Observable Consequences of Fundamental-Length Hypotheses

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The postulate of the existence of a fundamental length may be expressed in terms of a minimum uncertainty in position measurements, or equivalently as a minimum uncertainty in measurements of the gravitational field. The postulate is expressed mathematically by means of "indeterminate operators," whose properties are discussed. With their aid, it is shown that the postulate of a fundamental length has as a consequence a certain broadening of spectral lines. In the case of a fundamental length of the order of 10^{-13} cm, the predicted broadening is much larger than the widths of nuclear gamma transitions already observed. It is concluded that this fundamental length is already in serious contradiction with experiment. In the case of a fundamental length due to gravitational effects of the order of 10^{-33} cm, as previously suggested by the author, the broadening is too small to have been observed in any experiments done to date. A modified Mössbauer experiment is suggested which should be capable of detecting this small broadening, if it is present. The experiment is difficult, but appears to be possible with presently available techniques. The gravitational-field uncertainties calculated by DeWitt, Peres, and Rosen, and others, lead to a still smaller broadening, which our proposed method would be incapable of detecting.

I. INTRODUCTION AND SUMMARY

IN a previous article,[1] the author has discussed the possible existence of a fundamental length, that is, of an absolute limitation on the possible accuracy of measurements of positions of particles and of relative readings of clocks. The existence of a fundamental length l is expressed by postulating that the outcome of a measurement of the position of an elementary particle can never be predicted with greater accuracy than $\pm l$. There is some freedom in what postulate one makes about uncertainties in the position of the center of mass of a macroscopic body, as the uncertainties associated with its different constituent elementary particles might partially cancel. A wide class of fundamental-length postulates may be encompassed in the requirement

$$\Delta x \gtrsim l\beta(R/l), \tag{1}$$

where Δx is the minimum uncertainty in the position of the center of mass of a particle (elementary or composite) of radius[2] R. $\beta(y)$ is a nonincreasing function which is of the order of unity when y is of the order of unity and need not be defined for $y \ll 1$. The assumption that one makes about the possible improved accuracy with composite particles is expressed through the behavior of β. Equation (1) is to be thought of as applying also to the reading of a clock of radius R. (We are using natural units.) Equation (1) applies independently of the properties (except R) of the particle being measured, and does not rule out the possibility of stronger limitations for certain particles. The reader should consult Ref. 1 for further details.

It has been shown (Ref. 1, Sec. VI) that the postulate (1) is equivalent to the postulate

$$\Delta g_{ik} \gtrsim (l/R)\beta(R/l), \tag{2}$$

in which Δg_{ik} is the minimum uncertainty of a component of the metric tensor averaged over a cubic space-time region of side R, in a coordinate system which is nearly Lorentzian on the average. Equation (2) applies separately to all components of the metric tensor which are taken as independent. The posutlates (1) and (2) are equivalent in the sense that either can be deduced from the other.

There are a number of possibilities for l and β, of which we mention three. The present author,[1] by considering the effect of gravitation on hypothetical position measurements, arrived at Eq. (1) with

$$l \sim \sqrt{G} = 1.6 \times 10^{-33}\ \text{cm}, \quad \beta(y) = 1, \tag{3}$$

where G is the gravitational constant in natural units. Several authors[3–6] have considered the problem of measuring the gravitational field, assuming that test particles are subject only to the known quantum-mechanical uncertainties. They arrive at Eq. (2) with

$$l \sim \sqrt{G}, \quad \beta(y) = y^{-1}. \tag{4}$$

In addition, the idea has frequently been mentioned of a fundamental length of the order of baryon Compton wavelengths. If one takes this as the fundamental length, and also postulates that position uncertainties for different elementary particles are independent but that the largest number that can be put into a sphere of radius R is of the order of $(R/l)^3$, then one finds Eq. (1) with

$$l \sim 10^{-13}\ \text{cm}, \quad \beta(y) = y^{-3/2}. \tag{5}$$

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[1] C. A. Mead, Phys. Rev. 135, B849 (1964).
[2] For a quantum-mechanical system R must be defined as some appropriate average, e.g., $R = \langle |\mathbf{r}| \rangle$.
[3] A. Peres and N. Rosen, Phys. Rev. 118, 335 (1960).
[4] J. L. Anderson, Rev. Mex. Fis. 3, 176 (1954).
[5] T. Regge, Nuovo Cimento 7, 215, (1958).
[6] B. S. DeWitt, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (John Wiley & Sons, Inc., New York, 1962).

Of course, other assumptions about β are also possible with this value of l.

The purpose of the present article is to express the fundamental length postulate in mathematical form, so that consequences can be deduced from it; and to show that it leads to a broadening of spectral lines which, depending on the parameters l and β, may be observable. For the benefit of the reader who does not wish to work through the mathematics of Secs. II and III, we give here a crude argument which leads to the correct result for the broadening.

First, let us postulate Eq. (1), and consider for simplicity a single elementary particle moving in a potential energy field $V(\mathbf{r})$. If the position of the particle is displaced by $\delta\mathbf{r}$, its potential energy will be shifted by an amount

$$\delta V=(\nabla V)\cdot\delta\mathbf{r}. \tag{6}$$

According to the fundamental-length postulate, each component of the displacement is in doubt by l, so that V is also uncertain by the amount

$$\Delta V\gtrsim|\nabla V|l. \tag{7}$$

The kinetic energy is unaffected, so the total energy of the state is in doubt by

$$\Delta E\gtrsim l\langle|\nabla V|\rangle,$$

where $\langle\ \rangle$ denotes an average. Under normal circumstances, for a nearly stationary bound state with average binding energy E_B and radius R, we have

$$\langle|\nabla V|\rangle\sim\left\langle\frac{\mathbf{r}\cdot\nabla V}{r}\right\rangle\sim\frac{|E_B|}{R}.$$

If there are a number of particles present, their average energies are roughly additive, but by hypothesis the uncertainties combine in such a way as to lead to a factor β. An energy level of a bound system of radius R thus has a minimum spread in energy given by

$$\Delta E\gtrsim E_B(l/R)\beta(R/l), \tag{8}$$

where E_B is now the total binding energy. One might conclude, then, that the frequency of a transition between two such states will be spread out by

$$\Delta\nu\gtrsim(l/R)\beta(R/l)E_B. \tag{9}$$

However, the transitions usually studied in atomic and nuclear systems involve mainly the outer particles, with inner closed shells remaining more or less undisturbed. A more conservative and realistic estimate of the broadening is therefore

$$\Delta\nu\gtrsim(l/R)\beta(R/l)\nu_0, \tag{10}$$

where ν_0 is the average frequency of the transition.[7]

[7] If there is no correlation between the energy deviations of the two states, one obtains Eq. (9); if they correlate so as to cancel the uncertainty as nearly as possible, the result is Eq. (10). Equation (10) therefore gives the minimum possible broadening.

The result (10) can be obtained much more directly if instead of Eq. (1) we postulate the equivalent Eq. (2). The observed frequency of a transition undergoes a gravitational shift given by

$$\delta\nu\sim\nu_0\langle\phi\rangle,$$

where $\langle\phi\rangle$ is the average gravitational potential in the region occupied by the emitter. According to (2), however, $\langle\phi\rangle=\frac{1}{2}(g_{00}-1)$ is in doubt by

$$\Delta\langle\phi\rangle\gtrsim(l/R)\beta(R/l).$$

We therefore immediately obtain Eq. (10) again. The fact that the postulates (1) and (2) lead to the same predicted broadening is an instructive illustration of their equivalence. Let us now examine the order of magnitude of the predicted broadening for some particular fundamental-length hypotheses, in the case of nuclear γ transitions.

First, if we postulate a fundamental length of the order of 10^{-13} cm, then we have $\beta(R/l)\sim1$ (since for nuclei $R\sim10^{-13}$ cm), and we find from (10)

$$\Delta\nu\gtrsim\nu_0(l/R)\sim\nu_0. \tag{11}$$

According to this hypothesis, therefore, the widths of nuclear γ transitions ought to be very large, of the same order of magnitude as the average frequency. Of course, observed widths are many orders of magnitude smaller than this. We conclude, therefore, that this fundamental length postulate is already very strongly contradicted by experiment.

If we postulate the present author's fundamental length (3) together with the strong-broadening assumption (9), then the widths are of the same order of magnitude as the narrowest Mössbauer widths observed to date,[8–10] i.e., around 10^4 sec^{-1}. For example, in the case of the 93-keV, 44-sec transition of Ag^{107}, we have $E_B\sim10^9$ eV, and Eqs. (3) and (9) lead to a width about 10^6 times larger than the "natural" width $1/t$ (t=mean lifetime). It is amusing to note that Bizina *et al.*[9] observe a width of just this order of magnitude. They calculate that under the conditions of their experiment the line ought to be broadened by a factor of 10^5 due to magnetic interactions, and the resonant cross section they observe is about $\frac{1}{8}$ of what they calculate on this basis. Their result could therefore be explained by this fundamental-length hypothesis, but this is probably fortuitous as there are other effects present. Also, as mentioned before, the smaller broadening (10) seems much more reasonable than (9). Nevertheless, it would be worthwhile to refine these measurements.

If we make the more reasonable postulate of (3)

[8] P. P. Craig, D. E. Nagle, and D. R. F. Cochran, Phys. Rev. Letters 4, 561 (1960).

[9] G. E. Bizina, A. G. Beda, N. A. Burgov, and A. V. Davydov, Rev. Mod. Phys. 36, 358 (1964); Zh. Eksperim. i Teor. Fiz. 45, 1408 (1963) [English transl.: Soviet Phys.—JETP 18, 973 (1964)].

[10] H. Frauenfelder, *The Mössbauer Effect* (W. A. Benjamin, Inc., New York, 1963).

together with (10), we find for nuclear transitions

$$\Delta\nu \gtrsim (l/R)\nu_0 \sim 10^{-20}\nu_0. \tag{12}$$

For $\nu_0 \sim 10^5$ eV, Eq. (12) predicts a width $\Delta\nu \sim 10^{-15}$ eV ~ 1 sec^{-1}. This is much smaller than the narrowest widths observed so far. Its experimental detection would consist of measuring the width of a gamma transition with a lifetime of several seconds or more (and with all other sources of broadening excluded), and finding a measured width greater than $1/t$, and of the order of magnitude predicted by Eq. (12). In Sec. IV, an experiment is suggested which should be capable of detecting this broadening if it is present.

If one postulates Eq. (4) instead of (3), the predicted broadening is reduced by another factor of 10^{-20} due to the factor $\beta(R/l)$. There does not seem to be any way of detecting this very small broadening at the present time.

The argument just given for the existence of the broadening is obviously not rigorous or even very convincing. For example, it is quite normal in quantum mechanics for the potential energy to be unknown but for the total energy nevertheless to have a sharp value. Our first task, therefore, is to make this argument more rigorous; and this, in turn, requires that we express the fundamental-length postulate itself in mathematical form. Accordingly, in Sec. II we define "indeterminate operators," which have the property that they can never be diagonalized but are always subject to a minimum uncertainty. In Sec. III, the fundamental length postulate is expressed by using indeterminate operators to represent x (or g_{ik}), and the minimum spread in energy is derived. As in this section, two derivations are given, starting from the two equivalent ways of stating the fundamental-length hypothesis. There is no attempt to make the treatment relativistically covariant, as we are dealing with systems in which relativistic effects are not important. In Sec. IV, a modified Mössbauer experiment is described which should be capable of detecting the broadening predicted by (12) if it is present, thus providing an experimental test of the author's fundamental-length theory. The experiment is quite difficult, but does not appear to be impossible with presently available techniques.

The paper has been organized in such a way that the understanding of the experimental part (Sec. IV) does not depend on any knowledge of the mathematical part (Secs. II, III). Accordingly, if the reader is willing to accept (provisionally) the broadening formulas (10), (12) on the basis of the crude arguments just given, he may skip directly to Sec. IV, where the proposed experimental test of Eq. (12) is discussed.

There are two appendices: Appendix A briefly discusses the question of relativistic covariance, and also contains some remarks about the "stochastic" theory of Ingraham.[11] It is concluded that Ingraham's theory is not a fundamental-length theory in the sense meant here; that is, it is not equivalent to Eq. (1) or (2). Appendix B discusses some mathematical properties of the indeterminate operators, and their time dependence.

[11] R. L. Ingraham, Nuovo Cimento 24, 1117 (1962); 27, 303 (1963); 32, 323 (1964).

II. INDETERMINATE OPERATORS

The problem to be treated in this section is: How can we give mathematical expression to a requirement such as (1) without doing too much violence to the operator formalism of quantum mechanics? At first glance this seems impossible (see, e.g., Ref. 1, Sec. VII), since the existence of eigenfunctions corresponding to sharp values of all operators plays an important role in the development of quantum theory. In fact, however, the requirement can be expressed quite simply.

Suppose there is some physical quantity A for which we wish to require

$$\Delta A \gtrsim \lambda.$$

If A is to be represented by an operator, this means that for any normalized ket $|\alpha\rangle$, we must have

$$\langle\alpha|A^2|\alpha\rangle - \langle\alpha|A|\alpha\rangle^2 \gtrsim \lambda^2.$$

Clearly, the operator A must not have any eigenfunctions. This property can be guaranteed in the following way: Let there exist a Hermitian operator Q with the following properties: (a) Q is bounded, e.g.,

$$-1 \leq Q \leq 1. \tag{13}$$

Equation (13) means that all the eigenvalues of Q lie in the indicated range, or that $\langle\alpha|Q|\alpha\rangle$ is always in the range. (b) Q fails to commute with A, e.g.,

$$[A,Q] = i\lambda. \tag{14}$$

Now because of (14) and the uncertainty principle,

$$\Delta A \Delta Q \gtrsim \lambda, \tag{15}$$

while because of (13),

$$\Delta Q \lesssim 1. \tag{16}$$

Equations (15) and (16) immediately combine to give

$$\Delta A \gtrsim \lambda, \tag{17}$$

which is the desired result. An operator such as A, whose commutator with some bounded operator is a nonzero c number, will be called an "indeterminate operator."[12]

Although an indeterminate operator has no eigenfunctions, and therefore no eigenvalues, it is clear that the moments of the probability distribution of A can always be found; that is, one can evaluate $\langle\alpha|A^2|\alpha\rangle$, $\langle\alpha|A^3|\alpha\rangle$, etc. While the above properties will be sufficient for the purposes of the following section, the rigorous self-consistent definition of indeterminate

[12] This could obviously be generalized by requiring that the absolute value of the commutator have a nonzero lower bound, instead of being necessarily a c number.

operators requires a discussion of such things as domains of definition. These matters are discussed in Appendix B, along with the time dependence of the Q operators in the Heisenberg picture. These considerations will permit us to answer the question of the possible results of a single measurement of an indeterminate operator.

These operators may seem somewhat strange, but in fact we have all (perhaps unconsciously) used at least one such operator ever since we first began learning quantum mechanics: the momentum of a particle in a box is an indeterminate operator. For a particle in a one-dimensional box of length L, we have

$$[p,x]=-i,$$

and

$$0\leq x\leq L.$$

Therefore,

$$\Delta p\gtrsim 1/L.$$

In the next section, we express the postulates (1) and (2) by using indeterminate operators to represent x (or g_{ik}), i.e., by postulating the existence of appropriate Q operators. The uncertainty in energy is then derived by showing that the Hamiltonian also fails to commute with certain operators containing Q. No physical interpretation is given to the Q operators here, as we want our results to depend only on the fundamental-length hypothesis, not on any particular physical interpretation of the origin of the fundamental length. For the purposes of this article, the Q operators are to be thought of simply as a mathematical device which permits us to express the fundamental-length postulate in precise form and draw conclusions from it.

III. DERIVATION OF LINE BROADENING

A. Fundamental-Length Approach

Equation (1) means that for any state of a system under consideration, the position of the center of mass of a body of radius R is always in doubt by at least $l\beta(R/l)$. Therefore, if the position is represented by a Schrödinger-picture operator, that operator must be indeterminate. Accordingly, we postulate the existence of a vector operator $\mathbf{Q}$ such that

$$[r_\alpha,Q_\beta]=il\beta(R/l)\delta_{\alpha\beta}, \tag{18}$$

(where Greek suffixes label the three space coordinates), and

$$-1\leq Q_\alpha\leq 1. \tag{19}$$

The commutation rule (18) is satisfied if we simply write

$$\mathbf{Q}=l\beta(R/l)\mathbf{p},$$

so the question naturally arises of whether the fundamental-length postulate can be interpreted simply as requiring the boundedness of $\mathbf{p}$. The bound that one gets for $\mathbf{p}$ in this case is

$$|\mathbf{p}|\lesssim[l\beta(R/l)]^{-1},$$

which means that the velocity v satisfies

$$|\mathbf{v}|\lesssim[ml\beta(R/l)]^{-1},$$

where m is the mass.

It is now clear that this will not do, since the theory is supposed to apply to macroscopic as well as microscopic bodies. For macroscopic bodies, m is roughly proportional to R^3, so this would lead to velocity limitations contrary to everyday experience unless $\beta(y)$ decreases at least as y^{-3}. This is not the case for the fundamental-length theories of Eqs. (3), (4), or (5). We conclude, therefore, that $\mathbf{Q}$ must be a new operator, perhaps having something to do with the measuring apparatus, but at any rate not one of the familiar dynamical variables of the system.

Now let us consider a single elementary particle moving in a potential field. The Hamiltonian is

$$\mathfrak{H}=p^2/2m+V(\mathbf{r}), \tag{20}$$

and (18) takes the form

$$[r_\alpha,Q_\beta]=il\delta_{\alpha\beta}. \tag{21}$$

Using (20), (21) and the usual commutator between $\mathbf{r}$ and $\mathbf{p}$, we can evaluate the commutator[13]

$$[\mathfrak{H},(\mathbf{r}\cdot\mathbf{Q})]=il\mathbf{r}\cdot\nabla V-(i/m)\mathbf{p}\cdot\mathbf{Q}. \tag{22}$$

It now follows from (22) and the uncertainty principle that

$$\Delta\mathfrak{H}\Delta(\mathbf{r}\cdot\mathbf{Q})\gtrsim|l\langle\mathbf{r}\cdot\nabla V\rangle-(1/m)\langle\mathbf{p}\cdot\mathbf{Q}\rangle|. \tag{23}$$

From (19) it is clear that

$$\Delta(\mathbf{r}\cdot\mathbf{Q})\lesssim R. \tag{24}$$

We now find, therefore,

$$\Delta\mathfrak{H}\gtrsim\frac{1}{R}\left|l\langle\mathbf{r}\cdot\nabla V\rangle-\frac{1}{m}\langle\mathbf{p}\cdot\mathbf{Q}\rangle\right|. \tag{25}$$

For stationary or nearly stationary bound states, the first term on the right-hand side of (25) is always positive because of the virial theorem. The only question is whether it can be cancelled by the second term. A moment's reflection shows, however, that, while such a cancellation might take place for some states, it cannot take place for all states of a complete set because the operator $\mathbf{p}\cdot\mathbf{Q}$ has zero trace. It follows that if its expectation value is positive for some states in a complete set, permitting partial or complete cancellation, there must be other states in the set for which it is negative, leading to reinforcement. Also, it is shown in Appendix B that the Hamiltonian is not "maximal" from which it follows that its eigenvalue problem is insoluble.

The situation can be understood in somewhat more detail as follows: We can express the position operator $\mathbf{r}$ as

$$\mathbf{r}=\mathbf{r}_0-l\mathbf{P}, \tag{26}$$

[13] Operators such as $\mathbf{r}\cdot\mathbf{Q}$ are understood to be symmetrized, so that they are Hermitian.

in which $\mathbf{r}_0$ commutes with $\mathbf{Q}$, and $\mathbf{P}$ is the momentum conjugate to $\mathbf{Q}$. The Hamiltonian can now be broken up into a zero-order part plus a perturbation as follows:

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}', \tag{27}$$

where

$$\mathcal{H}_0=p^2/2m+V(\mathbf{r}_0), \tag{28}$$

and

$$\mathcal{H}'=-l\mathbf{P}\cdot\nabla V(r_0)+\cdots. \tag{29}$$

The eigenfunctions of $\mathcal{H}_0$ are simply products $\psi(\mathbf{r}_0)\phi(\mathbf{Q})$. They are very highly degenerate, since $\mathcal{H}_0$ does not contain $\mathbf{P}$ or $\mathbf{Q}$ at all. We refer to these zero order energy levels as "multiplets." The vanishing of the trace of $\mathbf{p}\cdot\mathbf{Q}$ can easily be seen in the representation of the zero-order eigenfunctions.

Now suppose we try to construct eigenfunctions (or approximate eigenfunctions) of the full Hamiltonian $\mathcal{H}$. Under the perturbation, a multiplet may be split into many levels, but if l is small the splitting will be small compared with the energy difference between multiplets. In fact, because of parity conservation, there are no matrix elements of the first-order term in $\mathcal{H}'$ connecting states of the same multiplet. It follows that the splitting within a multiplet is at least of second order in l:

$$\delta E=O(l^2), \tag{30}$$

where δE is the difference in energy between any two perturbed states belonging to the same multiplet. Because of the tracelessness of $\mathbf{p}\cdot\mathbf{Q}$, there will always be some states for which

$$\langle\mathbf{p}\cdot\mathbf{Q}\rangle\leq 0. \tag{31}$$

We call these "broadened states," since from (25) and (31) they have

$$\Delta\mathcal{H}\gtrsim(l/R)\langle\mathbf{r}\cdot\nabla V\rangle. \tag{32}$$

Now if l is small, it is clear from (30) and (32) that the width of a broadened state is greater than the multiplet splitting. It follows that the energy distribution of a broadened state overlaps the energies of all states of the same multiplet, so that even the smallest perturbation will mix them. A state as normally prepared, therefore, will always contain large admixtures of broadened states satisfying (32), unless the entire multiplet contains no broadened states. This last circumstance is clearly rather exceptional.

The rest of the argument can be taken over virtually unchanged from that of the introduction. For most potentials we have

$$\langle\mathbf{r}\cdot\nabla V\rangle\sim|E_B|.$$

For composite systems, the contributions of different particles combine to give a factor of β. For a transition involving only an outer shell, then, we find as before for the spread in frequency:

$$\Delta\nu\gtrsim\nu_0(l/R)\beta(R/l), \tag{33}$$

which is the desired result.

B. Gravitational Approach

The derivation given in the previous subsection may be criticized as being rather artificial. The model used is that of a nonrelativistic particle moving in a force field, with the fundamental-length postulate incorporated; however, it is known that this particle picture does not work in any case for distances less than the Compton wavelength. Field theory must be used in order to allow for virtual pair creation, etc. It is therefore of more than academic interest that the argument starting from the postulate (2) be presented, despite its greater complexity. Another reason why the previous derivation is unsatisfactory is brought out in Appendix B.

To express the postulate (2) in mathematical form, it is necessary to represent the gravitational field by indeterminate operators. As a preliminary to this, it is necessary that the gravitational field be quantized. This involves a number of problems, but fortunately we do not need to solve them. It is sufficient for our purposes to quantize the field in linear, nonrelativistic approximation. That is, we only go to the first order in the gravitational constant G, and to second order in time derivatives (i.e., second order in c^{-1}, if we were using cgs units).

Following the usual procedure, then,[14] we represent the metric tensor as the sum of the Lorentz metric plus a small correction:

$$g_{ik}=g_{ik}{}^{(0)}+4\gamma_{ik},$$

where

$$g_{00}{}^{(0)}=-g_{\alpha\alpha}{}^{(0)}=1,\quad g_{ik}{}^{(0)}=0,\quad i\neq k,$$

and γ_{ik} is presumed to be small, so that terms of second or higher order in γ_{ik} may be neglected. We now define

$$\psi_{ik}\equiv\gamma_{ik}-\tfrac{1}{2}g_{ik}{}^{(0)}\gamma, \tag{34}$$

where

$$\gamma\equiv\gamma^i{}_i=g^{ik(0)}\gamma_{ik}. \tag{35}$$

In terms of the tensor ψ_{ik}, the field equations take the form

$$-\Box\psi_{ik}=\left(-\frac{\partial^2}{\partial t^2}+\nabla^2\right)\psi_{ik}=4\pi GT_{ik}, \tag{36}$$

where G is the gravitational constant and T_{ik} is the energy-momentum tensor of matter. There is also a subsidiary condition:

$$\partial_k\psi_i{}^k=0. \tag{37}$$

Since we do not require a completely covariant formulation, it is convenient to break up the tensor ψ_{ik} into scalar, vector, and tensor parts according to transformation properties under pure rotations:

$$\psi_{00}\equiv\phi;$$
$$\psi_{0\alpha}\equiv B_\alpha;$$
$$\psi_{\alpha\beta}\equiv A_{\alpha\beta}=A_\alpha{}^\beta. \tag{38}$$

[14] L. D. Landau and E. Lifshitz, *The Classical Theory of Fields*, translated by M. Hamermesh (Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1961), pp. 323–326.

Note that the three-dimensional space metric is now taken to be the ordinary Euclidian one, i.e., without the minus sign. It will presently be apparent that ϕ is the usual Newtonian gravitational potential. We also divide the energy-momentum tensor into parts as follows:

$$\begin{aligned} T_{00} &\equiv U; \\ T_{0\alpha} &\equiv J_\alpha; \\ T_{\alpha\beta} &\equiv \tau_{\alpha\beta}. \end{aligned} \tag{39}$$

For the energy-momentum tensor due to a mass density ρ and velocity field $\mathbf{v}$, we have, up to second order in $\mathbf{v}$:

$$\begin{aligned} U &= \rho(1+v^2); \\ \mathbf{J} &= -\rho\mathbf{v}; \\ \tau_{\alpha\beta} &= \rho v_\alpha v_\beta. \end{aligned} \tag{40}$$

From Eqs. (36), (38), and (39) we now have for the field equations:

$$\begin{aligned} -\Box\phi &= 4\pi G U; \\ -\Box\mathbf{B} &= 4\pi G\mathbf{J}; \\ -\Box A_{\alpha\beta} &= 4\pi G\tau_{\alpha\beta}, \end{aligned} \tag{41}$$

while the subsidiary condition (37) becomes

$$\begin{aligned} \dot{\phi} - \boldsymbol{\nabla}\cdot\mathbf{B} &= 0; \\ \partial\mathbf{B}/\partial t - \boldsymbol{\nabla}A &= 0. \end{aligned} \tag{42}$$

The Lagrangian density which leads to the field equations (36) is

$$\mathfrak{L} = \frac{1}{8\pi G}(\partial_j\psi_{ik}\partial^j\psi^{ik}) - \psi_{ik}T^{ik}.$$

We wish to represent our fields by Schrödinger-picture operators. To this end, we first make a Fourier expansion with periodic boundary conditions in a box of volume V as follows:

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa[\phi(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}) \\ &\qquad + \phi^\dagger(\boldsymbol{\kappa})\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})]; \\ \mathbf{B}(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa[\mathbf{B}(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}) \\ &\qquad + \mathbf{B}^\dagger(\boldsymbol{\kappa})\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})]; \\ A_{\alpha\beta}(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa[A_{\alpha\beta}(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}) \\ &\qquad + A^\dagger{}_{\alpha\beta}(\boldsymbol{\kappa})\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})]; \\ U(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa U(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}); \\ \mathbf{J}(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa \mathbf{J}(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}); \\ \tau_{\alpha\beta}(\mathbf{r}) &= \frac{1}{\sqrt{V}}\sum_\kappa \tau_{\alpha\beta}(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}). \end{aligned} \tag{43, 44}$$

The sums go over all wave number vectors $\boldsymbol{\kappa}$ satisfying the periodic boundary conditions.

After some straightforward manipulations, we arrive at the Hamiltonian

$$\begin{aligned} \mathfrak{IC} = \mathfrak{IC}^0 + \sum_\kappa \Big\{ & \frac{\kappa^2}{2\pi G} \\ & \times[\phi^\dagger(\boldsymbol{\kappa})\phi(\boldsymbol{\kappa}) - 2\mathbf{B}^\dagger(\boldsymbol{\kappa})\cdot\mathbf{B}(\boldsymbol{\kappa}) + A^\dagger{}_{\alpha\beta}(\boldsymbol{\kappa})A^{\alpha\beta}(\boldsymbol{\kappa})] \\ & + U(-\boldsymbol{\kappa})[\phi(\boldsymbol{\kappa}) + \phi^\dagger(-\boldsymbol{\kappa})] \\ & - 2\mathbf{J}(-\boldsymbol{\kappa})\cdot[\mathbf{B}(\boldsymbol{\kappa}) + \mathbf{B}^\dagger(-\boldsymbol{\kappa})] \\ & + \tau^{\alpha\beta}(-\boldsymbol{\kappa})[A_{\alpha\beta}(\boldsymbol{\kappa}) + A^\dagger{}_{\alpha\beta}(-\boldsymbol{\kappa})]\Big\}, \end{aligned} \tag{45}$$

with the commutation relations

$$\begin{aligned} [\phi(\boldsymbol{\kappa}), \phi^\dagger(\boldsymbol{\kappa}')] &= \frac{2\pi G}{\kappa}\delta_{\kappa\kappa'}; \\ [B_\alpha(\boldsymbol{\kappa}), B_\beta{}^\dagger(\boldsymbol{\kappa}')] &= -\frac{\pi G}{\kappa}\delta_{\alpha\beta}\delta_{\kappa\kappa'}; \\ [A_{\alpha\beta}(\boldsymbol{\kappa}), A^\dagger{}_{\gamma\epsilon}(\boldsymbol{\kappa}')] &= \frac{\pi G}{\kappa}(\delta_{\alpha\gamma}\delta_{\beta\epsilon} + \delta_{\alpha\epsilon}\delta_{\beta\gamma})\delta_{\kappa\kappa'}, \end{aligned} \tag{46}$$

all other commutators being zero. In Eq. (45), $\mathfrak{IC}^0$ is the Hamiltonian for matter alone, i.e., for everything in the system except the gravitational field.

It is easy to verify from the equations of motion that

$$\begin{aligned} \dot{\phi}(\mathbf{r}) = -i[\phi(\mathbf{r}), \mathfrak{IC}] = \frac{1}{\sqrt{V}}\sum_\kappa -i\kappa[\phi(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}) \\ - \phi^\dagger(\boldsymbol{\kappa})\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})], \end{aligned} \tag{47}$$

with similar relations holding for $\mathbf{B}$ and A. The subsidiary condition (42) cannot be satisfied as an operator identity, but must be treated as a condition on allowable state vectors. If for a given wave number $\boldsymbol{\kappa}$ we take the z axis in the direction of $\boldsymbol{\kappa}$, and make use of (43) and (47), we see that the subsidiary condition (42) can be expressed by the requirement

$$\begin{aligned} [\phi(\boldsymbol{\kappa}) + B_z(\boldsymbol{\kappa})]|\ \rangle &= 0; \\ [B_\alpha(\boldsymbol{\kappa}) + A_{\alpha z}(\boldsymbol{\kappa})]|\ \rangle &= 0, \end{aligned} \tag{48}$$

where $|\ \rangle$ is an allowable state vector.

In order to represent the postulate (2), we must make the field operators indeterminate. Accordingly, for each $\boldsymbol{\kappa}$ we postulate the existence of bounded scalar, vector, and tensor operators $Q(\boldsymbol{\kappa})$, $\mathbf{R}(\boldsymbol{\kappa})$, and $S_{\alpha\beta}(\boldsymbol{\kappa})$, and the commutation relations

$$\begin{aligned} [\phi(\boldsymbol{\kappa}), Q^\dagger(\boldsymbol{\kappa}')] &= il\xi(\kappa)\delta_{\kappa\kappa'}; \\ [B_\alpha(\boldsymbol{\kappa}), R_\beta{}^\dagger(\boldsymbol{\kappa}')] &= il\xi(\kappa)\delta_{\alpha\beta}\delta_{\kappa\kappa'}; \\ [A_{\alpha\beta}(\boldsymbol{\kappa}), S_{\gamma\epsilon}{}^\dagger(\boldsymbol{\kappa}')] &= il\xi(\kappa)(\delta_{\alpha\gamma}\delta_{\beta\epsilon} + \delta_{\alpha\epsilon}\delta_{\beta\gamma})\delta_{\kappa\kappa'}. \end{aligned} \tag{49}$$

The parameter $\xi(\kappa)$ will be determined later in such a

way as to give the result (2). Before doing this, however, we must make sure that the commutation laws (49) are compatible with the subsidiary condition (48). The commutation rules make the field operators indeterminate, so that they have no eigenvalues; according to (48), however, certain combinations of them must have the eigenvalue zero. Hence, our bounded operators must be defined in such a way that the operators appearing in square brackets in (48) commute with all the Q, R, S. It is easy to see that this will be guaranteed if we require

$$R_z(\kappa)=-Q(\kappa)=-\tfrac{1}{2}S_{zz}(\kappa);$$
$$R_\alpha(\kappa)=-S_{\alpha z}(\kappa),\quad \alpha\neq k. \tag{50}$$

The only one of the bounded operators that we will really need is the scalar part Q. By combining (49) and (50), we see that it satisfies the commutation rules

$$[\phi(\kappa),Q^\dagger(\kappa')]=il\xi(\kappa)\delta_{\kappa\kappa'};$$
$$[B_z(\kappa),Q^\dagger(\kappa')]=-il\xi(\kappa)\delta_{\kappa\kappa'}; \tag{51}$$
$$[A_{zz}(\kappa),Q^\dagger(\kappa')]=il\xi(\kappa)\delta_{\kappa\kappa'}.$$

All other commutators involving Q are zero except those derivable from (51) by taking Hermitian conjugates.

We are now ready to formulate the postulate (2). Define the field operator

$$Q(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_\kappa[Q(\kappa)\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}) +Q^\dagger(\kappa)\exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})], \tag{52}$$

and require that it be bounded between plus and minus unity at all points. It obviously follows that its average over a spherical region of radius R is also bounded:

$$-1\leq\langle Q(R)\rangle\leq 1. \tag{53}$$

Now Eq. (2) will follow for ϕ if

$$[\langle\phi(R)\rangle,\langle Q(R)\rangle]\sim il\beta(R/l). \tag{54}$$

It will simplify matters at this point if we specialize slightly by requiring that $\beta(y)=y^{-n}$, where n is some nonnegative number, not necessarily an integer. Now averaging over a region of radius $\sim R$ is essentially the same as limiting ourselves to wave numbers for which $\kappa\leq 1/R$, and this is also true for the cubic space-time region over which ϕ is to be averaged. Using (43) and (52), therefore, we arrive at the requirement

$$[\langle\phi(R)\rangle,\langle Q(R)\rangle]=\frac{2}{V}il\sum_{\kappa<1/R}\xi(\kappa)=i\left(\frac{l}{R}\right)^{n+1}. \tag{55}$$

Replacing summation by integration in the usual way, we find that (55) will be satisfied for all R if

$$\xi(\kappa)=(n+1)\pi^2 l^n\kappa^{n-2}. \tag{56}$$

Equations (51), (53), and (56) are equivalent to the postulate (2) for the potential ϕ. The other components of the metric tensor are handled in an analogous way using the operators R and S.

To derive the broadening, consider some position-dependent Hermitian operator $\eta(\mathbf{r})$, involving only the matter variables, represented as follows:

$$\eta(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_\kappa\eta(\boldsymbol{\kappa})\exp(i\boldsymbol{\kappa}\cdot\mathbf{r}), \tag{57}$$

and define an operator Γ by

$$\Gamma=\int\eta(\mathbf{r})Q(\mathbf{r})d^3r=\sum_\kappa\eta(-\boldsymbol{\kappa})[Q(\kappa)+Q^\dagger(-\kappa)]. \tag{58}$$

Now, using (45), (51), (58), and the equation of motion

$$i\dot{\eta}=[\eta,\mathfrak{IC}],$$

we find

$$[\mathfrak{IC},\Gamma]=\sum_\kappa\{-i\dot{\eta}(\kappa)[Q(\kappa)+Q^\dagger(-\kappa)]\}$$
$$-il\sum_\kappa\eta(-\kappa)\xi(\kappa)\left\{\frac{\kappa^2}{2\pi G}[\phi(\kappa)+\phi^\dagger(-\kappa)]\right.$$
$$+2U(\kappa)+\frac{\kappa^2}{\pi G}[B_z(\kappa)+B_z^\dagger(-\kappa)]$$
$$\left.+4J_z(\kappa)+\frac{\kappa^2}{2\pi G}[A_{zz}(\kappa)+A_{zz}^\dagger(-\kappa)]+2\tau_{zz}(\kappa)\right\}. \tag{59}$$

For reasons similar to those given in the previous subsection, the first summation is traceless and therefore does not affect the broadening on the average (though perhaps for some states). We now have to consider the other terms in (59). Let

$$y=(1/2\pi G)[\phi(\kappa)+\phi^\dagger(-\kappa)].$$

Then we have from the equation of motion (41):

$$\kappa^2 y+2U(\kappa)=-\ddot{y}.$$

To lowest order in time derivatives,

$$y=-\frac{2U(\kappa)}{\kappa^2},\quad \ddot{y}=-\frac{2\ddot{U}(\kappa)}{\kappa^2}.$$

Hence, up to second order in time derivatives,

$$\kappa^2 y+2U(\kappa)=\frac{2\ddot{U}(\kappa)}{\kappa^2}=\frac{2\ddot{\rho}(\kappa)}{\kappa^2},$$

in which we have used (40) and noted that the other term in U is already of second order in time derivatives; its second time derivative is of fourth order, therefore negligible under the circumstances we are considering.

The treatment of the terms involving B and A is similar. Here, however, J and τ are already at least of first order in time derivatives, so all contributions of these terms are negligible after the required two more time differentiations. Hence, on the average (leaving out traceless operators), and through the second order in time derivatives, we have

$$[\mathfrak{IC},\Gamma]=2il\sum_\kappa \eta(-\kappa)\xi(\kappa)\ddot{\rho}(\kappa)\kappa^{-2}, \tag{60}$$

or

$$[\mathfrak{IC},\Gamma]=2il\int \eta(\mathbf{r})Y(\mathbf{r})d^3r, \tag{61}$$

where

$$Y(\mathbf{r})=\frac{1}{\sqrt{V}}\sum_\kappa \xi(\kappa)\ddot{\rho}(\kappa)\kappa^{-2}\exp(i\kappa\cdot\mathbf{r}). \tag{62}$$

To evaluate the operator Y, note that for a system of particles

$$\rho(\kappa)=\frac{1}{\sqrt{V}}\sum_j m_j \exp(-i\kappa\cdot\mathbf{r}_j), \tag{63}$$

where the sum goes over all particles and m_j is the mass of the jth particle.

Combining (56), (62), and (63) we find after some algebra:

$$Y(\mathbf{r})=-l^n Z_n\frac{d}{dt}\sum_j\frac{\mathbf{p}_j\cdot(\mathbf{r}-\mathbf{r}_j)}{|\mathbf{r}-\mathbf{r}_j|^{n+1}}, \tag{64}$$

where

$$Z_n=\left(\frac{n+1}{2}\right)\int_0^\infty u^{n-1}\frac{d}{du}\left(\frac{\sin u}{u}\right)du \tag{65}$$

(with a convergence factor used if necessary).

So far, we have not specified the operator $\eta(\mathbf{r})$; it is now convenient to define it as follows[15]:

$$\begin{aligned}\eta(\mathbf{r})&=l^{-n}Y(\mathbf{r}), && r\leq R;\\ &=0, && r>R.\end{aligned} \tag{66}$$

Combining (61) and (66), we now find

$$\langle[\mathfrak{IC},\Gamma]\rangle=2il^{1-n}\int_{r\leq R}\langle Y^2(\mathbf{r})\rangle d^3r. \tag{67}$$

If the indicated time differentiation is carried out in (64), one of the resulting terms is seen to be

$$2l^n Z_n\sum_j\frac{T_j}{|\mathbf{r}-\mathbf{r}_j|^{n+1}} \tag{68}$$

(T=kinetic energy), while the others are of the same order of magnitude, on the average.[16] Combining this with (67) and the uncertainty principle, we find for a state with radius R

$$\Delta\mathfrak{IC}\Delta\Gamma\gtrsim 2l\frac{\langle T\rangle^2}{R^{2n-1}}. \tag{69}$$

(Since we are only interested in orders of magnitude, we need not distinguish between $\langle T^2\rangle$ and $\langle T\rangle^2$.) By inspection of (58), (66), and (68) it is also clear that

$$\Delta\Gamma\lesssim l^{-n}\frac{\langle T\rangle}{R^{n-2}}.$$

We now find

$$\Delta\mathfrak{IC}\gtrsim\langle T\rangle\left(\frac{l}{R}\right)^{n+1}\sim\langle T\rangle\frac{l}{R}\beta\left(\frac{R}{l}\right). \tag{70}$$

Except for the factor of β, Eq. (70) is the same as (32) because of the virial theorem. As before, for the frequency spread in a transition, we find

$$\Delta\nu\gtrsim\nu_0\frac{l}{R}\beta\left(\frac{R}{l}\right). \tag{71}$$

This derivation is not subject to the objection made against the one of the previous subsection, as it was not necessary to discuss precise localization of particles at all. The indeterminate quantity responsible for the broadening in this treatment is essentially the retardation correction to the Newtonian gravitational potential, averaged over the region occupied by the nucleus. The treatment does not depend on any assumptions about $\mathfrak{IC}^0$ in Eq. (45). $\mathfrak{IC}^0$ may be assumed to include all non-gravitational effects.

C. Conclusion

The considerations of this section justify the line broadening given by Eq. (10), obtained by crude arguments in Sec. I. The consequences of (10), already discussed in Sec. I, therefore follow immediately. In particular, the idea of a fundamental length of the order of 10^{-13} cm can be ruled out on the grounds of serious contradiction with experiment.

In the author's opinion, it should come as no surprise to find that this large fundamental length is in conflict with experiment. A fundamental length is, after all, a rather drastic change in our basic concepts. One would expect, therefore, that any attempt to use ordinary quantum mechanics in the region of the fundamental length should lead to dramatic conflicts between theory and experiment, such as one encounters in trying to apply classical mechanics to atomic systems. Hence, the fact that ordinary quantum mechanics has had even qualitative success in dealing with nuclear spectroscopy should have been taken as an indication that we were not yet approaching a fundamental length in the nuclear domain. It does not necessarily follow, however, that the length 10^{-13} cm plays no major role in physics, as what we mean by a fundamental length is something rather specific. Cf. Appendix A.

[15] The origin is now placed at the center of mass of the system of particles.

[16] Because Y is a time derivative, its average value must be zero for a stationary state (nearly zero for a nearly stationary state), from which it follows that the other terms, on the average, are of the same magnitude as the one shown. Of course, it does not follow that $\langle Y^2\rangle$ or $\langle YQ\rangle$ needs to be zero.

As already mentioned in Sec. I, the much smaller fundamental length considered by the author[1] is neither confirmed nor contradicted by any experiments done to date. In the next section, we suggest an experiment which appears to be feasible (though difficult) with presently available techniques, and which would be capable of detecting the line broadening due to this fundamental length, if it is present.

IV. SUGGESTED EXPERIMENTAL TEST

A. Principle of the Experiment

The broadening of nuclear gamma lines predicted by the author's fundamental-length theory is given in order of magnitude by Eq. (12). For typical gamma energies, the broadening is of the order of 1 sec^{-1}. In principle, then, the way to detect this broadening would be to measure the width of a gamma transition of lifetime several seconds or more by means of the Mössbauer effect. If the broadening is present, the observed width will be larger than the reciprocal of the lifetime.

A quick look at the orders of magnitude involved, however, reveals that the situation is not quite so simple. The width is ordinarily measured by measuring the relative velocity at which source and absorber are no longer in resonance due to the Doppler shift. A width of $10^{-20}\nu_0$ [as predicted by Eq. (12)], however, corresponds to a Doppler velocity of only 1 or 2 Å per min. Thus, even if other sources of broadening are eliminated, an impossible degree of precision in control of the velocity is required. Vibrations in the apparatus alone normally lead to velocity fluctuations several orders of magnitude larger than this.

For such narrow lines, therefore, it appears that something other than the Doppler effect must be used to shift source and absorber out of resonance with one another. The author would like to suggest that the gravitational red shift be used for this purpose. Let us see how this can be done. If two identical nuclei are separated from each other by a distance d parallel to a gravitational field, their resonance frequencies will differ by

$$\delta\nu = \nu_0 g d, \tag{72}$$

where for the gravitational field at the earth's surface, $g = 980$ cm sec^{-2} or 1.09×10^{-18} cm^{-1} in natural units. If the resonance width is Γ, the nuclei will be shifted out of resonance when

$$d \geq \Gamma/\nu_0 g \equiv h.$$

For the width predicted by Eq. (12), this is satisfied for $h\sim10^{-2}$ cm. It will be less, of course, for long-lived transitions if our predicted broadening is not present. The width of a line, therefore, can be measured by measuring h, the vertical separation at which two nuclei cease to be in resonance. Since it is no longer necessary to move source and absorber relative to one another, it becomes possible to have them both in the same crystal. This eliminates the problem with vibrations, since vibrations of one part of the crystal relative to another are just the acoustical vibrations of the crystal, and their effects are already included in the standard theory of the Mössbauer effect.[10]

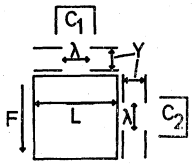

FIG. 1. Schematic diagram of proposed experimental arrangement. Photons from a cubic crystal of side L pass through pairs of slits of width λ to counters C_1 (located above the crystal) and C_2 (located to the side). F denotes the direction of the earth's gravitational field.

The proposed experimental setup is indicated schematically in Fig. 1. A cubic crystal of side L contains nuclei of a suitable isotope, some of which are excited. There is a gamma counter C_1 above (or below) the crystal and one to the side, C_2. The photons enter the counters through pairs of slits of width λ. What happens will be discussed in more detail in Subsec. D, but the qualitative situation is clear. If h is less than L, the photons emitted in the vertical direction will be in resonance only with those absorbers within a vertical distance h of their emitter; those emitted in the horizontal direction are in resonance with all the absorbers between the emitter and the counter. The horizontally emitted photons thus have a higher probability of being absorbed (or scattered) than the vertical ones, resulting in a higher counting rate in the counter C_1 than in C_2. The extent of this effect is obviously dependent on the width of the line; in particular, if the line is so broad that $h\gg L$ (which is the case in all experiments done up to now), there will be no effect at all.

In the following subsections, we discuss the selection of an appropriate isotope, the elimination of other sources of broadening, and finally the actual prediction of the experimental result. The role of the slit width λ will then be made clear. We confine ourselves to indicating what needs to be done, without pretending to solve all the experimental problems (e.g., preparation of a sufficiently pure sample). There is no doubt that the problems are difficult but it is felt that they can probably be overcome if a sufficient effort is made.

B. Selection of Isotope

We first list some desirable properties that a gamma-emitting isotope should have in order to be useful for our proposed experiment:

(a) Its "natural" width $1/t$ must be less than the predicted width due to fundamental-length broadening. According to Eq. (12), this means that the mean lifetime t must be at least a few seconds. This requirement, obviously, is absolutely essential.

(b) It should have at least a fairly large Mössbauer fraction (fraction of photons emitted without recoil); this means a low value of ν_0, and a high Debye temperature for the crystal.

(c) α, the ratio of internal conversion electrons to gamma photons, should not be too high.

(d) If the isotope has a magnetic moment, and these are randomly oriented, the resulting broadening of the line will swamp the effect we are looking for. It would be helpful, therefore, for the isotope to have spin zero.

(d′) If the spin is not zero, it is necessary to line up the nuclear spins. This can be done if the substance used is paramagnetic, in which case the nuclear spins are aligned by their hyperfine interaction with the electronic magnetic moments.[17]

(e) The isotope in its ground state should be stable, or at least very long lived; otherwise its decay products will build up as impurities and broaden the line.

(f) For economic reasons, it is desirable that the isotope occur naturally, and with a high natural abundance.

A survey of the table of isotopes[18] indicates that there are none that satisfy all these requirements. The best prospect appears to be the isotope $_{45}Rh^{103}$. Its excited isomer $_{45}Rh^{103m}$ $(\frac{7}{2}+)$ decays to the ground state $(\frac{1}{2}-)$ by internal conversion, or by emission of a 40-keV gamma photon. We now discuss its properties in relation to the requirements listed above:

(a) Published values for the half-life range from 45 min to 58 min.[18] We will use the last figure in our calculations, as it is the most recent quoted in the table. In any case, the lifetime is quite long enough for our purpose, and use of another figure would only affect minor details.

(b) As mentioned above, the energy of the transition is 40 keV. The Debye temperature of rhodium in the range 10–14 deg K has been measured as 450 deg K.[19] (It decreases at higher temperatures, but our experiment must be done at very low temperatures, so this is the appropriate value to use.) For temperatures small compared with the Debye temperature (corresponding to the conditions of this experiment), the Mössbauer fraction f is given by[10]

$$f=\exp(-2w),$$

where

$$w=\tfrac{3}{4}R/k\Theta_D, \quad R=E_\gamma^2/2Mc^2.$$

Inserting the values $E_\gamma=40$ keV, $\Theta_D=450$ deg K, $M=103$ amu, we find $f=0.9968$, i.e., nearly all the gamma emissions are recoil-free. This requirement, therefore, is exceptionally well satisfied.

(c) Avignon, Michalowicz, and Bouchez[20] find $\alpha_K=40$, $\alpha_L=470$, and can account for all the decays with these. The total value of α, therefore, is 510. This is far from ideal, but acceptable.

(d) This requirement is not satisfied. The spin of the ground state is $\frac{1}{2}$, the magnetic moment is -0.0879 nm.[18,21]

(d′) Rhodium is paramagnetic. The configuration of the ground state of the free atom (apart from closed shells) is $4d^85s$: $^4F_{9/2}$. According to the Landé formula, the free atom should have a magnetic moment of 6 Bohr magnetons. In the solid metal, the magnetic moment is believed to be due entirely to the spins of the two uncompensated d electrons.[22] At very low temperatures, this leads to a magnetic moment of 2 Bohr magnetons.

(e) Rh^{103} is the only known stable (or long-lived) isotope of rhodium.[18]

(f) The isotope occurs naturally, with an abundance of 100%.

There are a number of other isotopes which satisfy requirement (a), but all encounter difficulty with one or more of the other requirements, making their use apparently more difficult than Rh^{103}. We do not wish to state categorically that this is the only possible isotope, but it does seem to be the best, according to data available to the author. Hence, all the following calculations will be based on the assumption that Rh^{103} is being used. The reader who wishes to consider some other isotope will find it easy to generalize the theory of the experiment.

C. Elimination of Other Sources of Broadening

The "natural" width of the line in question (half-width at half-height in angular units, in the absence of fundamental length or other broadening) is $\Gamma_0=\frac{1}{2}t$ $=1.0\times10^{-4}$ sec$^{-1}=3.3\times10^{-15}$ cm^{-1}. The broadened width predicted by Eq. (12) is in order of magnitude $\Gamma\sim10^{-20}\nu_0\sim2\times10^{-11}$ cm$^{-1}\sim10^4\Gamma_0$. The fundamental-length broadening will therefore be the dominant effect if other sources of broadening can be eliminated, or reduced sufficiently. The criterion which we use in this subsection is that broadening due to other effects may be no greater than Γ_0. From the orders of magnitude involved, it is clear that this requirement could be relaxed by two or three factors of 10, without destroying the possibility of detecting the fundamental-length broadening.

Now consider a crystal of very pure rhodium metal at very low temperature. At absolute zero, the electronic spins are all aligned by their mutual dipole-dipole

[17] C. J. Gorter, Physica **14**, 504 (1958).
[18] D. Strominger, J. M. Hollander, and G. T. Seaborg, Rev. Mod. Phys. **30**, 585 (1958).
[19] F. Clusius and C. G. Losa, Z. Naturforsch. **10a**, 545 (1955).
[20] P. Avignon, A. Michalowicz, and R. Bouchez, J. Phys. Radium **16**, 404 (1955).
[21] H. Kuhn and G. C. Woodgate, Proc. Phys. Soc. (London) **64A**, 1090 (1951).
[22] T. I. Kakushadze, Ann. Physik **8**, 366 (1961).

interactions, and the nuclear spins by their hyperfine interaction with the electronic spins. If the crystal is perfect, therefore, the local magnetic fields seen by the different nuclei will all be of exactly the same magnitude (though not necessarily the same direction), and all nuclei will be oriented in the same way relative to the local field. The magnetic shift of the resonance frequency will therefore be the same for all the nuclei, and they will remain in perfect resonance with one another. The same applies to shifts due to local electric fields. For an actual crystal at small but finite temperature, there will be a small number of randomly distributed impurities, dislocations, and wrongly oriented spins, causing a slight variation in local fields and consequent broadening of the line. We must require that these broadenings not be too large. We consider in turn the various interactions present:

(1) Electron-nuclear interactions: Apart from local effects confined to the immediate neighborhood of the impurity, the main broadening effect due to an impurity, dislocation, or wrongly oriented electronic spin is through the magnetic dipole-dipole interaction between electronic and nuclear spins. If $\mathfrak{N}$ is the density of the crystal in atoms/cm^3 and F_e is the fraction of sites occupied by impurities, dislocations, or wrongly oriented electronic spins, then the broadening is[23]

$$\Gamma_{n-e}\cong\mu_n\mu_e\mathfrak{N}F_e.$$

In our case, as mentioned before, $\mu_n=0.0879$ nm, $\mu_e=2$ Bohr magnetons. At room temperature, $\mathfrak{N}=7.26\times10^{22}$ cm^{-3}, and we will use this figure since there seems to be no experimental value for low temperatures. If we use these numbers and require $\Gamma_{n-e}\lesssim\Gamma_0$, we find

$$F_e\lesssim2\times10^{-10}. \tag{73}$$

As a limitation on the number of impurities and dislocations, Eq. (73) is a requirement on the purity and perfection of the crystal. As a limitation on wrongly oriented electronic spins, it is a requirement on the temperature. The electronic spins are oriented chiefly by their mutual near-neighbor dipolar interactions. The energy of this interaction is

$$W_{e\text{-}e}\cong\mu_e^2\mathfrak{N}.$$

The fraction of wrong orientations is

$$F_e=\exp(-W_{e\text{-}e}/kT_e),$$

where T_e is the temperature of the electronic spin system. Equation (73) will now be satisfied for wrong orientations if

$$T_e\lesssim8\times10^{-3}\ \text{degK}. \tag{74}$$

(2) Internuclear interactions: There is also a broadening due to isotopic impurities and wrongly oriented nuclei; and it should be remembered that the excited isomer Rh^{103m} is itself an "isotopic impurity" for this purpose. If F_n is the fraction of such impurities, the broadening is

$$\Gamma_{n\text{-}n}\cong\mu_n^2\mathfrak{N}F_n.$$

Here we use $\mu_n=1$ nuclear magneton, since it may be the moment of an impurity or of the excited state. Requiring $\Gamma_{n-n}\lesssim\Gamma_0$ gives

$$F_n\lesssim6\times10^{-8}. \tag{75}$$

As a limit on impurities, Eq. (75) chiefly limits the number of nuclei that we can initially excite, and hence the counting rate. As a limit on wrong orientations, it gives another temperature requirement. The nuclear spins are oriented chiefly by the hyperfine interaction with the electronic magnetic moment associated with the same site. For the free atom, the hyperfine interaction energy $W_{\rm hf}$ has been found[21] to be 0.144 cm^{-1} (in checking this reference, the reader should bear in mind that our units differ by a factor of 2π from the ones used there). We can estimate $W_{\rm hf}$ in the metal by assuming that the hyperfine interaction is of the form $B\mathbf{\mu}_e\cdot\mathbf{\mu}_n$, and that the coupling constant B is the same in the metal as in the free atom. This assumption, together with some elementary angular-momentum theory and the properties mentioned in Subsec. B under property (d′), gives $W_{\rm hf}=0.065$ cm^{-1}. Now

$$F_n=\exp(-W_{\rm hf}/kT_n),$$

where T_n is the nuclear-spin temperature. Equation (75) will be satisfied if

$$T_n\lesssim9\times10^{-4}\ \text{degK}. \tag{76}$$

(3) "Resonance" broadening: This is the effect of transfer of excitation from one nucleus to another via electrostatic interaction. Its broadening effect is less than the natural linewidth if the nearest-neighbor distance is greater than a wavelength, as in the present case.

It is believed that the effects discussed above constitute the most important sources of broadening under the conditions of this experiment. If conditions (73)–(76) are satisfied, therefore, one would expect the observed width to be $\sim\Gamma_0$, unless some hitherto unknown broadening mechanism, such as our fundamental-length broadening, is present. It appears that the purity requirement (73) will be hardest to satisfy (a mass spectrometer might have to be used). Temperatures considerably lower than those of (74) and (76) have been attained, though there may be some difficulty in maintaining such temperatures while the experiment is being carried out.

D. Predictions of Experimental Results

We must now calculate the counting rates in the two counters C_1 and C_2 of Fig. 1, as functions of the linewidth Γ and slit width λ. For definiteness, we assume

[23] A. Abragam, *The Principles of Nuclear Magnetism* (Clarendon Press, Oxford, 1961), pp. 126–8.

that L in Fig. 1 is 1 mm, that conditions (73)–(76) are satisfied, and that the distance Y between slits is $\sim L$.

We confine ourselves here to a rough order-of-magnitude calculation, for the following reasons: First, the theory to be tested only makes order-of-magnitude predictions; second, many details depend on the detailed experimental geometry. We also use a simple particle picture for the photons. It is easy to convince oneself that a more careful calculation would not affect the qualitative results.

The width of the Mössbauer resonance absorption is twice the linewidth. If the linewidth is simply Γ_0, therefore, the cross section will be reduced to half its peak value at a vertical separation h_0, determined by

$$\nu_0 g h_0 = 2\Gamma_0.$$

Putting in the appropriate numerical values for our problem, we find

$$h_0 = 3.0\times 10^{-6}\text{ cm}. \tag{77}$$

In general, if the width is Γ, this is replaced by

$$h = h_0(\Gamma/\Gamma_0). \tag{78}$$

The average absorption cross section at resonance is[10]

$$\sigma = \sigma_0(\Gamma_0/\Gamma), \tag{79}$$

where

$$\sigma_0 = \frac{\pi}{(2I+1)(\alpha+1)\nu_0^2}. \tag{80}$$

Here I is the spin of the ground state, $\frac{1}{2}$ in our case. There is no corresponding factor for the excited state, since the degeneracy is lifted by the hyperfine interaction. Putting in the numerical values ($\alpha=510$), we get

$$\sigma_0 = 7.47\times 10^{-22}\text{ cm}^2. \tag{81}$$

Now suppose we start with a certain fraction F^* [limited by (75)] of the nuclei excited. We simplify by assuming that the cross section is given by (79) if the vertical separation of emitter and absorber is less than h, and is zero otherwise. It is also necessary to notice that all the absorbers are in their ground hyperfine level, while only half the emitters decay directly to this level. Hence half of the emitted photons are not subject to absorption at all, since they are shifted out of resonance by the hyperfine interaction.

From Fig. 1 it is clear that the only photons counted will be those emitted from a layer of thickness λ, and at an angle $\lesssim\lambda/L$ relative to the horizontal (in the case of C_2) or the vertical plane determined by the slits (in the case of C_1).

Of the photons emitted in the vertical direction, half cannot be absorbed (as mentioned above), and the other half are in resonance with the intervening absorbers only up to a distance h. The absorption probability for a vertically emitted photon is therefore

$$P_{\text{vert}} = \tfrac{1}{2}[1-\exp(-U_{\text{vert}})],$$

where

$$U_{\text{vert}} = \sigma\mathfrak{N}h = \sigma_0\mathfrak{N}h_0 = 1.6\times 10^{-4}.$$

Therefore

$$P_{\text{vert}} = 0.8\times 10^{-4}. \tag{82}$$

Regardless of Γ, therefore, very few of these will be absorbed.

For photons emitted in the horizontal direction, we must distinguish between two cases. If $\lambda<2h$, the photon is in resonance with all the intervening absorbers unless it is one of the fraction of photons which are out of resonance. For this case, then, we find on the average

$$\bar{U}_{\text{hor}} \cong \sigma\mathfrak{N}L/2. \tag{83}$$

If $\lambda>2h$, only a fraction of about $(2h/\lambda)$ of the intervening absorbers, on the average, will be in resonance with the emitted photon, so we find

$$\bar{U}_{\text{hor}} \cong \frac{\sigma NL}{2}\left(\frac{2h}{\lambda}\right) = \sigma_0 N h_0 \frac{L}{\lambda} = U_{\text{vert}}\frac{L}{\lambda}. \tag{84}$$

The situation may be summed up as follows: If $\lambda=L$, the two counting rates are equal. As λ is decreased, the ratio of the rate in C_1 to that in C_2 increases until $\lambda\sim 2h$, after which there is essentially no further change. Thus, the experiment yields an approximate value for h, and hence for the width.

Let us now compare in more detail the predictions which one would make with and without the fundamental-length hypothesis. If there is no fundamental length (or other source of broadening), we have $\Gamma=\Gamma_0$, $h=h_0$. For small slit width, we find from (77), (81), (83):

$$\bar{U}_{\text{hor}} = 2.71. \tag{85}$$

Since this is fairly large, we must take into account the fact that the effective absorber thickness depends on the distance of the emitter from the edge. We have

$$P_{\text{hor}} = \tfrac{1}{2}\left\{1-\frac{1}{L}\int_0^L \exp\left(-2\bar{U}_{\text{hor}}\frac{x}{L}\right)dx\right\}$$
$$= 0.41. \tag{86}$$

In this case, then, we would predict that the ratio of counting rates should increase with decreasing λ, eventually attaining a value of about 5/3 for $\lambda\sim 6\times 10^{-6}$ cm, and then remaining approximately constant as λ is decreased further.

On the other hand, if there is a fundamental length with broadening given by (12), then $h\sim 10^{-2}$ cm, $\Gamma\sim 10^4\Gamma_0$. Using (79), (82), and (83), we now find for small λ:

$$U_{\text{hor}} \sim 2U_{\text{vert}},$$
$$P_{\text{hor}} \sim 2\times 10^{-4}.$$

We would now predict that the ratio of rates should increase with decreasing λ only until $\lambda\sim 10^{-2}$ cm, at

which point the two rates differ by only about one part in 10^4. Further decrease of λ should not appreciably affect the ratio.

It is clear that the predictions of the two theories are quite different, so that the outcome of this experiment will be strongly affected by the presence of a fundamental length.

We must finally say a word about the counting rate. It will clearly be smallest when λ is smallest, i.e., for $\lambda=6\times10^{-6}$ cm, the smallest value we need. According to (75), the fraction F^* of excited nuclei can be as large as 6×10^{-8}. The number of excited nuclei in the layer which is counted is $\mathfrak{N}F^*\lambda L^2$. The solid angle gives a factor $\sim\lambda/6L$. We must also multiply by $1/t$ to get the rate of decay, and divide by $(1+\alpha)$ to get the gamma counting rate. This gives

$$\text{Rate}\cong\frac{NF^*L\lambda^2}{6(1+\alpha)t}\cong1\ \text{sec}^{-1},$$

which is large enough to measure without difficulty. It will be larger, of course, for larger slit width.

E. Conclusion

The proposed experiment should be capable of detecting the broadening predicted by the author's fundamental-length theory, thus providing an experimental test of the theory. The smaller broadening predicted by Eq. (4) could not be detected in this way. Therefore, if the large difference in counting rates predicted by Eq. (86) is found, it would show that the author's theory is wrong, but would not invalidate the work of Refs. 3–6. The experiment is difficult, but it appears that, with sufficient effort, it could be performed in the foreseeable future.

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APPENDIX A: COVARIANCE; REMARKS ON "STOCHASTIC" THEORY

Although we have not employed a manifestly covariant notation in this article, it is clear from the fact that the postulates (1) and (2) refer to the outcomes of *measurements* that the physical content of the theory is capable of being expressed in covariant language. For the outcome of a measurement can always be expressed as a scalar (perhaps involving the degrees of freedom of the measuring apparatus).

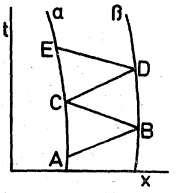

FIG. 2. World lines of two observers, α and β, communicating by means of light signals. AB, BC, CD, and DE are paths of light signals.

As an example, Fig. 2 shows schematically the world lines of two observers, α and β, both equipped with clocks. The line $ABCDE$ represents light signals exchanged between the two observers. If the observers are to be Lorentz observers, they should be moving with constant velocity (more generally, in free fall). If their observations are to relate to the same Lorentz frame, their relative velocity should be zero (more generally, a determination of their relative velocity should have given the result zero, with the minimum possible uncertainty). One way of measuring the distance L between the two observers is to measure the time required for a light signal to propagate back and forth. Thus, observer α could say that L (at t_B, the time of event B) is one-half the time between A and C, as measured by his clock. L as defined by this particular measurement is therefore

$$L=\frac{1}{2}\int_A^C ds_\alpha. \tag{A1}$$

It is obvious that L is an invariant; Eq. (1) would postulate that the value of this invariant can never be predicted precisely in advance.

The synchronization of clocks can be treated by considering the time interval between events B and D, as measured by the two observers. For observer β, the time of B is just τ_B, the reading of his clock at event B, and similarly for D. His value of the time difference is therefore

$$t_D(\beta)-t_B(\beta)=\int_B^D ds_\beta. \tag{A2}$$

For observer α, the time of B is measured as halfway between the times of A and C, as measured by his clock:

$$t_B(\alpha)=\tau_A+\frac{1}{2}\int_A^C ds_\alpha=\tau_C-\frac{1}{2}\int_A^C ds_\alpha. \tag{A3}$$

Similarly,

$$t_D(\alpha)=\tau_C+\frac{1}{2}\int_C^E ds_\alpha=\tau_A+\int_A^C ds_\alpha+\frac{1}{2}\int_C^E ds_\alpha. \tag{A4}$$

The time difference between the two events as measured by α is therefore

$$t_D(\alpha)-t_B(\alpha)=\frac{1}{2}\int_A^E ds_\alpha, \tag{A5}$$

and the difference between the time intervals measured

by the two observers is

$$T=\int_B^D ds_\beta-\frac{1}{2}\int_A^E ds_\alpha. \tag{A6}$$

T is again a scalar, and in fact a combination of scalars of exactly the same type as L in Eq. (A1). Thus the first term on the right-hand side of (A6) could be interpreted as twice the value of L as measured by β, while the second could be considered to be the sum of two successive measurements of L by α. If L is subject to a minimum uncertainty, therefore, T must be also. Observer α could synchronize his clock with that of β as follows: After receiving the light signal at C, carrying with it the reading τ_B of β's clock at B, he resets his clock so that he agrees on the time of B. Referring to Eq. (A3), then, he sets his clock so that

$$\tau_C=\tau_B+\frac{1}{2}\int_A^C ds_\alpha.$$

The difference T is now a measure of the loss of synchronization of the two clocks between events B and D. The minimum uncertainty in T means, therefore, that clocks cannot be known to be synchronized below a minimum uncertainty.

These considerations lead to two conclusions: first, the fundamental-length postulate (1) is expressible in covariant form; second, since clocks can be used to measure distances, it is inconsistent to require a minimum uncertainty in distance measurements without requiring the same minimum uncertainty for the synchronization of clocks. To study the transformation laws relating quantities measured by different observers in relative motion in the presence of a fundamental length, one must take into account the fact that the relative velocity of the observers cannot now be known precisely. The author hopes to treat this problem in a future publication.

It is now clear that the "stochastic" theory of Ingraham[11] is not a fundamental-length theory in our sense of the term. In Ingraham's theory, clocks can be synchronized precisely, while we have just seen that this is inconsistent with the minimum position uncertainty (1) in our kind of fundamental length theory. And in fact, his theory cannot be interpreted in terms of minimum position uncertainty of test particles. The physically meaningful quantities in his theory are not the field amplitudes $\phi(x)$, but averages of them over finite, *but precisely specified*, regions, e.g.,

$$\bar{\phi}(\mathbf{x})=\int\phi(\mathbf{x}+\xi)f(\xi^2)d^3\xi, \tag{A7}$$

where the integral goes over a spacelike surface, and f is a normalized weighting function. The average $\bar{\phi}(x)$, defined by (A7), is defined at all points x in Ingraham's theory. Since x, the center of the region over which the average is taken, is specified precisely, it is clear that a prescription such as (A7) is to be interpreted, not in terms of a fundamental length in our sense of the term, but rather in terms of an extended (though precisely localizable) test particle. In our theory, the precise location of the region in which the field has the measured average value would be in doubt, owing to the uncertain position of the test particle.

Even this interpretation of the stochastic theory leads to some difficulty, because of the fact that the average $\bar{\phi}$ is defined at all points. First, it is not clear physically how this can be done, since test particles would have to overlap in order to measure $\bar{\phi}$ at points sufficiently close to one another. Second, if $\bar{\phi}$ is indeed known at all points, ϕ can also be calculated simply by inverting the integral transform (A7). For example, in the case of the Gaussian weighting function used by Ingraham,

$$f(\xi^2)=[(2\pi)^{1/2}\lambda]^{-3}\exp(-\tfrac{1}{2}\xi^2/\lambda^2),$$

the Fourier transforms of ϕ and $\bar{\phi}$ are related by

$$\bar{\phi}(\mathbf{k})=\phi(\mathbf{k})\exp(-\tfrac{1}{2}k^2\lambda^2).$$

Hence, a knowledge of the averages implies a knowledge of the detailed behavior of the field, so it is not clear that the theory can be interpreted as a limitation on the possibilities of measurement at all. It might have to be interpreted simply in terms of nonlocal interaction.

There are, therefore, some problems connected with the physical interpretation of Ingraham's theory (as he himself states). The main point for our purposes, however, is that the stochastic theory is not a fundamental-length theory in our sense of the term. The absence of the large line broadening (11), therefore, is not necessarily evidence against Ingraham's theory.

APPENDIX B: MATHEMATICAL PROPERTIES OF INDETERMINATE OPERATORS

In this appendix, we discuss some mathematical problems inherent in the definition of indeterminate operators, using the momentum of the particle in a box as an example. The main reference for this is the book by von Neumann.[24]

Consider, then, the problem of the particle in a one-dimensional box of unit length, or rather, the Hilbert space of complex functions $\psi(Q)$ defined in the interval $0\leq Q\leq 1$. According to the formalism of Sec. II, the "momentum" operator

$$P=-i(d/dQ)$$

should be indeterminate, since we have

$$[P,Q]=-i,\qquad 0\leq Q\leq 1. \tag{B1}$$

[24] J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, translated by R. T. Beyer (Princeton University Press, Princeton, New Jersey, 1955), especially pp. 145–169.

By the reasoning of Sec. II, therefore, we should have

$$\Delta P \gtrsim 1. \tag{B2}$$

If P is to be Hermitian, however, its domain (the set of functions on which it is defined) must be restricted in some way, and there are many ways to do this.[25] For example, we might define an operator P^0 by

$$P^0 = -i(d/dQ)$$

in the domain of functions satisfying

$$\phi(1) = \phi(0) = 0. \tag{B3}$$

P^0 is indeed indeterminate: it has no eigenfunctions. However, its domain can be extended in an infinite number of ways. Thus for any θ, $0 \leq \theta \leq 2\pi$, we can define an operator P_θ by

$$P_\theta = -i(d/dQ)$$

in the domain of functions satisfying

$$\phi(1) = \phi(0)\exp(i\theta). \tag{B4}$$

The operators P_θ are easily shown to be Hermitian; moreover, they are all "extensions" of P^0, since for any θ, the domain defined by (B4) includes that of (B3).[24] Furthermore, each P_θ possesses a complete set of eigenfunctions. We have

$$\begin{aligned} P_\theta \phi_{n\theta}(Q) &= (2\pi n + \theta)\phi_{n\theta}(Q) \\ \phi_{n\theta}(Q) &= \exp[i(2\pi n + \theta)Q], \\ n &= \text{integer}. \end{aligned} \tag{B5}$$

These are not eigenfunctions of P^0, of course, since they do not lie in its domain. From the fact that the eigenvalue problem is soluble for P_θ, it follows[24] that P_θ is "maximal," i.e., its domain cannot be extended further without destroying its Hermitian character.

The existence of eigenfunctions appears to violate the uncertainty principle, according to the reasoning of Sec. II. The reason for this apparent contradiction is that the proof of the uncertainty relations[26] applies only to wave functions on which the commutator $(PQ-QP)$ is defined. This is not the case for the $\phi_{n\theta}(Q)$, since $Q\phi_{n\theta}(Q)$ does not satisfy (B4) and therefore is not in the domain of P_θ.

We conclude, then, that in order for the reasoning of Sec. II to be valid, the domain of the indeterminate operator must be restricted in such a way that the commutator is also defined throughout the domain. Indeterminate operators, therefore, in general are not maximal. Thus, P^0 defined above is indeterminate, but its maximal extensions P_θ are not. If it is also desired that all polynomials made up of P and Q be defined, the domain must be restricted still further by requiring

$$\frac{d^n}{dQ^n}\phi(0) = \frac{d^n}{dQ^n}\phi(1) = 0, \quad n = 0, 1, 2, \cdots. \tag{B6}$$

This may seem a rather arbitrary limitation of the domain, but the following points should be kept in mind: First, any extension of the domain (B6) makes it impossible to define certain polynomials. Thus, PQ is not defined for (B4), nor is P^2 for (B3). Second, the domain (B6) is still dense in the Hilbert space, which is all that one needs for most purposes. Third, the choice of any one of the extensions (B4) would also be rather arbitrary.

The operators to which we are accustomed in quantum mechanics are maximal and possess complete sets of eigenfunctions (more rigorously, their eigenvalue problems are soluble in the sense of von Neumann[24]). The physical interpretation of the use of operators to represent physical quantities is contained in the postulates:

(a) The possible measured values of an operator are its eigenvalues.

(b) If ϕ_k is the eigenfunction of operator P with eigenvalue k, and ψ the wave function representing the state of a physical system, then the probability of obtaining the result k when one measures P is

$$W(k) = |(\phi_k, \psi)|^2.$$

In the case of continuous eigenvalues, this is replaced by

$$W(k)dk \propto |(\phi_k, \psi)|^2 dk.$$

(c) If a measurement of the operator P has yielded the value k, then the wave function of the system immediately after the measurement is an eigenfunction of P with eigenvalue k.

We can generalize these postulates to include indeterminate operators by using the known properties of the particle in a box as a guide. Thus, we know that the possible observed values of momentum are just the real numbers. A glance at (B5), moreover, shows that the eigenvalues of the extensions P_θ comprise all the real numbers. A reasonable generalization of (a) is therefore:

(a′) The possible measured values of an indeterminate operator are the eigenvalues of its maximal extensions.

To get the probability of a particular value of the momentum (or a particular infinitesimal range of values), we take the inner product of the state function ψ with $\exp(ikQ)$, the eigenfunction of free-particle momentum. In the region $0 \leq Q \leq 1$, the only region that counts here, this is identical with the eigenfunction of the appropriate extension (n and θ chosen so that $2\pi n + \theta = k$). Postulate (b) may therefore be generalized as follows:

[25] Reference 24, pp. 151–154.
[26] Reference 24, pp. 230–234.

(b′) If ϕ_k is an eigenfunction of some maximal extension of the indeterminate operator P with eigenvalue k, and ψ is the state function of the system, then the probability that a measurement of P will yield the result k is given by

$$W(k) \propto |(\phi_k,\psi)|^2,$$

or

$$W(k)dk \propto |(\phi_k,\psi)|^2 dk$$

in the case of continuous eigenvalues. In using (b′) for the case of continuous eigenvalues, one must be sure that the ϕ_k are properly normalized, as otherwise a weighting factor must be inserted into the formula for the probability density. For ordinary operators with continuous eigenvalues, the weighting factor in (b) is constant if the ϕ_k satisfy

$$(\phi_k,\phi_{k'}) = A\delta(k-k'), \tag{B7}$$

where A is a constant. This cannot be required of the ϕ_k for an indeterminate operator, since they are not all eigenfunctions of the same operator and hence are not orthogonal. In the case of the particle in the box, we know that (b′) is correct as it stands (without the weighting factor) if we choose for the ϕ_k

$$\phi_k = \exp(ikQ),$$

which satisfy

$$(\phi_k,\phi_{k'}) = \frac{e^{i(k'-k)}-1}{i(k'-k)},$$

i.e., the inner product is a function only of the difference $(k-k')$. A reasonable generalization of (B7) is therefore to normalize the ϕ_k for an indeterminate operator in such a way that

$$(\phi_k,\phi_{k'}) = f(k-k'), \tag{B8}$$

and then use (b′) without a weighting factor.

A possible generalization of (c) is:

(c′) If a measurement of the indeterminate operator P has yielded the value k, and if P_k is that extension of P which has k as one of its eigenvalues, then the wave function of the system immediately after the measurement is an eigenfunction of P_k with the eigenvalue k.

It appears, therefore, that incorporation of indeterminate operators into the formalism of quantum mechaincs causes no insurmountable difficulty. The postulates (a′), (b′), and (c′) may not be unique (i.e., other sets of postulates may be possible), but they seem to be reasonable generalizations of (a), (b), and (c). They were chosen so as to yield the standard results when applied to the momentum of a particle in a box. Postulate (c′), like (c), should be thought of as applying to rather idealized measurements. It does not correctly describe the effect of actual measurements on the system in all cases.

We are now in a position to make some remarks about the Hamiltonian operator (27), discussed in Sec. III A. It may be written

$$\mathfrak{H} = p^2/2m + V(\mathbf{r}_0 - l\mathbf{P}).$$

Since P is indeterminate and hence not maximal, it also follows that $\mathfrak{H}$ is not maximal (its domain could be extended by extending that of P). Hence the eigenvalue problem for $\mathfrak{H}$ is insoluble,[24] confirming the earlier analysis. The eigenfunctions of the extensions of $\mathfrak{H}$ are easily seen to be

$$\psi_n(\mathbf{r}_0 - l\mathbf{k})\phi_k(\mathbf{Q}),$$

where ψ_n is the nth eigenfunction of $\mathfrak{H}_0$. The eigenvalues are obviously just those of $\mathfrak{H}_0$, since the term $l\mathbf{k}$ simply gives a uniform translation of the potential. The energy spread, therefore, is over a discrete set of possible values if the eigenvalues of $\mathfrak{H}_0$ are discrete. This might call into question some of the conclusions of Sec. IV, in which a continuous spread was tacitly assumed. However, this should not be taken literally, as the nucleus is always in interaction with various fields which possess continuous energy levels. In particular, this objection does not apply to the treatment of Sec. III B, which, as has already been pointed out, is the preferred treatment anyway.

Finally, we must say a very brief word about the time dependence of the bounded operators Q in the Heisenberg picture. In order that commutation relations such as Eq. (18) continue to hold at different times, it is clear that Q must change with time in the same way as other operators of the system:

$$Q(t) = \exp(i\mathfrak{H}t)Q(0)\exp(-i\mathfrak{H}t). \tag{B9}$$

Since the transformation (B9) is unitary, it will not of course affect the boundedness of Q. It is not necessary to include this in the analysis of the body of the paper, which was all carried out in the Schrödinger picture.