THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 144, NO. 4 29 APRIL 1966

Configuration-Space Photon Number Operators in Quantum Optics*

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(Received 8 November 1965; revised manuscript received 22 December 1965)

Some properties of the operator $\hat{n}_{V,t}$ representing the number of photons localized in a finite volume V at time t are investigated. To some extent these properties reflect the well-known difficulty of localizing photons in space-time. However, it is shown that, when the linear dimensions of V are large compared with the wavelength of any occupied mode of the field, the $\hat{n}_{V,t}$ operator acquires some simple properties. The commutator of $\hat{n}_{V,t}$ and the detection operator $\hat{\mathbf{A}}(\mathbf{x},t)$ is expressible in an interesting form. The commutator and relations derived from it become particularly simple for certain space-time regions which we label conjoint and disjoint. An orthogonal set of eigenstates of $\hat{n}_{V,t}$ is found, together with the corresponding eigenvalues, and it is shown that an arbitrary state is expressible in terms of these eigenstates. Some Nth-order correlations of the $\hat{n}_{V,t}$ operators are evaluated, and the results are used to calculate the probability distribution of eigenvalues of $\hat{n}_{V,t}$ for an arbitrary state of the field.

1. INTRODUCTION

THERE are many problems in quantum optics, particularly those concerned with photoelectric measurements of the field, which are most conveniently treated with the help of an operator $\hat{n}_{V,t}$ representing the number of photons localized in a finite volume V at time t. $\hat{n}_{V,t}$ can be expressed in terms of the detection operator[1–4] $\hat{\mathbf{A}}(\mathbf{x},t)$, which we define by

$$\hat{\mathbf{A}}(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{\{\mathbf{k},s\}}\hat{a}_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp i(\mathbf{k}\cdot\mathbf{x}-ckt)\,, \tag{1}$$

in the form

$$\hat{n}_{V,t}=\int_V \hat{\mathbf{A}}^\dagger(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x},t)d^3x\,. \tag{2}$$

Here $\hat{\mathbf{A}}_{\mathbf{k},s}$ is the annihilation operator for photons of wave-vector spin mode $\mathbf{k}$, s; $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ is the unit polarization vector satisfying the relation

$$\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s'}=\delta_{s,s'}\,; \tag{3}$$

L^3 is the normalization volume, and $\{\mathbf{k},s\}$ stands for the set of all modes of the field to which the detector responds. The effective number of modes in this set will in general be finite for a finite normalization volume. It can be shown that the number of counts registered by a photodetector is represented by $\hat{n}_{V,t}$, and that correlations between counts registered by several photodetectors are given by normally ordered products of $\hat{n}_{V,t}$ operators.[1,3,5] In practical situations, a surface of area S of the photodetector is usually exposed to normally incident plane waves for a time T, and we can then identify the volume V with the volume cTS swept out by the area S in a time T.

It can be seen from the definitions (1) and (2) that, when the volume V coincides with the total (normalization) volume L^3, $\hat{n}_{V,t}$ becomes identical with the number operator

$$\hat{n}=\sum_{\{\mathbf{k},s\}}\hat{a}_{\mathbf{k},s}{}^\dagger\hat{a}_{\mathbf{k},s}\,.$$

Moreover, the expectation values of $\hat{n}_{V,t}$ and $\hat{n}$ are very simply related for Fock states of type $|\{n_{\mathbf{k},s}\}\rangle$ by

$$\langle\hat{n}_{V,t}\rangle=(V/L^3)\langle\hat{n}\rangle\,. \tag{4}$$

In general, however, although $\hat{n}_{V,t}$ and $\hat{n}$ always commute, the eigenstates of $\hat{n}$ are not necessarily also eigenstates of $\hat{n}_{V,t}$, except in the trivial case where the number

* This research was sponsored by the U. S. Air Force Cambridge Research Laboratories, Office of Aerospace Research.

[1] See, for example, the review by L. Mandel and E. Wolf, Rev. Mod. Phys. **37**, 231 (1965). We adopt the convention used there of representing all operators by a caret sign.

[2] See, also, S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Harper & Row, New York, 1961), 1st ed., p. 172.

[3] R. J. Glauber, Phys. Rev. **130**, 2529 (1963); **131**, 2766 (1963).

[4] T. F. Jordan, Phys. Letters **11**, 289 (1964).

[5] L. Mandel, Phys. Rev. **136**, B1221 (1964).

of modes is unity. Furthermore, two operators $\hat{n}_{V_1,t_1}$ and $\hat{n}_{V_2,t_2}$ do not strictly commute even for disjoint space-time regions (V_1,t_1) and (V_2,t_2). These facts, which are manifestations of the difficulty of localizing photons in space-time, make $\hat{n}_{V,t}$ somewhat more awkward to handle than the number operator $\hat{n}$.

Nevertheless, as we show below, for volumes V whose linear dimensions are large compared with the wavelengths of all modes of the set $\{\mathbf{k},s\}$, the operator $\hat{n}_{V,t}$ acquires some simple properties. We shall see that, for certain space-time regions which we designate as disjoint and conjoint, the commutator of $\hat{n}_{V,t}$ and $\hat{\mathbf{A}}(\mathbf{x},t)$ is expressible in a simple form. This allows us to find the complete orthogonal set of eigenstates of $\hat{n}_{V,t}$ and the corresponding eigenvalues, all of which are degenerate to infinite order. The notation of disjoint and conjoint space-time regions is found to be useful also in the discussion of correlations of the operators $\hat{n}_{V,t}$, which are encountered in many problems in coherence theory.

2. EVALUATION OF THE COMMUTATORS

From the definition (1), and the well-known commutation rules obeyed by $\hat{a}_{\mathbf{k},s}$, we can immediately write down the commutator of $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{\mathbf{A}}^\dagger(\mathbf{x}',t')$. Thus

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{\mathbf{A}}^\dagger\mathbf{x}',t']=\frac{1}{L^3}\sum_{\{\mathbf{k},s\}}\boldsymbol{\varepsilon}_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\exp(i[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-ck(t-t')]);$$
$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{\mathbf{A}}(\mathbf{x}',t')]=[\hat{\mathbf{A}}^\dagger(\mathbf{x},t),\hat{\mathbf{A}}^\dagger(\mathbf{x}',t')]=0, \tag{5}$$

where the product of two vector operators is here to be understood as a tensor operator.

Consider now the commutator of $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{n}_{V,t'}$. From (2),

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\int_V[\hat{\mathbf{A}}(\mathbf{x},t),\hat{\mathbf{A}}^\dagger(\mathbf{x}',t')\cdot\hat{\mathbf{A}}(\mathbf{x}',t')]\,d^3x'$$
$$=\int_V[\hat{\mathbf{A}}(\mathbf{x},t),\hat{\mathbf{A}}^\dagger(\mathbf{x}',t')]\cdot\hat{\mathbf{A}}(\mathbf{x}',t)\,d^3x',$$

and with the help of (1) and (5), on interchanging orders of integration and summation, we obtain

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\frac{1}{L^{9/2}}\sum_{\{\mathbf{k},s\}}\sum_{\{\mathbf{k}',s'\}}\boldsymbol{\varepsilon}_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'})\hat{a}_{\mathbf{k}',s'}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)]\int_V\exp(i[(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'-c(k'-k)t'])\,d^3x'. \tag{6}$$

Let us take the volume of integration V to be in the form of a rectangular box with sides l_1, l_2, l_3 parallel to the three axes, and let $\mathbf{x}_0$ be the midpoint of this volume. Then

$$\int_V e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}'}d^3x'=Ve^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_0}\prod_{j=1}^{3}\frac{\sin[\frac{1}{2}(k_j'-k_j)l_j]}{\frac{1}{2}(k_j'-k_j)l_j}, \tag{7}$$

and

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\frac{V}{L^{9/2}}\sum_{\{\mathbf{k},s\}}\sum_{\{\mathbf{k}',s'\}}\hat{a}_{\mathbf{k}'s'}\boldsymbol{\varepsilon}_{\mathbf{k},s}(\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'})$$
$$\exp(i[(\mathbf{k}\cdot\mathbf{x}-ckt)+(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}_0-c(k'-k)t'])\prod_{j=1}^{3}\frac{\sin[\frac{1}{2}(k_j'-k_j)l_j]}{\frac{1}{2}(k_j'-k_j)l_j}. \tag{8}$$

We next consider the summation over $\mathbf{k}$. It is clear that the principal contributions will come from values of $\mathbf{k}$ in the neighborhood of $\mathbf{k}'$ such that

$$|k_1-k_1'|\lesssim 2/l_1,$$
$$|k_2-k_2'|\lesssim 2/l_2, \tag{9}$$
$$|k_3-k_3'|\lesssim 2/l_3.$$

If the lengths l_1, l_2, l_3 are of order 1 cm or longer, as is usually the case, while the wave numbers k belonging to the set $\{\mathbf{k},s\}$ are in the optical region, then the conditions (9) imply equality between the two vectors $\mathbf{k}$ and $\mathbf{k}'$ to a very good approximation. We shall take it for granted throughout that these conditions on the linear dimensions of V are satisfied. Accordingly, we may also write

$$\boldsymbol{\varepsilon}_{\mathbf{k},s}\approx\boldsymbol{\varepsilon}_{\mathbf{k}',s}, \tag{10}$$

so that

$$\boldsymbol{\varepsilon}_{\mathbf{k},s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\approx(\boldsymbol{\varepsilon}_{\mathbf{k}',s}{}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'})=\delta_{s,s'}. \tag{11}$$

With the help of (10) and (11) we can write (8) in the form

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\frac{V}{L^{9/2}}\sum_{\{\mathbf{k}\}}\sum_{\{\mathbf{k}',s'\}}\hat{a}_{\mathbf{k}',s'}\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\exp[i(\mathbf{k}'\cdot\mathbf{x}-ck't)]$$
$$\times\exp\left(i[(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{x}-\mathbf{x}_0)-c(k-k')(t-t')]\right)\prod_{j=1}^{3}\frac{\sin[\frac{1}{2}(k_j-k_j')l_j]}{\frac{1}{2}(k_j-k_j')l_j}. \quad (12)$$

We now replace the summation over $\{\mathbf{k}\}$ by an integral according to the usual rule

$$\frac{1}{L^3}\sum_{\{\mathbf{k}\}}\rightarrow\frac{1}{(2\pi)^3}\int_{\{\mathbf{k}\}}d^3k.$$

It is convenient to introduce a new variable $\mathbf{k}''=\mathbf{k}-\mathbf{k}'$. In view of the inequalities (9) we may write

$$\begin{aligned}k-k'&=[(\mathbf{k}'+\mathbf{k}'')^2]^{1/2}-k'\\&\approx k'\left(1+\frac{\mathbf{k}'\cdot\mathbf{k}''}{k'^2}\right)-k'\\&\approx\frac{\mathbf{k}'\cdot\mathbf{k}''}{k'},\end{aligned} \quad (13)$$

and this allows us to express (12) in the form

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\frac{V}{L^{3/2}}\sum_{\{\mathbf{k}',s'\}}\hat{a}_{\mathbf{k}',s'}\boldsymbol{\varepsilon}_{\mathbf{k}'s'}\exp[i(\mathbf{k}'\cdot\mathbf{x}-ck't)]$$
$$\times\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\exp\left(i\mathbf{k}''\cdot[(\mathbf{x}-\mathbf{x}_0)-c\mathbf{k}'(t-t')/k']\right)\prod_{j=1}^{3}\frac{\sin(\frac{1}{2}k_j''l_j)}{\frac{1}{2}k_j''l_j}d^3k'', \quad (14)$$

where we have replaced the finite limits of the k_1'', k_2'', k_3'' variables by infinite limits, since the principal contributions to the integral arise from small values of $|k_j''|$ such that $|k_j''|\lesssim 1/l_j$ $(j=1, 2, 3)$. The integrals have the form of the well-known Dirichlet integrals

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ik_j''y_j)\frac{\sin(\frac{1}{2}k_j''l_j'')}{\frac{1}{2}k_j''}dk_j''&=1, \quad \text{if} \quad |y_j|<\tfrac{1}{2}l_j,\\&=0, \quad \text{otherwise.}\end{aligned} \quad (15)$$

By using (15) and introducing the discontinuous function $U(\mathbf{x};V)$ defined by

$$\begin{aligned}U(\mathbf{x};V)&=1, \text{ if } \mathbf{x} \text{ lies within the volume } V,\\&=0, \text{ if } \mathbf{x} \text{ lies outside the volume } V,\end{aligned} \quad (16)$$

we can rewrite (14) in the form

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\frac{1}{L^{3/2}}\sum_{\{\mathbf{k},s\}}\hat{a}_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)]U[\mathbf{x}-c\mathbf{k}(t-t')/k;V]. \quad (17)$$

Certain conclusions can be drawn at once from Eq. (17). Let us call two space-time regions (V_1,t_1) and (V_2,t_2) disjoint if, when one of the volumes V_1 or V_2 is enlared by displacing the boundary outwards a distance $c|t_1-t_2|$, the two volumes do not overlap. Then, if $\mathbf{x}_1$ is any point in V_1 and $\mathbf{x}_2$ is any point in V_2, the events $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2)$ will have a space-like separation. Now if $(\mathbf{x},t)$ and (V,t') are disjoint, $U[\mathbf{x}-c\mathbf{k}(t-t')/k;V]$ vanishes for all $\mathbf{k}$, so that[6]

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=0, \text{ if } (\mathbf{x},t) \text{ and } (V,t') \text{ are disjoint.} \quad (18)$$

Sometimes two regions (V,t) and (V',t'), which are not disjoint, appear to be disjoint in relation to the

[6] Compare also the discussion by Schweber (Ref. 2) for a spinless field and for equal-time operators.

field. Thus, consider a field in the form of plane waves, traveling in one direction, and let V and V' be nonoverlapping volumes located side by side, so that no "ray" of the field passes through both. If $\mathbf{x}$ lies within V, it is clear that the function $U[\mathbf{x}-c\mathbf{k}(t-t')/k_jV']$ vanishes for any $\mathbf{k},s$ mode of the field which is occupied, so that the expectation value of the commutator

$$\langle[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V',t'}]\rangle=0. \qquad (19)$$

Let us call the space-time region (V_1,t_1) conjoint with (V_2,t_2), if the volume V_2 can be reduced to $V_2'>0$ by displacing its boundary inwards a distance $c|t_1-t_2|$, and if the volume V_1 lies entirely within V_2'. Then, if $\mathbf{x}_1$ is any point within V_1 and $\mathbf{x}_2$ is any point on the boundary of V_2, the events $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2)$ will have a space-like separation. Now if $(\mathbf{x},t)$ is conjoint with (V,t'),

$$U[\mathbf{x}-c\mathbf{k}(t-t')/k;V]=1,$$

for all k, and from (17) we see that the commutator

$$[\hat{\mathbf{A}}(\mathbf{x},t),\hat{n}_{V,t'}]=\hat{\mathbf{A}}(\mathbf{x},t),\quad \text{if } (\mathbf{x},t) \text{ is conjoint with } (V,t'). \qquad (20)$$

Again it is possible that two regions (V,t) and (V',t') which are not conjoint may appear to be conjoint in relation to the field. If the field is in the form of plane waves traveling in one direction, and if V' is large enough to enclose V and is located in line with and behind V, so that every ray traversing V also traverses V', then for any $\mathbf{x}$ within V, and a suitably chosen interval $(t'-t)$, we shall have $U[\mathbf{x}-c\mathbf{k}(t-t')/k_jV']=1$ for any $\mathbf{k},s$ mode of the field which is occupied.

Needless to say, disjointness and conjointness as here defined are not exhaustive properties, and a space-time region need not be either disjoint or conjoint with another. However in the special case of an event $(\mathbf{x},t)$ and an extended region (V,t), it can be seen that $(\mathbf{x},t)$ is either conjoint or disjoint with (V,t) according as $\mathbf{x}$ lies inside or outside V.[7]

3. EIGENSTATES OF $\hat{n}_{V,t}$

We can use the Hermitian conjugates of Eqs. (18) and (20),

$$[\hat{\mathbf{A}}^\dagger(\mathbf{x},t),\hat{n}_{V,t'}]=0,\quad \text{if } (\mathbf{x},t) \text{ is disjoint with } (V,t'), \qquad (21)$$

$$[\hat{\mathbf{A}}^\dagger(\mathbf{x},t),\hat{n}_{V,t'}]=-\hat{\mathbf{A}}^\dagger(\mathbf{x},t),\quad \text{if } (\mathbf{x},t) \text{ is conjoint with } (V,t'), \qquad (22)$$

to derive the eigenstates of $\hat{n}_{V,t}$. Thus, if $|\{0\}\rangle$ is the vacuum state of the field, then from (21)

$$\begin{aligned}\hat{n}_{V,t}\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle &=\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)\hat{n}_{V,t}|\{0\}\rangle,\\ &=0,\quad \text{if } (\mathbf{x}_1,t_1) \text{ is disjoint with } (V,t),\end{aligned} \qquad (23)$$

and from (22)

$$\begin{aligned}\hat{n}_{V,t}\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle &=\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)\hat{n}_{V,t}|\{0\}\rangle+\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle,\\ &=\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle,\quad \text{if } (\mathbf{x}_1,t_1) \text{ is conjoint with } (V,t).\end{aligned} \qquad (24)$$

It follows that $\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle$ is an eigenstate of $\hat{n}_{V,t}$ belonging to the eigenvalue 0 or 1 according as $(\mathbf{x}_1,t_1)$ is disjoint or conjoint with (V,t). By operating on both sides of (23) or (24) with $\hat{\mathbf{A}}^\dagger(\mathbf{x}_2,t_2)$ on the left, and applying (21) and (22) we can similarly show that

$$\hat{n}_{V,t}\hat{\mathbf{A}}^\dagger(\mathbf{x}_2,t_2)\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle =m\hat{\mathbf{A}}^\dagger(\mathbf{x}_2,t_2)\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle, \qquad (25)$$

where m is 0, 1, or 2 according as $(\mathbf{x}_1,t_1)$ and $(\mathbf{x}_2,t_2)$ are both disjoint with (V,t), one is disjoint and one conjoint with (V,t), or both are conjoint with (V,t). The product of several vector operators is to be understood as a tensor operator. By continuing in this way we can readily see that any state[6]

$$\hat{\mathbf{A}}^\dagger(\mathbf{x}_N,t_N)\cdots\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)|\{0\}\rangle\equiv|\mathbf{x}_N,t_N,\cdots,\mathbf{x}_1,t_1\rangle \qquad (26)$$

is an eigenstate of $\hat{n}_{V,t}$ if $(\mathbf{x}_1,t_1),\cdots,(\mathbf{x}_N,t_N)$ are either conjoint or disjoint with (V,t), and that the corresponding eigenvalue is equal to the number of events $(\mathbf{x}_1,t_1),\cdots,(\mathbf{x}_N,t_N)$ conjoint with (V,t). The states defined by (26) are to be regarded as tensors. In particular, if $t_1=t_2=\cdots=t_N=t$, then, apart from a set of measure zero,[7] any state of the type

$$|S\rangle\equiv\hat{\mathbf{A}}^\dagger(\mathbf{x}_N,t)\cdots\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t)|\{0\}\rangle \qquad (27)$$

is an eigenstate of $\hat{n}_{V,t}$, and the corresponding eigenvalue is equal to the number of points $\mathbf{x}_1,\cdots,\mathbf{x}_N$ lying within V. Moreover it is clear that each eigenvalue is degenerate to infinite order (at least as $L\rightarrow\infty$) since we may have an unlimited number of points outside V in Eq. (27) without affecting the eigenvalue.

Despite the infinite degeneracy it is not difficult to show that states of the general type (27) form an orthogonal set. Consider two different states

$$|S\rangle\equiv\hat{\mathbf{A}}^\dagger(\mathbf{x}_N,t)\cdots\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t)|\{0\}\rangle$$

and

$$|S'\rangle\equiv\hat{\mathbf{A}}^\dagger(\mathbf{x}_{N'}',t)\cdots\hat{\mathbf{A}}^\dagger(\mathbf{x}_1',t)|\{0\}\rangle.$$

If the states are different, the points $\mathbf{x}_1$, $\mathbf{x}_2,\cdots$, etc. and $\mathbf{x}_1'$, $\mathbf{x}_2',\cdots$, etc. cannot all coincide. Suppose that $\mathbf{x}_r$ does not coincide with any of the points $\mathbf{x}_1',\cdots,\mathbf{x}_{N'}'$. Then it is possible to choose a volume V such that $\mathbf{x}_r$ lies within V, while all the points $\mathbf{x}_1',\mathbf{x}_2',\cdots$, etc. lie outside V.[7] It follows from the foregoing that $|S'\rangle$ is an

[7] It would appear that points $\mathbf{x}$ lying neither inside nor outside the volume V form a set of measure zero. However, this is not strictly correct, since, as we have noted, it is not meaningful to localize the position where the photon is absorbed to better than about a wavelength. Accordingly, there is a small region (of volume much less than V) in the vicinity of the boundary of V, where we cannot strictly distinguish between points lying inside and outside V.

eigenstate of $\hat{n}_{V,t}$ belonging to the eigenvalue 0, while $|S\rangle$ is an eigenstate of $\hat{n}_{N,t}$ belonging to the eigenvalue 1 or higher. Since eigenstates of a Hermitian operator belonging to different eigenvalues are orthogonal, we see that the different states of type (27) form an orthogonal set.

The approximations involved in the derivation of the commutator (17) are reflected also in the orthogonality property, and it is apparent that difficulties will arise with the foregoing argument if the separations between corresponding points $\mathbf{x}_1$ and $\mathbf{x}_1'$, $\mathbf{x}_2$ and $\mathbf{x}_2'$, etc. are *all* of the order of, or less than, a wavelength of the set $\{\mathbf{k},s\}$. These difficulties are again connected with the impossibility of localizing the position of a photon to this accuracy. The orthogonality property therefore fails for states which are sufficiently close.[7]

In order to prove that states of the type (27) form a complete set, we will now show that any Fock state $|\{n_\lambda\}\rangle(\lambda\equiv\mathbf{k},s)$ can be expanded in terms of these states. We first note that the Hermitian conjugate of Eq. (1) can readily be inverted to read

$$\hat{a}_{\mathbf{k},s}{}^\dagger=\frac{1}{L^{3/2}}\int\hat{\mathbf{A}}^\dagger(\mathbf{x},t)\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{x}-ckt)]d^3x, \quad (28)$$

where the integral extends over the whole normalization volume. Now since every Fock state is expressible in the form

$$|\{n_\lambda\}\rangle=\prod_\lambda\frac{(\hat{a}_\lambda{}^\dagger)^{n_\lambda}}{\sqrt{(n_\lambda!)}}|\{0\}\rangle, \quad (29)$$

it follows from (28) that we can write

$$|\{n_\lambda\}\rangle=\prod_\lambda\frac{1}{L^{3n_\lambda/2}}\int\cdots\int\hat{\mathbf{A}}^\dagger(\mathbf{x}_{1_\lambda},t)\cdot\boldsymbol{\varepsilon}_\lambda\hat{\mathbf{A}}^\dagger(\mathbf{x}_{2_\lambda},t)\cdot\boldsymbol{\varepsilon}_\lambda\cdots\hat{\mathbf{A}}^\dagger(\mathbf{x}_{n_\lambda},t)\cdot\boldsymbol{\varepsilon}_\lambda|\{0\}\rangle$$
$$\times\frac{1}{\sqrt{(n_\lambda!)}}\exp(i[\mathbf{k}_\lambda\cdot(\mathbf{x}_{1_\lambda}+\cdots+\mathbf{x}_{n_\lambda})-ck_\lambda n_\lambda t])d^3x_{1_\lambda}\cdots d^3x_{n_\lambda}. \quad (30)$$

Since all the $\mathbf{x}_{1_\lambda},\cdots,\mathbf{x}_{n_\lambda}$, etc., apart from a set of measure zero,[7] must lie either inside or outside V, we see that Eq. (30) is an expansion of $|\{n_\lambda\}\rangle$ in terms of the eigenstates of $\hat{n}_{V,t}$.

We conclude therefore that any state of the field can be expanded in terms of the eigenstates (27) of $\hat{n}_{V,t}$, and that these states form a complete orthogonal set. However, the states (27) are not normalized to unity, and the projectors for these unnormalized states do not appear to offer a very simple resolution of the unit operator.

4. MOMENTS AND CORRELATIONS OF $n_{V,t}$

We will now apply the foregoing results to the evaluation of some moments of the $\hat{n}_{V,t}$ operators, which differ in certain interesting respects from the moments of $\hat{n}$.

We first note that, whether (V_1,t_1) is conjoint or disjoint with (V_2,t_2), the operators $\hat{n}_{V_1,t_1}$ and $\hat{n}_{V_2,t_2}$ commute. For, from Eq. (18) and its Hermitian conjugate (21), or from Eq. (20) and its conjugate (22), we find

$$\begin{aligned}\hat{n}_{V_1,t_1}\hat{n}_{V_2,t_2}&=\int_{V_1}\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)\cdot\hat{\mathbf{A}}(\mathbf{x}_1,t_1)\hat{n}_{V_2,t_2}\,d^3x_1,\\&=\int_{V_1}\hat{n}_{V_2,t_2}\hat{\mathbf{A}}^\dagger(\mathbf{x}_1,t_1)\cdot\hat{\mathbf{A}}(\mathbf{x}_1,t_1)\,d^3x_1,\\&=\hat{n}_{V_2,t_2}\hat{n}_{V_1,t_1},\end{aligned} \quad (31)$$

and similarly for the higher order products of operators. However, no obvious conclusion can be drawn if (V_1,t_1) is neither conjoint nor disjoint with (V_2,t_2). On the other hand, when $t_1=t_2=t$ the operators $\hat{n}_{V_1,t}$ and $\hat{n}_{V_2,t}$ always commute, since one can be expressed as the sum of operators conjoint and disjoint with the other.[7]

Consider now the problem of calculating the correlation between the number of photons in (V_1,t_1), $(V_2,t_2),\cdots$, (V_N,t_N), which, as is now well known,[1,3–5] is given by the expectation value of the normally ordered product

$$\begin{aligned}\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}:\rangle&=\sum_{i_1}\cdots\sum_{i_N}\int_{V_1}\cdots\int_{V_N}\langle\hat{A}_{i_1}{}^\dagger(\mathbf{x}_1t_1)\cdots\hat{A}_{i_N}{}^\dagger(\mathbf{x}_N,t_N)\hat{A}_{i_N}(\mathbf{x}_N,t_N)\cdots\hat{A}_{i_1}(\mathbf{x}_1,t_1)\rangle d^3x_1\cdots d^3x_N\\&=\sum_{i_1}\cdots\sum_{i_{N-1}}\int_{V_1}\cdots\int_{V_{N-1}}\langle\hat{A}_{i_1}{}^\dagger(\mathbf{x}_1,t_1)\cdots\hat{A}_{i_{N-1}}{}^\dagger(\mathbf{x}_{N-1},t_{N-1})\hat{n}_{V_N,t_N}\\&\qquad\times\hat{A}_{i_{N-1}}(\mathbf{x}_{N-1},t_{N-1})\cdots\hat{A}_{i_1}(\mathbf{x}_1,t_1)\rangle d^3x_1\cdots d^3x_{N-1}.\end{aligned} \quad (32)$$

We may use Eq. (17) or its Hermitian conjugate to move the $\hat{n}_{V_N,t_N}$ operator successively to the right or left. If the regions $(V_1,t_1),\cdots,(V_N,t_N)$ are all disjoint, we find

$$\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}:\rangle=\sum_{i_1}\cdots\sum_{i_{N-1}}\int_{V_1}\cdots\int_{V_{N-1}}\langle\hat{A}_{i_1}{}^\dagger(\mathbf{x}_1,t_1)\cdots\hat{A}_{i_{N-1}}{}^\dagger(\mathbf{x}_{N-1},t_{N-1})\times\hat{A}_{i_{N-1}}(\mathbf{x}_{N-1},t_{N-1})\cdots\hat{A}_{i_1}(\mathbf{x}_1,t_1)\hat{n}_{V_N,t_N}\rangle d^3x_1\cdots d^3x_{N-1},$$
$$=\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_{N-1},t_{N-1}}:\hat{n}_{V_N,t_N}\rangle. \quad (33)$$

The same result holds also if $(V_1,t_1),\cdots,(V_N,t_N)$ are not disjoint, but if the volumes $V_1,\cdots,V_N$ in which photon counts are to be correlated are located side by side in a field of plane waves, so that no ray traverses more than one volume. In practice these conditions usually apply to correlation measurements with several photoelectric detectors. From (33) it follows by recursion that

$$\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}:\rangle=\langle\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}\rangle, \quad (34)$$

if $(V_1,t_1),\cdots,(V_N,t_N)$ are all disjoint.

The situation is very different if the regions $(V_1,t_1),\cdots,(V_N,t_N)$ cannot be treated as disjoint. Suppose that we are again dealing with a field of plane waves and with identical volumes V, which are arranged in a straight line so that every ray traversing one volume traverses all. Moreover, let the separations between successive volumes V_1, $V_2,\cdots,V_N$ be $c(t_2-t_1)$, $c(t_3-t_2),\cdots,c(t_N-t_{N-1})$. Under these conditions every event in one space-time region has corresponding events in all the others lying on the same light cone. In principle it is possible to set up a correlation experiment with N photodetectors to which these conditions apply. Hence, if $(\mathbf{x}_r,t_r)$ belongs to any one of the regions, $(V_1,t_1),\cdots,(V_N,t_N)$, we have from (17)

$$\langle[\hat{\mathbf{A}}(\mathbf{x}_r,t_r),\hat{n}_{V_s,t_s}]\rangle=\langle\hat{\mathbf{A}}(\mathbf{x}_r,t_r)\rangle, \quad (35)$$

and $(\mathbf{x}_r,t_r)$ appears to be conjoint with (V_s,t_s) in relation to the field. This result can be used in (32) to move the $\hat{n}_{V_N,t_N}$ operator repeatedly to the right. Thus

$$\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}:\rangle=\sum_{i_1}\cdots\sum_{i_N}\int_{V_1}\cdots\int_{V_{N-1}}\langle\hat{A}_{i_1}{}^\dagger(\mathbf{x}_1,t_1)\cdots\hat{A}_{i_{N-1}}{}^\dagger(\mathbf{x}_{N-1},t_{N-1})\times\hat{A}_{i_{N-1}}(\mathbf{x}_{N-1},t_{N-1})\cdots\hat{A}_{i_1}(\mathbf{x}_1,t_1)(\hat{n}_{V_N,t_N}-N+1)\rangle d^3x_1\cdots d^3x_{N-1},$$
$$=\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_{N-1},t_{N-1}}:(\hat{n}_{V_N,t_N}-N+1)\rangle. \quad (36)$$

From (36) it follows by recursion that

$$\langle:\hat{n}_{V_1,t_1}\cdots\hat{n}_{V_N,t_N}:\rangle=\langle\hat{n}_{V_1,t_1}(\hat{n}_{V_2,t_2}-1)\cdots(\hat{n}_{V_N,t_N}-N+1)\rangle. \quad (37)$$

Equations (34) and (37) are to be compared with the corresponding formula for the total number operator

$$\langle:\hat{n}^N:\rangle=\langle\hat{n}(\hat{n}-1)\cdots(\hat{n}+N-1)\rangle, \quad (38)$$

where disjointness does not arise.

When the times $t_1,t_2,\cdots,t_N$ are all equal, it is not difficult to treat the more general situation, where the regions $(V_1,t_1),\cdots,(V_N,t_N)$ are not necessarily either conjoint or disjoint. We first recall that any operator $\hat{n}_{V,t}$ can be expressed as the sum of operators conjoint or disjoint with another operator $\hat{n}_{V',t}$, since V can always be expressed as the sum of volumes falling inside and outside V'. Accordingly, any normally ordered correlation (with $t_1=t_2=\cdots=t_N$) can be written as the sum of correlations $\langle:\hat{n}_{V_1,t}\cdots\hat{n}_{V_N,t}:\rangle$, such that every volume V_r lies wholly inside or wholly outside every other volume $V_{r'}(r'>r)$.[7] If this is done, we can use the basic relation (32) together with Eqs. (18) and (20), as appropriate, to move the $\hat{n}_{V_N,t}$ operator repeatedly to the right. We then obtain

$$\langle:\hat{n}_{V_1,t}\cdots\hat{n}_{V_N,t}:\rangle=\langle:\hat{n}_{V_1,t}\cdots\hat{n}_{V_{N-1},t}:[\hat{n}_{V_N,t}-\mathfrak{N}(V_1,\cdots,V_{N-1};V_N)]\rangle, \quad (39)$$

where $\mathfrak{N}(V_1,\cdots,V_{N-1};V_N)$ is the number of volumes $V_1,\cdots,V_{N-1}$ which lie within V_N. From (39) it follows by recursion that

$$\langle:\hat{n}_{V_1,t}\cdots\hat{n}_{V_N,t}:\rangle=\hat{n}_{V_1,t}[\hat{n}_{V_2,t}-\mathfrak{N}(V_1;V_2)]\cdots\times[\hat{n}_{V_N,t}-\mathfrak{N}(V_1,\cdots,V_{N-1};V_N)]. \quad (40)$$

This relation reduces to (34) or (37) under the appropriate restriction on the V's.

5. DISTRIBUTION OF EIGENVALUES OF $\hat{n}_{V,t}$

Let us now look briefly at the problem of determining the distribution of the number of photons $n_{V,t}$ localized in the volume V at time t. This problem is most conveniently tackled by way of the characteristic function $\langle\exp iy\hat{n}_{V,t}\rangle$. We first observe that the expectation value of the normally ordered operator $:\exp(\hat{n}_{V,t}x):$ is given by

$$\langle :\exp(\hat{n}_{V,t}x):\rangle=\sum_{r=0}^{\infty}(x^r/r!)\langle :\hat{n}_{V,t}{}^r:\rangle, \tag{41}$$

where the rth-order correlations $\langle :\hat{n}^r{}_{V,t}:\rangle$ are of the type given by Eq. (37). With the help of (37) the above sum can be evaluated, provided the use of the relation (37) remains valid as $r\rightarrow\infty$. Since the derivation of this relation was based on the commutator (17), which involved some (good) approximations, the question of the convergence of the expansion (41) when (37) is used ought to be investigated. However, we will not go into this question here, but merely note that, when the use of (37) in (41) is valid

$$\begin{aligned}\langle :\exp(\hat{n}_{V,t}x):\rangle &=\sum_{r=0}^{\infty}(x^r/r!)\langle \hat{n}_{V,t}(\hat{n}_{V,t}-1)\cdots(\hat{n}_{V,t}-r+1)\rangle\\ &=\langle (1+x)^{\hat{n}_{V,t}}\rangle,\end{aligned} \tag{42}$$

or, when x is replaced by $(e^{iy}-1)$,

$$\langle \exp(iy\hat{n}_{V,t})\rangle=\langle :\exp[\hat{n}_{V,t}(e^{iy}-1)]:\rangle. \tag{43}$$

By expressing the density operator $\hat{\rho}$ of the field in the basis formed by the eigenstates $|\{v_{\mathbf{k},s}\}\rangle$ of $\hat{\mathbf{A}}(\mathbf{x},t)$ in the general form found by Sudarshan,[8]

$$\hat{\rho}=\int\phi(\{v_{\mathbf{k},s}\})|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|\,d^2\{v_{\mathbf{k},s}\}, \tag{44}$$

where $\phi(\{v_{\mathbf{k},s}\})$ is a generalized function, so that

$$\langle\exp(iy\hat{n}_{V,t})\rangle=\mathrm{Tr}\int\phi(\{v_{\mathbf{k},s}\}):\exp[\hat{n}_{V,t}(e^{iy}-1)]:\;|\{v_{\mathbf{k},s}\}\rangle\langle\{v_{\mathbf{k},s}\}|\,d^2\{v_{\mathbf{k},s}\},$$

we see that the operators $\hat{\mathbf{A}}(\mathbf{x},t)$ and $\hat{\mathbf{A}}^\dagger(\mathbf{x},t)$ in the definition (2) of $\hat{n}_{V,t}$ can be replaced by their right and left eigenvalues $\mathfrak{V}(\mathbf{x},t)$ and $\mathfrak{V}^*(\mathbf{x},t)$. Hence

$$\langle\exp iy\hat{n}_{V,t}\rangle=\int\phi(\{v_{\mathbf{k},s}\})\exp[U(e^{iy}-1)]\,d^2\{v_{\mathbf{k},s}\}, \tag{45}$$

where

$$U=\int_V\mathfrak{V}^*(\mathbf{x},t)\cdot\mathfrak{V}(\mathbf{x},t)\,d^3x,$$

and

$$\mathfrak{V}(\mathbf{x},t)=\frac{1}{L^{3/2}}\sum_{\mathbf{k},s}v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp i(\mathbf{k}\cdot\mathbf{x}-ckt).$$

Equation (45) gives the characteristic generating function for the distribution $p(n_{V,t})$ of the number of photons $n_{V,t}$ in V at time t. The Fourier transform of this function with respect to y is $p(n_{V,t})$. However, the inversion can be performed at once, if we recall that $\exp[U(e^{iy}-1)]$ is the characteristic function of a Poisson distribution with parameter U. Then Eq. (45) leads immediately to

$$p(n_{V,t})=\int\phi(\{v_{\mathbf{k},s}\})\frac{U^{n_{V,t}}}{n_{V,t}!}e^{-U}\,d^2\{v_{\mathbf{k},s}\}. \tag{46}$$

This relation is identical in form to that found previously[9,1] for the total photon number within the normalization volume. It is also similar to the formula obtained for the distribution of counts registered by an illuminated photoelectric detector, when the interaction with the detector is treated in some detail.[10–12]

While the operator $\hat{n}_{V,t}$ appears to be the appropriate operator for characterizing the number of photons localized in (V,t), in the sense of a photoelectric measurement of the field, it is not yet clear whether the eigenstates of $\hat{n}_{V,t}$ are equally useful. Although states of this type have proved to be very valuable in some branches of statistical mechanics, the question remains to be answered by applications to problems of practical interest.

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