,, & -\frac{1}{2}\pi<\arg\lambda<\frac{1}{2}\pi\\ e^{\lambda\pi i}J_\lambda(Z)/(i\sin\lambda\pi)\,, & \frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi\end{cases}$$

We then easily obtain (1) for all k.

We note that this asymptotic form of $S(\lambda,s)$ is independent of V_0, a, and s.

The same asymptotic form for $S(\lambda,s)$ is obtained for cut-off Coulomb potentials. The proof is similar and will not be repeated. In fact, if we take any potential which has the power series expansion $\sum_{n=-1} a_n r^n$ at the origin so that the wave function is meromorphic for all l, and have the potential cutoff at $r=a$, one can prove that (1) always follow trivially from (3), (4), and (5).

III. POTENTIAL OF YUKAWA TYPE

The radial Schrödinger equation reads

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-V(r)\right]\Psi(k,l,r)=0\,, \tag{6}$$

where $rV(r)$ has power series expansion at the origin.

The wave function $\Psi(k,l,r)$ is defined by the boundary condition

$$\Psi(k,l,r)\to r^{l+1}\,,\quad r\to 0\,. \tag{7}$$

We note that for $\mathrm{Re}\,l<-\frac{1}{2}$, (7) is insufficient to determine a unique solution for (6). Instead we have to analytically continue the solution from the right half

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[1] H. Cheng, following paper, Phys. Rev. 144, 1237 (1966).

plane. To avoid this difficulty we define

$$Z=\ln r,\quad \phi(k,l,Z)=r^{-1/2}\Psi(k,l,r);\tag{8}$$

then we have

$$\left[\frac{d^2}{dZ^2}+k^2e^{2Z}-\lambda^2-e^{2Z}V(e^Z)\right]\phi(k,l,Z)=0,\tag{9}$$

where $\lambda=l+\frac{1}{2}$. From (7) the boundary condition for $\phi(k,l,Z)$ is

$$\phi(k,l,Z)\to e^{\lambda Z},\quad \text{Re}Z\to-\infty,\tag{10}$$

which strongly resembles the scattering problem of the one-dimensional Schrödinger equation, with λ playing the role of ik. If we take a path such that

$$\text{Re}(\lambda Z)\leq 0$$

as $\text{Re}Z\to-\infty$, then (10) is a well defined boundary condition since $|e^{\lambda Z}|\ll|e^{-\lambda Z}|$ in this limit. The WKB method developed for the high-energy scattering problem of the one-dimensional Schrödinger equation[2] can almost be directly applied.

A. Superposition of Yukawa Potentials

We first take the potential

$$V(r)=V_0e^{\mu}\int_{\mu}^{\infty}\frac{e^{-\mu' r}}{r}e^{-\mu'}d\mu'=V_0\frac{e^{-\mu r}}{r(r+1)}.$$

Then we have

$$\left[\frac{d^2}{dZ^2}+k^2e^{2Z}-\lambda^2-V_0\frac{e^{-\mu e^Z}e^Z}{(e^Z+1)}\right]\phi(k,l,Z)=0.\tag{11}$$

As $|\lambda|\to\infty$, the WKB solutions

$$\phi_{\pm}(k,l,Z)=\exp\left[\pm\int_0^Z\{\lambda^2-k^2e^{2t}+V_0e^{-\mu e^t}e^t/(e^t+1)\}^{1/2}dt\right]\Big/\left[\lambda^2-k^2e^{2Z}+V_0e^{-\mu e^Z}e^Z/(e^Z+1)\right]^{1/4}\tag{12}$$

hold, excluding the neighborhoods of $Z_n=(2n+1)\pi i$, $n=0,\pm1,\pm2,\cdots$, where the potential has a pole, and the turning points. Near Z_n, we have

$$\left[\frac{d^2}{dZ^2}-\lambda^2-\frac{V_0e^{\mu}}{\xi_n}\right]\phi(k,l,Z)\approx0,\tag{13}$$

where $\xi_n=Z-Z_n$. We may solve (13) to obtain

$$\phi(k,l,Z)=-2\lambda\xi_ne^{\lambda Z}\Psi\left(1-\frac{V_0e^{\mu}}{2\lambda},2;2e^{-i\pi}\lambda\xi_n\right),\tag{14}$$

where $\Psi(a,c,x)$ is a solution of the confluent hypergeometric equation with parameters a and c.[3] This particular solution is chosen since[3]

$$\Psi(a,c,x)\to x^{-a},\quad |x|\to\infty,\quad -\tfrac{3}{2}\pi<\arg x<\tfrac{3}{2}\pi,$$

and as a result, (14) is consistent with the boundary condition (10). We note at this point that[3] $\Psi(a,c,x)$ has a branch point at $x=0$.

To get the S matrix, one has to obtain the solution at $|r|=\infty$, and in terms of Z, at $Z=\infty$. To continue the WKB solution from $\text{Re}Z=-\infty$ to $Z=\infty$, we have to take a path which stays on the correct sheet. Let us first consider $\frac{1}{2}\pi<\arg\lambda<\pi$. Then in the shaded region of the figure, $\arg(\lambda Z)\leq0$, $\text{Re}Z<0$, and $\phi(k,l,Z)$ can be approximated by

$$\phi(k,l,Z)=\phi_+(k,l,Z),\tag{15}$$

which satisfies the boundary condition (10). Since we are not allowed to pass through the branch cut, to get to $Z=\infty$ we have to pass the imaginary axis somewhere between $i\pi$ and $-i\pi$. We have to match (14) and (15) at $Z=i\pi$. Now, from (14) and the asymptotic form for $\Psi(a,c,x)$ when $\arg x\geq\frac{3}{2}\pi$, we obtain, when Z is in the right half plane

$$\phi(k,l,Z)\to e^{\lambda Z}-(\pi iV_0e^{\mu}/\lambda)e^{2\pi i\lambda}e^{-\lambda Z},\tag{16}$$

and hence

$$\phi(k,l,Z)=a_1\phi_+(k,l,Z)+a_2\phi_-(k,l,Z),$$

where a_1 and a_2 can be determined from (16) and (12). We may observe, however, that when $\xi_n\gg V_0e^{\mu}/\lambda^2$, the potential can always be neglected and $\phi_{\pm}$ are simply the WKB approximation for $J_{\pm\lambda}(kr)$. Thus we conclude that

$$r^{-1/2}\psi(k,l,r)\to(\tfrac{1}{2}k)^{-\lambda}\Gamma(1+\lambda)J_{\lambda}(kr)-\pi iV_0e^{\mu}e^{2\pi i\lambda}(\tfrac{1}{2}k)^{\lambda}\lambda^{-1}\Gamma(1-\lambda)J_{-\lambda}(kr).\tag{17}$$

From the asymptotic form for $J_{\pm\lambda}(Z)$ as $|Z|\to\infty$, and the definition of the Jost function $f(l,k)$

$$\psi(k,l,r)\to(2ik)^{-1}[f(l,k)e^{ikr}-f(l,ke^{-\pi i})e^{-ikr}],\quad r\to\infty,$$

we get, as $|l|\to\infty$, $\frac{1}{2}\pi<\arg l<\frac{3}{2}\pi$,

$$f(l,k)\to2\Gamma(1+\lambda)(\tfrac{1}{2}k)^{-l}\pi^{-\frac{1}{2}}e^{-\frac{1}{2}i\pi l}+2\pi^{1/2}V_0e^{\mu}e^{2\pi i\lambda}\lambda^{-1}(\tfrac{1}{2}k)^{l+1}\Gamma(1-\lambda)e^{\frac{1}{2}i\pi l},\tag{18}$$

$$f(l,ke^{-\pi i})\to2\Gamma(1+\lambda)(\tfrac{1}{2}k)^{-l}\pi^{-1/2}e^{\frac{1}{2}i\pi l}-2\pi^{1/2}V_0e^{\mu}e^{2\pi i\lambda}\lambda^{-1}(\tfrac{1}{2}k)^{l+1}\Gamma(1-\lambda)e^{-\frac{1}{2}i\pi l},\tag{19}$$

and

$$S(\lambda,s)=f(l,k)e^{il\pi}/f(l,ke^{-\pi i})\to\frac{1+\frac{1}{2}e^{2\pi i\lambda}V_0e^{\mu}(\frac{1}{2}k)^{2\lambda}\Gamma^2(-\lambda)}{1+\frac{1}{2}V_0e^{\mu}(\frac{1}{2}k)^{2\lambda}\Gamma^2(-\lambda)}\to e^{2\pi i\lambda}.\tag{20}$$

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[2] T. T. Wu (to be published).

[3] Bateman Manuscript Project, *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, Eqs. 6.5(2), 6.13.1(1), and 6.7.1(13).

The last step of (20) is justified as long as the term 1 both in the numerator and in the denominator can be neglected. This is true as long as $|\lambda|\to\infty$ in the direction $\frac{1}{2}\pi-\epsilon<\arg\lambda\le\pi$ and for all k^2. It should be noticed that ϵ can be arbitrarily small but cannot be zero. In fact, we shall show in Sec. IV that there are infinitely many Regge poles in the direction $\arg\lambda=\frac{1}{2}\pi$ and $\arg\lambda=\frac{3}{2}\pi$.

We may repeat the same argument and obtain

$$S(\lambda,s)\to\frac{1+\frac{1}{2}V_0e^{\mu}(\frac{1}{2}k)^{2\lambda}\Gamma^2(-\lambda)}{1+\frac{1}{2}e^{-2\pi i\lambda}V_0e^{\mu}(\frac{1}{2}k)^{2\lambda}\Gamma^2(-\lambda)}\to e^{2\pi i\lambda},\quad(21)$$

valid for $|\lambda|\to\infty$, $\pi<\arg\lambda<\frac{3}{2}\pi$, and for all s.

The Y function, defined by

$$S(\lambda,s)=\frac{Y(\lambda,s)+s^{\lambda}e^{i\pi\lambda}}{Y(\lambda,s)+s^{\lambda}e^{-i\pi\lambda}},$$

takes the asymptotic form, as $|\lambda|\to\infty$

$$Y(\lambda,s)\to e^{-\pi i\lambda}e^{-\mu}\frac{2^{2\lambda+1}}{V_0\Gamma^2(-\lambda)},\quad\frac{1}{2}\pi<\arg\lambda<\pi,\quad(22)$$

$$\to e^{\pi i\lambda}e^{-\mu}\frac{2^{2\lambda+1}}{V_0\Gamma^2(-\lambda)},\quad\pi<\arg\lambda<\frac{3}{2}\pi.\quad(23)$$

We note that the right-hand sides of (22) and (23) are independent of s.

B. Single Yukawa Potential

Next, we take the potential

$$V(r)=V_0(e^{-\mu r}/r),$$

then in place of (11), we have

$$\left[\frac{d^2}{dZ^2}+k^2e^{2Z}-\lambda^2-V_0e^{-\mu e^Z}e^Z\right]\phi(k,l,Z)=0.\quad(24)$$

The potential has no poles and the WKB solutions are valid everywhere excluding the neighborhood of turning points. This problem can be treated in a way similar to the case of the one-dimensional Gaussian potential.[2] There are two sets of turning points. The first set occurs at

$$k^2e^{2Z}\approx\lambda^2,$$

with $V_0e^{-\mu e^Z}e^Z$ small. For example, when $k>0$ and λ in the left-hand plane, they are

$$e^Z\approx-\lambda/k.$$

These turning points have nothing to do with the potential. They exist even when $V(r)=0$ and matching $\phi_{\pm}(k,l,Z)$ at those points gives us the known asymptotic form for $J_{\pm}(kr)$ at large r. The other set of turning points is at

$$\lambda^2+V_0e^{-\mu e^Z}e^Z\approx0,$$

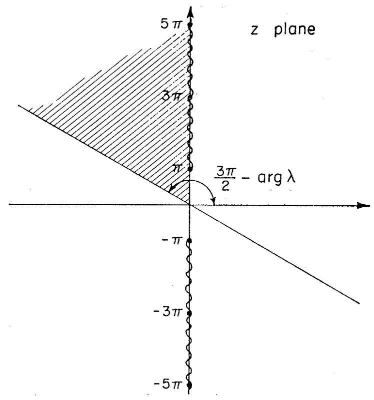

FIG. 1. Position of the poles of the potential and the branch cuts of the solution.

or

$$e^Z\approx-\frac{1}{\mu}\ln\left(\frac{-\lambda^2}{V_0}\right)-\frac{2n\pi i}{\mu},\quad n=0,\ \pm1,\ \pm2,\ \cdots.$$

We plot the curves

$$e^Z=-\frac{1}{\mu}\ln\left(-\frac{\lambda^2}{V_0}\right)-\frac{2t\pi i}{\mu}$$

in Fig. 2. The turning points correspond to $t=n$ at these curves.

Near the turning point Z_n, we have

$$\left[\frac{d^2}{d\xi_n{}^2}+\lambda^2\ln(\lambda^2)\xi_n\right]\phi(k,l,Z)\approx0,\quad(25)$$

where $\xi_n=Z-Z_n$. Solving (25) we get

$$\phi(k,l,Z)=\xi_n{}^{1/2}H_{1/3}{}^{(1)}\left(\tfrac{2}{3}(\lambda^2\ln\lambda^2)^{1/2}\xi_n{}^{3/2}e^{-i\pi}\right).\quad(26)$$

Equation (26) satisfies (25) as well as the boundary condition (10). We shall first concentrate on the region $\frac{1}{2}\pi<\arg\lambda<\pi$. Then in the shaded region of Fig. 2, $\phi_+(k,l,Z)$ is a good approximation to the wave function. To continue this solution to the region $Z=\infty$, we pass by the turning points where the WKB solutions fail, and we should match (26) with the WKB solutions at

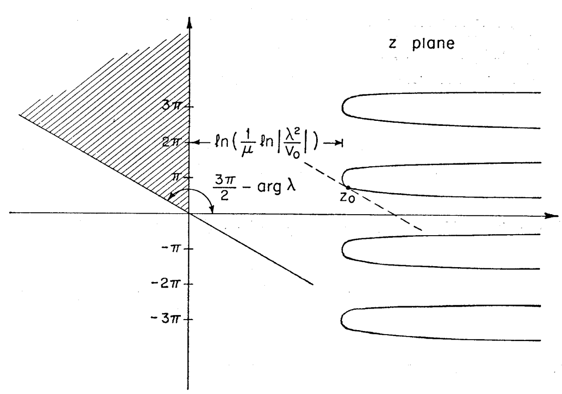

FIG. 2. Position of the turning points for single Yukawa potential.

each of the turning points. We shall only do this at the turning point which gives the largest correction, the others being smaller as $|\lambda|\to\infty$. This occurs at the point Z_0 where the tangent to the curve is parallel to the line $\arg Z=\frac{3}{2}\pi-\arg\lambda$, since $|e^{\lambda Z}|$ takes the largest value at this point of the curve. We have

$$Z_0\approx\ln\left(\frac{1}{\mu}\ln|\lambda^2/V_0|\right)+i(\arg\lambda-\pi).$$

Now, as $\xi\to\infty$, (26) gives

$$\phi(k,l,Z)\to\exp[-i\tfrac{2}{3}(\lambda^2\ln\lambda^2)^{1/2}\xi^{3/2}]+i\exp[i\tfrac{2}{3}(\lambda^2\ln\lambda^2)^{1/2}\xi^{3/2}]. \quad (27)$$

We obtain

$$\phi(k,l,Z)=\phi_+(k,l,Z)+i\times\exp\left[2\int_0^{Z_0}(\lambda^2+V_0e^{-\mu e^t}e^t-k^2e^{2t})^{1/2}dt\right]\times\phi_-(k,l,Z), \quad (28)$$

where

$$\phi_\pm(k,l,Z)=(\lambda^2+V_0e^{-\mu e^Z}e^Z-k^2e^{2Z})^{-1/4}\times\exp\left[\pm\int_0^Z(\lambda^2+V_0e^{-\mu e^t}e^t-k^2e^{2t})^{1/2}dt\right].$$

Again, $\phi_\pm(k,l,Z)$ may be regarded as $J_{\pm\lambda}(kr)$, and, estimating

$$\int_0^{Z_0}(\lambda^2+V_0e^{-\mu e^t}e^t-k^2e^{2t})^{1/2}dt\approx\lambda Z_0\approx\lambda\ln\ln\lambda+i\lambda(\arg\lambda-\pi),$$

we obtain, for $|l|\to\infty$

$$f(l,k)\to2\pi^{-1/2}(\tfrac{1}{2}k)^{-l}\Gamma(1+\lambda)e^{-i\pi l/2}+2\pi^{-1/2}(\tfrac{1}{2}k)^{l+1}\times\Gamma(1-\lambda)(\ln|\lambda|)^{2\lambda}e^{2i\lambda\arg\lambda}e^{-3i\pi l/2}, \quad (29)$$

$$f(l,ke^{-\pi i})\to2\pi^{-1/2}(\tfrac{1}{2}k)^{-l}\Gamma(1+\lambda)e^{i\pi l/2}-2\pi^{-1/2}(\tfrac{1}{2}k)^{l+1}\times\Gamma(1-\lambda)(\ln|\lambda|)^{2\lambda}e^{2i\lambda\arg\lambda}e^{-5i\pi l/2}, \quad (30)$$

valid for $\frac{1}{2}\pi<\arg l<\pi$ and all k. Similar expression can be obtained for $\pi<\arg l<\frac{3}{2}\pi$.

The S matrix again approaches

$$S(\lambda,s)\to e^{2i\lambda\pi},$$

valid for $\frac{1}{2}\pi<\arg\lambda<\frac{3}{2}\pi$, and for all s, as can be easily obtained from (30).

IV. THE ASYMPTOTIC BEHAVIOR FOR THE REGGE-POLE PARAMETERS

The Jost function differs slightly in each separate case, as we have seen. However, the qualitative features of the Regge pole parameters are the same in all cases. We shall therefore base our discussion on (20) and (21).

The Regge poles in the upper half-plane take the asymptotic form

$$\alpha_n\to n\pi i(\ln n)^{-1}\left[1-\frac{\ln(2\pi k^{-1}e^{-1}e^{-\frac{1}{2}i\pi})}{\ln n}+\frac{\ln\ln n}{\ln n}\right]+O\left(\frac{n}{(\ln n)^3}\right),\quad\text{as}\quad n\to+\infty, \quad (31)$$

and

$$\mathrm{Res}S(\lambda,s)|_{\lambda=\lambda_n}\to(2\ln n)^{-1}\left[1-\frac{\ln(2\pi k^{-1}e^{-\frac{1}{2}\pi i})}{\ln n}+\frac{\ln\ln n}{\ln n}\right]+O\left(\frac{n}{(\ln n)^3}\right),\quad\text{as}\quad n\to+\infty, \quad (32)$$

and in the lower half-plane, take the asymptotic form

$$\alpha_n\to-n\pi i(\ln n)^{-1}\left[1-\frac{\ln(2\pi k^{-1}e^{-1}e^{3i\pi/2})}{\ln n}+\frac{\ln\ln n}{\ln n}\right]+O\left(\frac{n}{(\ln n)^3}\right),\quad n\to\infty \quad (33)$$

and

$$\mathrm{Res}S(\lambda,s)|_{\lambda=\lambda_n}\to e^{2\pi i\alpha_n}(2\ln n)^{-1}\left[1-\frac{\ln(2\pi k^{-1}e^{3i\pi/2})}{\ln n}+\frac{\ln\ln n}{\ln n}+O\left(\frac{1}{(\ln n)^2}\right)\right],\quad n\to\infty \quad (34)$$

which blows up exponentially as $n\to\infty$.

We also see that as $n\to\infty$

$$\arg\alpha_n\to\tfrac{1}{2}\pi,\ \text{upper half-plane},$$
$$\to\tfrac{3}{2}\pi,\ \text{lower half-plane}.$$

V. SUMMARY

We have succeeded in establishing Eq. (1) for two potentials of Yukawa type. Although our calculation may appear involved, it is actually rather simple. For the purpose of clarification we shall reiterate the few key arguments by applying them to a general potential of Yukawa type.

Let us consider a potential of the form

$$V(r)=\int_\mu^\infty\frac{e^{-\mu' r}}{r}\rho(\mu')d\mu'.$$

There are two possibilities: (1) $\rho(\mu)$ vanishes faster than any exponential function $e^{-\mu a}$ as $\mu\to\infty$; then $rV(r)$ has no singularity in the entire r plane. (2) $\rho(\mu)$ vanishes no faster than $e^{-\mu a}$ for some a; then $rV(r)$ has singularities in the complex r plane.

In the first case, $r^2V(r)$ is not bounded as $r\to\infty$; otherwise, the entire function $r^2V(r)$ will be a constant. Therefore, turning points, given by $r^2V(r)\approx-\lambda^2$, can be found. There may be many turning points. Let r_0 be one of these points, and $Z_0=\ln r_0$.

As before, we shall consider the Schrödinger equation in the form (9), with the boundary condition

$$\phi \to e^{\lambda Z} \tag{35}$$

as $|Z|\to\infty$, in the direction $\mathrm{Re}(\lambda Z)\leq 0$.

When $|\lambda|\to\infty$, the WKB solution ϕ_+, satisfying (35), is a good approximation, where

$$\phi_\pm=\frac{1}{\sqrt{p(Z)}}\exp\left(\pm\int_0^Z p(t)dt\right), \tag{36}$$

with

$$p(Z)=[\lambda^2+e^{2Z}V(e^Z)-k^2e^{2Z}]^{1/2}.$$

The solution ϕ_+ continues to be a good approximation in the region $\mathrm{Re}(\lambda Z)\leq 0$, until a turning point Z_0 is encountered. In the neighborhood of Z_0, Eq. (9) is approximated by

$$[d^2/d\xi^2+a\xi]\phi\approx 0, \tag{37}$$

where $\xi=Z-Z_0$, and

$$a=-(d/dZ)[e^{2Z}V(e^Z)]|_{Z=Z_0}.$$

Equation (37), just like Eq. (25) can be solved to give

$$\phi=\xi^{1/2}H_{1/3}(\tfrac{2}{3}a^{1/2}\xi^{3/2}e^{-i\pi}), \tag{38}$$

which satisfies (35) in the limit $\xi\to\infty$, and we obtain an equation similar to (28):

$$\phi\approx\phi_+ + ie^{2\lambda Z_0}\phi_-. \tag{39}$$

If there are many turning points, we should match the solution at each turning point. As a first approximation, however, we only need to do so at the turning point which gives the maximum contribution. From (39), this turning point occurs at the point $\mathrm{Re}(\lambda Z_0)$ is maximum, while satisfying $\mathrm{Re}(\lambda Z_0)\leq 0$.

Afterwards, $\phi_\pm$ can be replaced by $(\frac{1}{2}k)^{\mp\lambda}\Gamma(1\pm\lambda)\times J_{\pm\lambda}(kr)$, and we obtain

$$\phi\to(\tfrac{1}{2}k)^{-\lambda}\Gamma(1+\lambda)J_\lambda(kr) + ie^{2\lambda Z_0}(\tfrac{1}{2}k)^{\lambda}\Gamma(1-\lambda)J_{-\lambda}(kr). \tag{40}$$

If $|Z_0|\ll|\lambda|$, as $|\lambda|\to\infty$, then (40) shows that ϕ is essentially proportional to $J_{-\lambda}(kr)$ in the region $kr\gg|\lambda|$. Therefore the asymptotic form (1) holds. The single Yukawa potential belongs to this case.

In the second case, $V(r)$ has singularities in the complex r plane, and the WKB solution fails in the neighborhood of these points. We may match the solution at each singular point. However, as in the first case, we only need to do so at the singular point which gives the maximum contribution. If the singularity is a simple pole, it is identical to the case presented in Sec. IIIA, and (1) is verified. If the singularity is a double pole, the differential equation in the neighborhood of the singularity Z_0 takes the form

$$\left[\frac{d^2}{d\xi^2}-\lambda^2-\frac{a}{\xi^2}\right]\phi\approx 0, \tag{41}$$

where $\xi=Z-Z_0$. Equation (41) can be solved to give

$$\phi\approx\xi^{1/2}H_{(1/4+a)^{1/2}}(-i\lambda\xi). \tag{42}$$

As $\xi\to+\infty$, (42) is matched by

$$\phi=\phi_+-2ie^{2\lambda Z_0}\cos(\pi(\tfrac{1}{4}+a)^{1/2})\phi_-. \tag{43}$$

As before, $\phi_\pm$ can be replaced by $(\frac{1}{2}k)^{\mp\lambda}\Gamma(1\pm\lambda)J_{\pm\lambda}(kr)$, thus

$$\phi=(\tfrac{1}{2}k)^{-\lambda}\Gamma(1+\lambda)J_\lambda(kr)-2ie^{2\lambda Z_0}\times\cos(\pi(\tfrac{1}{4}+a)^{1/2})\Gamma(1-\lambda)(\tfrac{1}{2}k)^{\lambda}J_{-\lambda}(kr) \tag{44}$$

and (1) is again obtained.

The asymptotic form (1) is therefore rather general. It is essentially due to the fact that $J_{\pm\lambda}(kr)$ is related to $r^{\pm\lambda}$ by a gamma function, which goes faster than exponentially as $|\lambda|\to\infty$. However, Eq. (1) may fail to hold in a few exceptional cases. One such case is given elsewhere.[4] Another example is provided by (44); if

$$\pi(\tfrac{1}{4}+a)^{1/2}=(n+\tfrac{1}{2})\pi,\quad n=0,\ \pm1,\ \pm2,\ \cdots,$$

then the second term in the right side of (44) vanishes, instead of being the dominant term. It may happen that the coefficient of $J_{-\lambda}(kr)$ vanishes exactly for some potential; in this case (1) would fail. This phenomenon is related to the reflectionless potential: if for some potential no scattering occurs, then $S(\lambda,k)=1$ for all λ.

ACKNOWLEDGMENT

One of the authors (H.C.) had numerous discussions with Dave Sharp on this subject during 1962.

[4] One example is given in H. Cheng (to be published).