Off-Shell Pion-Nucleon Scattering*

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The problem of off-shell pion-nucleon scattering is investigated under the assumption that the self-supporting dynamical system made up of the nucleon and $N^*(1238)$ continues to exist when one of the pions is virtual. Partial-wave amplitudes satisfying off-shell elastic unitarity are constructed from the off-shell $P_{3/2,3/2}$ nucleon exchange and $P_{1/2,1/2}$ $N^*(1238)$ exchange Born terms, although ambiguities arise in handling the high-energy behavior of the amplitudes. The reciprocal bootstrap requirement then leads to explicit expressions for the off-shell pionic form factors of the πNN and πNN^* vertexes.

1. INTRODUCTION

THE one-pion-exchange (OPE) model[1] has been generally accepted as a moderately successful phenomenological scheme for explaining a number of high-energy reactions. However, the failure of the model to account satisfactorily for certain features of these reactions has stimulated the theoretical investigation of two kinds of modifications of the basic OPE model. These are:

(i) Form factor corrections due to the off-shell nature of the exchanged meson. The principal result[2] of this investigation, a prescription for incorporating an empirical form factor into the analysis of pion-production data, has been applied, for example, to analyses of single-[3] and double-[4]pion production in nucleon-nucleon collisions, off-shell pion-pion scattering and pion production in pion-nucleon collisions,[5] double-pion production in proton-antiproton collisions,[6] and double-pion production in photon-nucleon and photon-nucleus collisions.[7] For further applications and references see also Ref. 8.

(ii) Absorption effects in the initial and final states due to the existence of strongly competing inelastic channels.[9] Such corrections have met with considerable success in a number of cases.[10]

This paper is concerned with reopening the question of the first of the above-mentioned modifications, for the following reasons:

(i) While absorption effects have been found to be of major importance in explaining the failures of the unmodified OPE model, the possibility of significant off-shell corrections or form factors is not excluded. One may now argue, in fact, that the new respectability given to the OPE model by the success of absorption effects makes it highly desirable that quantitative predictions of off-shell corrections be found and incorporated into the model.

(ii) An appealing variation, motivated by its success in the on-shell pion-nucleon problem, of the approach developed by Ferrari and Selleri[2] who made off-shell modifications of the one-dimensional dispersion relation solution of the pion-nucleon problem, lies in the use of partial-wave dispersion relations implicitly based on the assumed existence of a Mandelstam representation for off-shell scattering. This variation is exploited here in a way that resembles that of Ferrari and Selleri.

(iii) The success of certain kinds of bootstrap calculations[11] suggests a systematic program for predicting off-shell effects. The idea will be illustrated in this paper by considering the case of double-pion production on the basis of the Drell model,[12] and isolating the pion-nucleon elastic-scattering part of the appropriate Drell diagram.

Since the higher order corrections to the pion propagator and the $\gamma\pi^+\pi^-$ vertex exactly cancel,[13] the effect of the off-shell nature of the pion is entirely contained in treating the pion-nucleon problem with one pion off-shell. The particular bootstrap in mind is that which exists between N, the nucleon, and $N^*(1238)$. Here we appeal to the fact that the existence of one particle in a channel seems to provide the principal dynamics for explaining the existence of the other particle in a crossed channel. We conjecture that this reciprocal relationship is maintained even when a pion goes off-shell. When the principal dynamics in a particular channel are identified, it is a systematic task to determine the consequences of off-shell modifications which are essentially kinematical. Thus, the $P_{3/2,3/2}$ pion-nucleon scattering amplitude will depend in a definite way on the mass of the off-shell pion when the kinematics of the nucleon-exchange Born

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[11] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963).

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[13] F. E. Low, Phys. Rev. **110**, 974 (1958).

amplitude is modified to take into account that a pion is virtual.

The input off-shell Born amplitudes are calculated in the usual way within the framework of perturbation theory with the N^* treated as an elementary particle. These amplitudes are not yet consistently off-shell since they are calculated with the coupling constants evaluated conventionally with all particles on-shell. Upon inserting unknown form factors to describe the off-shell extrapolation of the coupling constants and applying the crossing relations, a set of linear equations is established which is easily solved, giving the form factors and allowing one to write down consistent off-shell Born amplitudes.

In Sec. 2 kinematical quantities are defined and the isotopic spin crossing relations and forms of the off-shell partial-wave elastic-scattering amplitudes are discussed. The unitarization of the Born amplitudes and the closely related problem of the high-energy behavior of the partial-wave amplitudes are discussed in Sec. 3. In the next section, the off-shell Born amplitudes for nucleon and N^* exchange are constructed and partial waves are projected out. The $P_{3/2,3/2}$ and $P_{1/2,1/2}$ partial-wave projections of the nucleon and N^* exchange amplitudes, respectively, are singled out for closer scrutiny because of the role they play in the nucleon, N^* bootstrap. In Sec. 5, we obtain the form factors mentioned in the preceding paragraph by comparing the partial-wave Born amplitudes with the corresponding pole amplitudes they are supposed to generate in the bootstrap sense. Finally, in the last section, our results are summarized and the use to which they may be put in further work is briefly discussed.

2. KINEMATICS

We consider the process $\pi_e+N\to\pi+N$ which may be thought of as a part of an OPE diagram where π_e is the exchanged or off-shell pion. Let p_1 and p_2 be the four-momenta of the initial and final nucleons, respectively, and q_1 and q_2 be the four-momenta of π_e and the final pion, respectively. We define the usual scalar invariants

$$s=(p_1+q_1)^2,\quad u=(q_1-p_2)^2, \tag{2.1}$$

with

$$p_1^2=p_2^2=M^2,\quad q_1^2=\Delta^2,\quad \text{and}\quad q_2^2=1.$$

If the total energy in the s channel is $W=\sqrt{s}$, then

$$k_1^2=[(W+M)^2-\Delta^2][(W-M)^2-\Delta^2]/4W^2 \tag{2.2}$$

and

$$k_2^2=[(W+M)^2-1][(W-M)^2-1]/4W^2 \tag{2.3}$$

are the squares of the center-of-mass momenta in the initial and final states, respectively. The energies of the initial and final nucleons are

$$E_1=(W^2+M^2-\Delta^2)/2W \tag{2.4}$$

and

$$E_2=(W^2+M^2-1)/2W, \tag{2.5}$$

respectively. We further find that

$$u=M^2+\tfrac{1}{2}(1+\Delta^2-W^2)+(M^2-1)(M^2-\Delta^2)/2W^2-2k_1k_2\cos\theta, \tag{2.6}$$

where θ is the center-of-mass scattering angle of the pion.

The two-dimensional reduction of the off-shell invariant amplitude,

$$\bar{u}(p_2)[A+\tfrac{1}{2}B\gamma\cdot(q_1+q_2)]u(p_1), \tag{2.7}$$

gives the off-shell physical amplitudes[2]

$$f_1=[(E_1+M)(E_2+M)]^{1/2}[A+(W-M)B]/8\pi W \tag{2.8}$$

and

$$f_2=[(E_1-M)(E_2-M)]^{1/2}\times[-A+(W+M)B]/8\pi W, \tag{2.9}$$

where A and B have two components of isotopic spin.

If A and B are the amplitudes in the u channel, then the corresponding amplitudes in the s channel are

$$XA \quad\text{and}\quad -XB, \tag{2.10}$$

respectively, where

$$X=\frac{1}{3}\begin{pmatrix}1 & 2\\ 4 & -1\end{pmatrix}, \tag{2.11}$$

the familiar isotopic spin crossing matrix. Here we have assumed that the two-component forms of the amplitudes are

$$A=\begin{pmatrix}A^{3/2}\\ A^{1/2}\end{pmatrix}\quad\text{and}\quad B=\begin{pmatrix}B^{3/2}\\ B^{1/2}\end{pmatrix}, \tag{2.12}$$

where the superscripts are the isotopic spins in either the s or u channels. Since $(\pm X)^2=1$, (2.10) applies as well in taking the s-channel amplitudes back into the u channel. Note that the above relations hold independently of crossing symmetry which is of course not possessed by the Born amplitudes considered here.

The partial-wave amplitude projections are

$$f_{l\pm}(W,\Delta^2)=\frac{1}{2}\int_{-1}^{1}(f_1P_l+f_2P_{l\pm1})d\cos\theta. \tag{2.13}$$

It is easily seen that the MacDowell[14] reflection symmetry,

$$f_{l+}(-W,\Delta^2)=-f_{(l+1)-}(W,\Delta^2), \tag{2.14}$$

still holds for the off-shell amplitudes.

[14] S. W. MacDowell, Phys. Rev. 116, 774 (1959).

3. OFF-SHELL UNITARITY AND ANALYTICITY

A partial-wave amplitude may be written in the form

$$f(W,\Delta^2)=\rho'(W,\Delta^2)F'(W,\Delta^2)\,, \tag{3.1}$$

where ρ' is a kinematical factor chosen to absorb kinematical zeros and singularities; e.g., $F'(W,\Delta^2)$ approaches a nonzero constant as $W\to M+1$. The indices describing the angular momentum, parity, and isotopic spin of the amplitude have been omitted. We then write

$$F'(W,\Delta^2)=N(W,\Delta^2)/D(W,\Delta^2)\,, \tag{3.2}$$

where, in the usual way, $N(W,\Delta^2)$ carries only the dynamical cuts and $D(W,\Delta^2)$ has only the unitary cuts in the complex W plane. According to the prescription[15] for incorporating elastic unitarity into the off-shell amplitude, D is determined by the requirement that its phase on the unitary cuts is the same as the on-shell phase shift. The form of D is given by its phase representation,[16]

$$D(W,\Delta^2)=P_1(W,\Delta^2)D_0(W)/P_2(W,\Delta^2) \tag{3.3}$$

where

$$D_0(W)=\exp\left\{-(W/\pi)\int_U dW' \times\delta(W')[W'(W'-W)]^{-1}\right\}, \tag{3.4}$$

$\delta(W')$ is the on-shell elastic-scattering phase shift defined always such that $0\leq\delta\leq\pi$, $\int_U$ denotes the integral over both unitary cuts, and P_1 and P_2 are arbitrary, real polynomials in W. The phase just above the left-hand unitary cut is actually $\pi-\delta_p$ where δ_p is the physical phase shift of the partial-wave amplitude associated with the left-hand cut according to (2.14). Since we may always define an elastic physical phase shift to lie between 0 and π, it follows that we may consistently require $0\leq\delta\leq\pi$ over both unitary cuts in (3.4).

The presence of the polynomial P_1 in (3.3) permits the insertion of poles or bound states into the amplitude. In case the amplitude in question is the one with the quantum numbers of the nucleon, it must possess a pole at $W=M$.

Unlike the bootstrap calculation where the position of the nucleon pole, or equivalently, the zero of the conventional denominator function is regarded as a derived quantity, the position of the zero in our calculation must be inserted in $D(W,\Delta^2)$ as part of the input information which generates, among other things, the variation with Δ^2 of the residue at the pole. It is clear that the position of the pole should be independent of Δ^2 since the production amplitude for a process involving a final pion and nucleon of combined total energy W will have a pole at $W=M$ regardless of the values of the other invariants describing the process. It is also clear that the amplitude should have no other poles since they would correspond to nonexistent bound states. Thus, $P_1(W,\Delta^2)$ must factor into $(W-M)Q(\Delta^2)$ where the function $Q(\Delta^2)$ may be absorbed by $N(W,\Delta^2)$ without changing the singularity structure of the latter.

The only other amplitude under consideration is the one which communicates with the N^*. Since the N^* pole is on the unphysical sheet of the complex W plane, it cannot appear in P_1. [Rather, it makes its presence known by causing $D_0(W)$ to become purely imaginary when W equals the mass of the N^*.] $P_1(W,\Delta^2)$ for this amplitude then reduces to a function of Δ^2 only and may be absorbed by $N(W,\Delta^2)$. Thus, we have $P_1(W,\Delta^2)=P_1(W)=W-M$ for the 1, 1 amplitude while $P_1(W)=1$ for the 3, 3 amplitude.

Finally we observe that the polynomial $P_2(W,\Delta^2)$ is superfluous since it may also be absorbed by $N(W,\Delta^2)$ without changing the latter's singularity structure.

The above considerations enable us to rewrite (3.2) in the form

$$F'(W,\Delta^2)=N(W,\Delta^2)/D(W)\,, \tag{3.5}$$

where

$$D(W)=P_1(W)D_0(W)\,. \tag{3.6}$$

Let $f^B(W,\Delta^2)$ be the partial-wave Born amplitude and, just as for the complete amplitude, let us factor out the kinematical singularities and zeros, defining an amplitude

$$F'^B(W,\Delta^2)=f^B(W,\Delta^2)/\rho'(W,\Delta^2)\,, \tag{3.7}$$

which, like $N(W,\Delta^2)$, contains only dynamical singularities. We now write

$$F'^B(W,\Delta^2)+C(W,\Delta^2)=N(W,\Delta^2)/D(W)\,, \tag{3.8}$$

where $C(W,\Delta^2)$, in part a continuum contribution, is a term which must be added to the Born term to give the correct amplitude in an exact theory; i.e., in a diagrammatric approach it would correspond to the sum of all other diagrams contributing to the dynamics which drive the partial-wave amplitude.

Consider the Cauchy integral for $C(W,\Delta^2)D(W)$, setting aside temporarily the question of convergence. We find

$$C(W,\Delta^2)D(W)=\int_V dW'D(W')\,\mathrm{Disc}[C(W',\Delta^2)][2\pi i(W'-W)]^{-1} -\int_U dW'F'^B(W',\Delta^2)\,\mathrm{Im}D(W')[\pi(W'-W)]^{-1}, \tag{3.9}$$

[15] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1957).
[16] M. Sugawara and A. Tubis, Phys. Rev. **130**, 2127 (1963).

where the first integral is over all the dynamical cuts V, and the discontinuity of $C(W',\Delta^2)$ on V is denoted by Disk[$C(W',\Delta^2)$], the details of which will be of no concern to us except in one important respect discussed near the end of this section.

Using (3.5) and (3.8) in (3.9), we obtain

$$F'(W,\Delta^2)=F'^B(W,\Delta^2)+[\pi D(W)]^{-1}\left\{\int_V dW'D(W')\,\mathrm{Disc}[C(W',\Delta^2)][2i(W'-W)]^{-1} - \int_U dW'F'^B(W',\Delta^2)\,\mathrm{Im}D(W')(W'-W)^{-1}\right\}. \quad (3.10)$$

In particular, the on-shell partial-wave amplitude is

$$f(W,1)=f^B(W,1)+[\rho'(W,1)/\pi D(W)]\left\{\int_V dW'D(W')\,\mathrm{Disc}[C(W',1)][2i(W'-W)]^{-1} - \int_U dW'f^B(W',1)\,\mathrm{Im}D(W')[(W'-W)\rho'(W',1)]^{-1}\right\}. \quad (3.11)$$

At this point we are faced with an arbitrary choice in handling the high-energy behavior of (3.10) or (3.11). Since the partial-wave Born amplitude diverges asymptotically for the exchange of systems with spin $>\frac{1}{2}$, the high-energy behavior of the above expressions may diverge in a highly unsatisfactory way unless there should turn out to be cancellations between the Born amplitude and the integral terms. This possibility seems unlikely in a practical calculation, assuming some reasonable approximation is made for $C(W',\Delta^2)$, in view of the special conditions necessary for such a cancellation. A way around this difficulty is to modify the Born amplitude so that it displays a realistic high-energy behavior. If this is accomplished by a simple cutoff, the dependence of the cutoff on the mass of the off-shell pion may be ambiguous. We may simply choose the cutoff to be independent of Δ^2, an arbitrary choice, and hope that the results are not sensitive to the cutoff in the low-energy resonance region. On the other hand, a Regge behavior for the exchanged system would also accomplish the desired result and would furthermore cause the cutoff to depend on Δ^2 in a nontrivial way.

To take a definite stand, with simplicity uppermost in mind, we introduce a Δ^2-independent cutoff factor and replace the Born amplitude for the exchange of a system of spin $J>0$ by

$$f_a{}^B(W,\Delta^2)=Z_{2J}(W,a)f^B(W,\Delta^2), \quad (3.12)$$

where

$$Z_{2J}(W,a)=\{[1+(M+1)^2/a^2]/[1+W^2/a^2]\}^{J-\frac{1}{2}}. \quad (3.13)$$

This is the minimum cutoff in the form of poles required to make the amplitude go to zero at high energy. The normalization is chosen so that $Z_{2J}=1$ at the physical threshold. The cutoff corresponds to poles at $W=\pm ia$ if $a^2>0$, an appealing feature since it satisfies the requirements that (1) the real-symmetric property of the amplitude should be preserved, (2) the new singularities should lie on the imaginary axis which already contains a dynamical cut along its entirety, and (3) it should be possible to choose the parameter a so that the new singularities are situated far enough from the interesting physical region to correspond only to some untenable short-range dynamics. Hopefully the end results generated by this arbitrary cutoff will not be greatly sensitive to the actual value of the parameter which plausibly should have a value of the order of, or a few times larger than, the mass of the nucleon.

Now that the Born amplitude has been modified to display a realistic asymptotic behavior there must be in addition some guarantee that the asymptotic behavior of the integral terms in (3.11) does not dominate over the behavior of $f_a{}^B(W,1)$. Again we suppose that in an exact calculation this would occur through delicate hidden cancellations between the two integrals of (3.11) if they individually diverged. It is therefore desirable to exhibit explicitly the high-energy behavior of the integrals by making subtractions, consistent with the assumed cancellations, in such a way that the apparent asymptotic behavior of the integrals matches or at least does not overshadow that of the (modified) Born amplitude. Thus, with

$$g=\rho'(W,1)/D(W)f^B(W,1)W^{2(1-J)},$$

we define L to be the smallest non-negative integer greater than or equal to

$$\lim_{W\to\pm\infty}\frac{W}{g}\frac{dg}{dW}. \quad (3.14)$$

The significance of L is that it is the smallest number of subtractions necessary to ensure the requirement stated above. It is easily shown that L has the same value on both the left-hand and right-hand unitary cuts. Taking the limit as $W\to\pm\infty$ is therefore not an apparently inconsistent procedure and, accordingly, both cuts are treated in a uniform manner in the following paragraphs. The value of L will clearly depend on whatever high-energy behavior is assumed for the on-shell phase shift.[17]

Consider the identity

$$h(W)\equiv\int dx\rho(x)/(x-W)$$
$$=W^{-L}\int dxx^{L}\rho(x)/(x-W)-\sum_{K=1}^{L}W^{-K}\int x^{K-1}\rho(x)dx. \quad (3.15)$$

Relation (3.15) shows that $h(W)\sim W^{-L-1}$ for large W if the zeroth to $(L-1)$th moments of $\rho(x)$ vanish. [The -1th moment of $\rho(x)$ is defined to be zero.] Again, choosing $\pm ia$ as subtraction points, if the zeroth to $(L-1)$th moments vanish, we may write

$$h(W)=(1+W^2/a^2)^{-\frac{1}{2}L}\int dx(1+x^2/a^2)^{\frac{1}{2}L}\times\epsilon(L,x,W)\rho(x)/(x-W), \quad (3.16)$$

where $\epsilon(L,x,W)$ is 1 if L is even. If L is odd,

$$\epsilon(L,x,W)=(x^2+a^2)^{-1/2}\times\{x+a^2[W+(W^2+a^2)^{1/2}]^{-1}\}. \quad (3.17)$$

Since the cancellation of the divergent parts of the integrals in (3.11) is equivalent to the vanishing of the zeroth to $(L-1)$th moments of the integrand, we may perform L subtractions of the form (3.16) to obtain

$$f(W,1)=f_a{}^B(W,1)+\rho'(W,1)[\pi D(W)(1+W^2/a^2)^{\frac{1}{2}L}]^{-1}\times\left\{\int_V dW'(1+W'^2/a^2)^{\frac{1}{2}L}\epsilon(L,W',W)D(W')\,\mathrm{Disc}[C(W',1)][2i(W'-W)]^{-1}-\int_U dW'(1+W'^2/a^2)^{\frac{1}{2}L}\epsilon(L,W',W)f_a{}^B(W',1)\,\mathrm{Im}D(W')[(W'-W)\rho'(W',1)]^{-1}\right\}. \quad (3.18)$$

The choice of $\pm ia$ for the subtraction points conveniently keeps the results thus far in one-parameter form.

For brevity we define

$$\rho(W,\Delta^2)=\rho'(W,\Delta^2)(1+W^2/a^2)^{-\frac{1}{2}L}/P_1(W), \quad (3.19)$$
$$F(W,\Delta^2)=f(W,\Delta^2)/\rho(W,\Delta^2), \quad (3.20)$$
$$F_a{}^B(W,\Delta^2)=f_a{}^B(W,\Delta^2)/\rho(W,\Delta^2), \quad (3.21)$$

and the off-shell extension of (3.18) becomes

$$F(W,\Delta^2)=F_a{}^B(W,\Delta^2)+[\pi D_0(W)]^{-1}\left\{\int_V dW'(1+W'^2/a^2)^{\frac{1}{2}L}\,\epsilon(L,W',W)D(W')\,\mathrm{Disc}[C(W',\Delta^2)][2i(W'-W)]^{-1}-\int_U dW'\epsilon(L,W'W)F_a{}^B(W',\Delta^2)\,\mathrm{Im}D_0(W')(W'-W)^{-1}\right\}. \quad (3.22)$$

In accordance with the discussion in Sec. 1 concerning the conjecture of the identification of the most important off-shell dynamics, we now maintain that $C(W',\Delta^2)$ is not strongly dependent on Δ^2, at least in comparison with the dependence of $F_a{}^B(W,\Delta^2)$, and hence we set

$$\Delta C(W',\Delta^2)=C(W',\Delta^2)-C(W',1)=0. \quad (3.23)$$

Using the notation defined in (3.23), we may write the change that occurs in (3.22), when the virtual pion goes off-shell, as

$$\Delta F(W,\Delta^2)=\Delta F_a{}^B(W,\Delta^2)-[\pi D_0(W)]^{-1}\times\int_U dW'\epsilon(L,W',W)\,\mathrm{Im}D_0(W')\times\Delta F_a{}^B(W',\Delta^2)(W'-W)^{-1}. \quad (3.24)$$

At this point it is worthwhile to consider the question of the convergence of the unitary integral in (3.24). We find that the integrand behaves asymptotically as

$$(\mathrm{Im}D_0/D_0)(W'^{L-2}/g). \quad (3.25)$$

If g does not go to zero at infinity, we have $g\sim W'^L$, apart from logarithmic factors, and the integral clearly converges. If $g\rightarrow 0$, then $L=0$ and (3.25) becomes

$$P_1(W')\sin\delta(W')|D_0(W')|\ln W'/W'^4. \quad (3.26)$$

Here we have used the facts that the cutoff Born amplitude $f_a{}^B\sim\ln W'/W'$ and that $\rho'\sim W'^2$ as will be seen in the next section. Inspection of (3.4) shows that $|D_0(W')|\sim|W'|^{(\delta_R-\delta_L)/\pi}$, apart from logarithmic factors, if the phase of D_0 approaches a constant $-\delta_R$ on

[17] Using the expression for ρ_{33}' found in Sec. 4, the parametric form $\delta_{33}(W)=\tan^{-1}[k_2{}^3/(a_0+a_1k_2{}^2+a_2k_2{}^4)]$ due to J. M. McKinley, Rev. Mod. Phys. 35, 788 (1963), and neglecting the right-hand unitary cut controlled by the small phase shift of the f_{2-}, isotopic spin-$\frac{3}{2}$ projection, we find $L=1$ for the 3,3 projection of the nucleon-exchange off-shell Born amplitude.

the right-hand unitary cut and a constant $-\delta_L$ on the left-hand cut. At worst, $\delta_R=\pi$, $\delta_L=0$, $P_1\sim W'$, and (3.26) reduces to W'^{-2} where again we have dropped unimportant logarithmic factors, and we conclude that the integral still converges.

After some further manipulations, we can bring (3.24) into the form

$$\Delta F(W,\Delta^2)=e^{i\delta(W)}\Big\{\Delta F_a{}^B(W,\Delta^2)\cos\delta(W)+[\pi\Lambda(W)]^{-1}P\int_U dW'\Lambda(W')\epsilon(L,W',W)\times\sin\delta(W')\Delta F_a{}^B(W',\Delta^2)/(W'-W)\Big\} \quad (3.27)$$

where

$$\Lambda(W)=|D_0(W)|=\exp\left[-(W/\pi)P\int_U dW'\delta(W')/W'(W'-W)\right] \quad (3.28)$$

and P denotes the principal value operator.

Equation (3.27) is quite similar to Eq. (52) of Ref. 2. The difference lies in the method of subtractions, the structure of $F_a{}^B$, and the existence of the left-hand unitary cut in (3.27).

Finally, introducing the factor $\rho(W,\Delta^2)$, we obtain for the off-shell partial-wave amplitude,

$$f(W,\Delta^2)=\rho(W,\Delta^2)\times[\Delta F(W,\Delta^2)+f(W,1)/\rho(W,1)]. \quad (3.29)$$

4. BORN AMPLITUDES

A. Nucleon Exchange

With the usual pseudoscalar pion-nucleon interaction

$$ig_1\bar{\psi}\gamma_5\tau\cdot\Phi\psi, \quad (4.1)$$

the nucleon-exchange Born amplitude turns out to be

$$A(s,u)=0,\qquad B(s,u)=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_1^2}{(u-M^2)}, \quad (4.2)$$

where $g_1^2/4\pi\approx 14$. The corresponding physical amplitudes are

$$f_1=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_1^2}{4\pi}\frac{[(E_1+M)(E_2+M)]^{1/2}}{2W}\frac{W-M}{u-M^2},$$

and (4.3)

$$f_2=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_1^2}{4\pi}\frac{[(E_1-M)(E_2-M)]^{1/2}}{2W}\frac{W+M}{u-M^2}.$$

We define

$$R_\pm=\{[(W\pm M)^2-1][(W\pm M)^2-\Delta^2]\}^{-1/2}. \quad (4.4)$$

Making use of the relations listed in Sec. 2, we find that the partial-wave projections are

$$f_{l\pm}{}^B(W,\Delta^2)=\begin{pmatrix}2\\-1\end{pmatrix}\frac{g_1^2}{8\pi}[(W-M)R_-Q_l(z_1)+(W+M)R_+Q_{l\pm1}(z_1)], \quad (4.5)$$

where

$$z_1=R_+R_-[(M^2-1)^2-W^2(W^2-2)+(\Delta^2-1)(W^2-M^2+1)], \quad (4.6)$$

and

$$Q_l(z_1)=\frac{1}{2}\int_{-1}^{1}dx\,P_l(x)/(z_1-x), \quad (4.7)$$

the Legendre function of the second kind.

We now choose

$$\rho_{33}(W,\Delta^2)=(4W^2R_+R_-)^{-1}(1+W^2/a_{33}{}^2)^{-\frac{1}{2}L_{33}}, \quad (4.8)$$

which has the same kinematical zeros as (4.5) for the $f_{1+}{}^B$ projection and includes the subtraction factor as defined in (3.19).

The high-energy asymptotic behavior of the partial waves (4.5) is found to be

$$f_{l\pm}{}^B(W,\Delta^2)\sim \ln W/W \quad (4.9)$$

as in the on-shell case.

B. N^* Exchange

We will calculate the pole diagram in the u channel, in which the N^* exists as an intermediate state, and then apply crossing to determine the exchange amplitude. The πNN^* vertex in the pole diagram is

$$g_3\bar{\psi}_{N^*}{}^\mu\psi_N(p_1+q_2)_\mu, \quad (4.10)$$

where g_3 is related to the width of the N^*. Our choice of propagator is[18]

$$P_{\mu\nu}=[3g_{\mu\nu}-\gamma_\mu\gamma_\nu-4K_\mu K_\nu/N^2+(\gamma_\mu K_\nu\gamma\cdot K+\gamma\cdot KK_\mu\gamma_\nu)/N^2]\times(\gamma\cdot K+N)/(K^2-N^2+iN\Gamma), \quad (4.11)$$

where $K=p_1-q_2$, N is the mass of the N^*, and Γ is the width of the N^*. The pole diagram amplitude is then

$$\bar{\psi}_N(p_2)[A^{(p)}(s,u)+\gamma\cdot(-q_1-q_2)B^{(p)}(s,u)]\psi_N(p_1)=\begin{pmatrix}1\\0\end{pmatrix}g_3{}^2\bar{\psi}_N(p_2)(p_2+q_1)^\mu P_{\mu\nu}(p_1+q_2)^\nu\psi_N(p_1). \quad (4.12)$$

The invariant pole amplitudes $A^{(p)}$ and $B^{(p)}$ are identified upon carrying out the reduction of (4.12). Crossing, according to (2.10), may then be applied and

[18] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) 18, 198 (1962).

the partial-wave projections of the N^* exchange Born amplitude are found to be

$$f_{l\pm}(W,\Delta^2)=\binom{1}{4}(g_3{}^2/48\pi N^2)\times[V_1\delta_{l,0}/R_+W^2+V_2\delta_{l\pm1,0}/R_-W^2+2Y_1R_-Q_l(z_3)+2Y_2R_+Q_{l\pm1}(z_3)], \quad (4.13)$$

with

$$V_1=A''(\Delta^2)+B''(\Delta^2)(W-M), \quad (4.14)$$

$$V_2=-A''(\Delta^2)+B''(\Delta^2)(W+M), \quad (4.15)$$

$$Y_1=A'(s,\Delta^2)+B'(s,\Delta^2)(W-M), \quad (4.16)$$

$$Y_2=-A'(s,\Delta^2)+B'(s,\Delta^2)(W+M), \quad (4.17)$$

$$A'(s,\Delta^2)=6sN^2(N+M)+2[N^5-2N^3(2M^2+1)-2MN^2(1+M^2)+2N(M^2-1)-M(M^2-1)^2]+(\Delta^2-1)[-2N^3-2MN^2+2N(M^2-2)+2M(M^2-1)], \quad (4.18)$$

$$A''(\Delta^2)=2N(N^2-M^2+1)+(\Delta^2-1)(N+M), \quad (4.19)$$

$$B'(s,\Delta^2)=-6sN^2-2[2N^4+2MN^3-4N^2+2M(M^2-1)N-(M^2-1)^2]+(\Delta^2-1)(4N^2+2MN), \quad (4.20)$$

$$B''(\Delta^2)=-4N^2+\Delta^2-1, \quad (4.21)$$

and

$$z_3=z_1+2W^2(M^2-N^2)R_+R_-. \quad (4.22)$$

It is further found that[11,18]

$$g_3{}^2=16\pi\Gamma N^5[(N-M)^2-1]^{-3/2}\times[(N+M)^2-1]^{-5/2}. \quad (4.23)$$

For the same reasons mentioned in justifying (4.8) we make the choice

$$\rho_{11}(W,\Delta^2)=(4W^2R_+R_-)^{-1}\times(1+W^2/a_{11}{}^2)^{-\frac{1}{2}L_{11}}(W-M)^{-1} \quad (4.24)$$

for the $f_{1-}{}^B$ projection.

On the other hand, the off-shell N^* exchange amplitude diverges as $W\ln W$ at high energy. This result and the corresponding one for the nucleon-exchange amplitude are the justifications for the implicit assumption that L is independent of Δ^2 in writing down (3.22). The consequences of L depending on Δ^2, possibly due to a Δ^2-dependent cutoff on the Born amplitude might seriously complicate the arguments leading to the subtracted Eq. (3.22).

5. COUPLED N, N^* EQUATIONS

Let us write the $P_{3/2,3/2}$ partial-wave projection of the nucleon-exchange Born amplitude as

$$f_{33}{}^B(W,\Delta^2)=g_1{}^2\Phi_1(\Delta^2)H_3{}^B(W,\Delta^2), \quad (5.1)$$

where $\Phi_1(\Delta^2)$ is the form factor describing the dependence of the πNN vertex on Δ^2. Similarly we write the $P_{1/2,1/2}$ partial-wave projection of the N^* exchange Born amplitude as

$$f_{11}{}^B(W,\Delta^2)=g_3{}^2\Phi_3(\Delta^2)H_1{}^B(W,\Delta^2)Z_3(W,a_{11}), \quad (5.2)$$

where Φ_3 is the pionic form factor associated with the πNN^* vertex. $H_1{}^B$ and $H_3{}^B$ are known functions already determined in Sec. 4. In (5.2) we have included the cutoff factor appropriate for a spin-$\frac{3}{2}$ exchange.

Equating the residue of the unitarized $P_{1/2,1/2}$ partial-wave projection of the N^* exchange amplitude at the pole $W=M$ with the residue of the off-shell nucleon-pole amplitude, we have, omitting some algebraic detail,

$$[(\Delta^2)^{1/2}\alpha/\pi]\{-3g_1{}^2/16M^2\beta+g_3{}^2\lambda_1\beta[\Phi_3(\Delta^2)j_1(\Delta^2)-j_1(1)]\}=-3(\Delta^2)^{1/2}g_1{}^2\Phi_1(\Delta^2)/16\pi M^2. \quad (5.3)$$

Similarly, equating the value of the unitarized $P_{3/2,3/2}$ partial-wave projection of the nucleon-exchange amplitude at $W=N$ with the value of the off-shell N^* pole amplitude at the same energy, we have

$$(i\sigma\tau\mu\nu/\pi)\{\tau^2g_3{}^2/8\Gamma N^4+g_1{}^2\lambda_3[\Phi_1(\Delta^2)j_3(\Delta^2)-j_3(1)]\}=i\sigma^2\tau^2\mu\nu g_3{}^2\Phi_3(\Delta^2)/8\pi\Gamma N^4. \quad (5.4)$$

In the above equations we have defined

$$\alpha=(4M^2-\Delta^2)^{1/2}, \quad (5.5)$$

$$\beta=(4M^2-1)^{1/2}, \quad (5.6)$$

$$\lambda_1=[4M^2(1+M^2/a_{11}{}^2)^{\frac{1}{2}L_{11}}]^{-1}\times\exp\left[\frac{M}{\pi}\int_U dW'\delta_{11}(W')/W'(W'-M)\right], \quad (5.7)$$

$$\lambda_3=[4N^2(1+N^2/a_{33}{}^2)^{\frac{1}{2}L_{33}}]^{-1}\times\exp\left[\frac{N}{\pi}P\int_U dW'\delta_{33}(W')/W'(W'-N)\right], \quad (5.8)$$

$$\sigma=[(N+M)^2-\Delta^2]^{1/2}, \quad (5.9)$$

$$\tau=[(N+M)^2-1]^{1/2}, \quad (5.10)$$

$$\mu=[(N-M)^2-\Delta^2]^{1/2}, \quad (5.11)$$

$$\nu=[(N-M)^2-1]^{1/2}, \quad (5.12)$$

$$j_1(\Delta^2)=\int_U \Lambda_1(W')\epsilon(L_{11},W',M)\sin\delta_{11}(W')\times H_1{}^B(W',\Delta^2)Z_3(W',a_{11})\times[(W'-M)\rho_{11}(W',\Delta^2)]^{-1}dW', \quad (5.13)$$

and

$$j_3(\Delta^2)=P\int_U \Lambda_3(W')\epsilon(L_{33},W',N)\times\sin\delta_{33}(W')H_3{}^B(W',\Delta^2)\times[(W'-N)\rho_{33}(W',\Delta^2)]^{-1}dW'. \quad (5.14)$$

The quantities δ_{11} and δ_{33} appearing in (5.7), (5.8), (5.13), and (5.14) are the physical phase shifts defined as discussed in the remarks following (3.4).

Equations (5.3) and (5.4) are readily solved for the two form factors. We get, using (4.23) to eliminate g_3^2,

$$\Phi_1(\Delta^2)=\{3g_1^2\alpha\nu^3\sigma\tau^5+256\pi\Gamma M^2N^5\alpha\beta^2\lambda_1[j_1(1)\sigma-j_1(\Delta^2)\tau] +128\Gamma g_1^2M^2N^4\alpha\beta^2\nu^3\tau^4\lambda_1\lambda_3 j_1(\Delta^2)j_3(1)\}/g_1^2\beta\nu^3\tau^4\gamma \quad (5.15)$$

and

$$\Phi_3(\Delta^2)=\{6\pi N\beta\tau^2+3g_1^2\nu^3\tau^5\lambda_3[\alpha j_3(\Delta^2)-\beta j_3(1)] +256\pi\Gamma N^5M^2\alpha\beta^2\lambda_1\lambda_3 j_1(1)j_3(\Delta^2)\}/2\pi N\beta\gamma, \quad (5.16)$$

where

$$\gamma=3\sigma\tau+128\Gamma M^2N^4\alpha\beta\lambda_1\lambda_3 j_1(\Delta^2)j_3(\Delta^2). \quad (5.17)$$

We may observe that the on-shell reduction of (5.15) and (5.16) yields, as it should, $\Phi_1(1)=1$ and $\Phi_3(1)=1$. This actually amounts to nothing more than a check on our algebraic manipulations since such a reduction is inherently built into the calculation.

6. SUMMARY

The important results of this paper are:

(i) the expression (3.29) for the unitarized off-shell partial-wave amplitude;

(ii) the input off-shell Born amplitudes for nucleon and N^* exchange, Eqs. (4.5) and (4.13), respectively; and

(iii) the form factors (5.15) and (5.16) which, when combined with the input Born amplitudes according to (5.1) and (5.2) and the unitarization procedure is carried out, finally yield the off-shell partial-wave amplitudes.

The assumptions which led to these results are:

(i) the basic conjecture that the N and N^* exchanges are the principal forces responsible for off-shell effects and that their self-supporting relationship continues to exist when a pion becomes virtual. This conjecture was appealed to in setting $\Delta C(W,\Delta^2)=0$ in Sec. 3 and again in Sec. 5 when the pole amplitudes were equated to the appropriate Born amplitudes;

(ii) the existence of an exact theory in which cancellations occur between the diverging parts of (3.11). This assumption was implemented in two steps: (1) the Born amplitudes were multiplied by cutoff factors, and (2) subtractions on the integrals were performed so that their asymptotic behavior matched the modified Born amplitudes.

Since the off-shell effects are partially obscured by the integrals arising from the unitarization procedure, it is difficult to see how our results can be put to immediate use in their present analytic form. Also, the answer to the question of how sensitively the form factors depend on the arbitrary parameters a_{11} and a_{33} is not readily apparent upon casually inspecting (5.15) and (5.16). In particular, since λ_1 and λ_3 seem to depend not insensitively on these parameters, serious difficulties may arise which can only be clarified by further investigation. This suggests the importance of carrying out a numerical evaluation of our results which may shed some light on the above question, and which entails facing up to the problems of the choice of parametric forms for the input phase shifts, and the evaluation of principal part integrals, the subjects of forthcoming work.