PHYSICAL REVIEW VOLUME 143, NUMBER 4 MARCH 1966

Relativistic Schrödinger Equations for Particles of Arbitrary Spin

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(Received 16 August 1965)

Relativistic wave equations in the Schrödinger form $i\partial\psi/\partial t=H\psi$ for particles of nonzero mass and arbitrary spin are investigated. The wave function ψ is taken to transform according to the representation $D(0,s)\oplus D(s,0)$ of the homogeneous Lorentz group, a unique spin s for the particle being thereby assured without the aid of any supplementary condition. It is shown that the requirement that the Schrödinger equation be invariant under the operations of the Poincaré group, as well as under space and time inversions and charge conjugation, restricts the possible choices of H (as a function of the operators representing the above symmetry operations) to a well-defined class which shrinks to a unique possibility (coinciding with the Hamiltonian derived by Weaver, Hammer, and Good) when a further regularity condition of a physical nature is imposed: namely, that the Hamiltonian have a unique finite limit in the rest system of the particle. In this process, an ambiguity which exists initially in the definition of operators representing time reversal and charge conjugation gets eliminated. The Hamiltonian itself is obtained in explicit form for particles of any spin.

I. INTRODUCTION

RELATIVISTIC wave equations suitable for the description of free particles of arbitrary mass and spin have long been known: the Dirac equation for the spin-$\frac{1}{2}$ case and its generalizations[1] to higher spin, the Kemmer equation[2] for spins 0 and 1, the Fierz-Pauli equations[3] for arbitrary spin, and others.[4] The formulations of all these equations are strongly conditioned by two basic requirements: that the wave function ψ be locally covariant[5] and that the wave equation be manifestly covariant. Local covariance of the wave function enables us to specify its behavior under transformations of the homogeneous proper orthochronous Lorentz group[6] by saying that it transforms according to an irreducible representation $D(m,n)$ of the group[7] or according to a direct sum of such representations: we may thus write $\psi=\sum\oplus\psi^{(m,n)}$. It is well-known that under the operation of space inversion $\psi^{(m,n)}\to\psi^{(n,m)}$; hence in any theory invariant under this operation, the wave function must contain, along with every part $\psi^{(m,n)}$ also a part $\psi^{(n,m)}$, so that we must have[8] $\psi=\sum\oplus[\psi^{(m,n)}\oplus\psi^{(n,m)}]$. This wave function is, in general, reducible under rotations into parts which transform according to different representations $D(j)$ of the rotation group, and therefore does not describe particles of a unique spin unless suitable restrictions are imposed on ψ. In the case of the conventional wave equations,[9] the restrictions are either implicit in the equations themselves as in Dirac's equations[1] for particles of arbitrary spin and in the Kemmer equation–both first-order differential equations—or may be imposed as a supplementary condition[10] ln addition to a (Klein-Gordon type) wave equation, as in the Fierz-Pauli formalism. The presence of redundant components and the necessity for supplementary conditions to eliminate them are serious drawbacks: they not only make it difficult to keep track of the really independent components, but also make a consistent introduction of interactions difficult or impossible. However these can be avoided in one way—and only one, as long as we insist on local covariance—and that is by considering only wave functions ψ transforming as $\psi^{(0,s)}\oplus\psi^{(s,0)}$. Such functions exhibit a unique spin s under rotations, and therefore no supplementary conditions are needed. Conventional treatments do not take advantage of this attractive possibility, however. They could not, for the simple reason that except in the very special case $s=\frac{1}{2}$ (Dirac equation for the electron) the requirement of *manifest covariance* of a first-order wave equation is incompatible with having the above transformation

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1 P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936).

2 N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939).

3 M. Fierz, Helv. Phys. Acta **12**, 3 (1939); M. Fierz and W. Pauli, Helv. Phys. Acta **12**, 297 (1939), and Proc. Roy. Soc. (London) **A173**, 211, 1939.

4 See, for example, E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations* (Hafner Publishing Company, New York, 1953) for details of these and other types of equations like Bhabha's multi-mass equation.

5 We propose to call a wave *function* "locally covariant" if, under a Lorentz transformation L which changes the coordinates of a given space-time point from $x=(x^0,x^1,x^2,x^3)$ to $x'=Lx$, the description of the wave function *at that point* changes from $\psi(x)$ to $\psi'(x')=\Lambda(L)\psi(x)$, where Λ is a purely numerical matrix, independent of differential operators (and of course of the coordinate itself, by homogeneity of space-time). "Manifest covariance" of a wave *equation* is a statement about the linear operator acting on the wave function in the equation: that it has an obviously invariant form.

6 This will hereafter be referred to simply as the Lorentz group. The inhomogeneous group containing also the operations of time- and space-translations will be called the Poincaré group, following the terminology in recent literature.

7 See, for instance, Ref. 4, Sec. 17; also Ref. 18 below. The representation $D(m,n)$ is $(2m+1)(2n+1)$-dimensional. It is reducible with respect to the subgroup consisting of pure rotations, being the direct sum of all the representations $D(s)$ with $s=(m+n)$, $(m+n-1)$, $\cdots$, $|m-n|$.

8 Operators to represent charge conjugation and time reversal can also be defined over the space of such functions.

9 See, for example, Ref. 4, Sec. 28.

10 A thorough investigation of the role of supplementary conditions in extracting, from a reducible or irreducible representation of the homogeneous Lorentz group, an irreducible representation of the Poincaré group, has been carried out by D. L. Pursey, Ann. Phys. (New York) **32**, 157 (1965).

property for the wave function itself. To construct an invariant first-order differential operator one has to form the scalar product of the gradient four-vector, which belongs to $D(\frac{1}{2},\frac{1}{2})$, with a set of matrix operators which must also belong to $D(\frac{1}{2},\frac{1}{2})$. But such matrices, operating on $\psi^{(m,n)}$, would lead[11] to $\psi^{(m',n')}$ where $|m'-m|=|n'-n|=\frac{1}{2}$, and therefore if the wave function in a manifestly covariant equation of this type contains a part $\psi^{(m,n)}$, it must contain at least one other part $\psi^{(m',n')}$. In the Dirac equation for the electron, the $\psi^{(0,\frac{1}{2})}$ and $\psi^{(\frac{1}{2},0)}$ parts supplement each other in this respect and therefore no new parts need be introduced, but clearly this will not work for any other spin.

It was Foldy[12] who for the first time constructed wave equations without supplementary conditions for particles of arbitrary spin. This was achieved at the cost of surrendering manifest covariance, but as he pointed out, manifest covariance is really a luxury: All we need for the relativistic invariance of the theory is that the solutions of the wave equation should form a representation space for the appropriate irreducible representation of the Poincaré group.[13] Foldy's wave functions in the canonical representation are not locally covariant either. Weaver, Hammer and Good[14] succeeded in obtaining relativistic wave equations involving locally covariant wave functions of the type $\psi^{(s,0)}\oplus\psi^{(0,s)}$. The advantage of having simple transformation properties for the wave function are obvious, particularly when attempting generalizations to include interactions. In fact, rules for constructing an invariant S matrix from (second-quantized) fields of the above type have been given recently by Weinberg,[15] though he was not interested in the question of wave equations satisfied by such fields. Weaver, Hammer, and Good (WHG) construct their wave equation in the Schrödinger form $i\partial\psi/\partial t=H\psi$ by assuming the form of the Hamiltonian in the rest system of the particle and then passing to an arbitrary frame of reference with the help of a Lorentz transformation operator obtained by generalization from the spin-$\frac{1}{2}$ case. This procedure leaves it an open question as to whether or not there are other wave equations of this type. A related question is: "To what extent is a relativistic wave equation determined purely by considerations of covariance, and to what extent do other physical notions play a part?" These are the main questions to which we address ourselves in this paper. More specifically, we seek the class of relativistic equations of the form $i\partial\psi/\partial t=H\psi$ which are suitable for describing particles having a unique spin s (without the aid of supplementary conditions), it being required that ψ be locally covariant.

Our approach to the problem will be as follows: Knowing the operator representing the generators of the Poincaré group on wave functions transforming according to $D(0,s)\oplus D(s,0)$, we construct out of these and the operators representing the discrete symmetry operations, the most general operator H which has the same transformation character as $i\partial/\partial t=-P_0$ where P_0 is the generator of time translation. In other words, we ensure relativistic invariance of the equation $i\partial\psi/\partial t=H\psi$ by constructing H such that H and $-P_0$ have identical commutation relations with the generators of the Poincaré group and with the operators representing space inversion, time reversal, and charge conjugation. The latter operators are themselves not uniquely determined by their commutation relations,[16] and there is a corresponding ambiguity in H. Thus, despite our restriction regarding the transformation property of the wave function, considerations of covariance alone do not suffice to determine a unique relativistic wave equation for a particle of given mass and spin. We show, however, that with an additional condition of an essentially physical nature, the possible choices narrow down to just the WHG equation. We also obtain an explicit expression for the Hamiltonian for particles of arbitrary spin.

II. THE GENERATORS OF THE POINCARÉ GROUP; DISCRETE SYMMETRIES

The commutation relations which define the abstract algebra of the infinitesimal generators[17] of the Poincaré group are well known. We present them here in order to establish notation and for ease of reference in later sections. If P_0 and $P_i(i=1, 2, 3)$ are the generators of time and space translations, and J_i, K_i those of pure rotations and pure Lorentz transformations ("boosts"),

[11] See, for instance, H. J. Bhabha, Rev. Mod. Phys. **21**, 451 (1949).

[12] L. L. Foldy, Phys. Rev. **102**, 568 (1956).

[13] For the classification of irreducible representations of the Poincaré group, see E. P. Wigner, Ann. of Math. **40**, 149 (1939); V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. U. S. **34**, 211 (1948).

[14] D. L. Weaver, C. L. Hammer, and R. H. Good, Jr., Phys. Rev. **135**, B241 (1964).

[15] S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[16] Foldy, in Ref. 12, has already pointed out the nonuniqueness in the determination of the discrete operators. Unlike the present case, however, the different possibilities there were all consistent with a preassigned Hamiltonian. It appears likely that the multiplicity of choices would disappear if we require that these operators be equivalent to momentum-independent operators in a representation in which the wave function is locally covariant; but we have not investigated this point.

[17] If $L(\epsilon)$ is a transformation of the Poincaré group, associated with a real parameter ϵ, we define its infinitesimal generator G by $L(\epsilon)=1+i\epsilon G$, where $\epsilon>0$ is chosen to be infinitesimal and 1 is the identity transformation.

respectively, then[18]

$$[P_0,P_i]=0, \tag{1a}$$

$$[P_i,P_j]=0, \tag{1b}$$

$$[J_i,P_0]=0, \tag{1c}$$

$$[J_i,P_j]=i\epsilon_{ijk}P_k, \tag{1d}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{1e}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \tag{1f}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k, \tag{1g}$$

$$[P_i,K_j]=-i\delta_{ij}P_0, \tag{1h}$$

$$[P_0,K_i]=-iP_i. \tag{1i}$$

On wave functions

$$\psi(\mathbf{x},t)=\begin{pmatrix}\psi^{(0,s)}(\mathbf{x},t)\\ \psi^{(s,0)}(\mathbf{x},t)\end{pmatrix} \tag{2}$$

these generators are represented by the following operators[19]:

$$P_0=p_0\equiv -i\partial/\partial t, \tag{3a}$$

$$\mathbf{P}=\mathbf{p}\equiv -i\nabla, \tag{3b}$$

$$\mathbf{J}=\mathbf{x}\times\mathbf{p}+\mathbf{S},\qquad \mathbf{S}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & \mathbf{s}\end{pmatrix}, \tag{3c}$$

$$\mathbf{K}=t\mathbf{p}+\mathbf{x}p_0+i\boldsymbol{\lambda},\quad \boldsymbol{\lambda}=\begin{pmatrix}\mathbf{s} & 0\\ 0 & -\mathbf{s}\end{pmatrix}=\rho_3\mathbf{S}, \tag{3d}$$

where the matrices $(s_1,s_2,s_3)=\mathbf{s}$ are a $(2s+1)$-dimensional representation of the angular-momentum operators, and ρ_3 is the third of the Pauli matrices. Incidentally, it may be observed that in the special case $s=0$, $D(0,s)$ and $D(s,0)$ are not distinct; our discussion based on Eqs. (3) will not apply to this case. Another point to be noted is that the generators (3) do not lead to a representation which is unitary in the usual sense, i.e. with respect to the scalar product $\int\psi^\dagger\psi d^3x$. The only exception is the case $s=\frac{1}{2}$. In general, the Lorentz-invariant scalar product will have the form $\int\psi^\dagger M\psi d^3x$ with $M\neq 1$. We shall defer the discussion of the "metric" operator M to a future publication. We shall also defer till then the justification of the requirement we make below that the operator P representing space inversion be unitary and operators T and C representing time reversal and charge conjugation be antiunitary in the usual sense.

We now turn to the determination of T, C, and P from the information available about them. For P this consists in the following relations[20]:

$$Pp_0=p_0P, \tag{4a}$$

$$P\mathbf{p}=-\mathbf{p}P, \tag{4b}$$

$$P\mathbf{J}=\mathbf{J}P, \tag{4c}$$

$$P\mathbf{K}=-\mathbf{K}P, \tag{4d}$$

$$P^2\sim 1. \tag{4e}$$

The sign $\sim$ indicates equality to within a phase factor.[21] A unitary transformation having the above properties is given by

$$P\psi(\mathbf{x},t)=\sigma\psi(-\mathbf{x},t), \tag{5}$$

where σ is a unitary matrix satisfying $\sigma\rho_3=-\rho_3\sigma$. The latter relation arises from the requirements

$$\sigma\mathbf{S}=\mathbf{S}\sigma, \tag{6a}$$

$$\sigma\boldsymbol{\lambda}=-\boldsymbol{\lambda}\sigma, \tag{6b}$$

which follow from the use of (3c) and (3d) in (4c) and (4d). The choice of σ is thus restricted to a linear combination of ρ_1 and ρ_2 which is unitary and satisfies $\sigma^2\sim 1$. The most general form of this kind is $e^{i\chi_s}(\rho_1\cos\theta+\rho_2\sin\theta)$, but the arbitrary phase factor $e^{i\chi_s}$ is of no significance in a discussion of free particles and can be ignored, and the angle θ can be reduced to zero by a unitary transformation which merely accomplishes a redefinition of the relative phase of the $D(0,s)$ and $D(s,0)$ parts of the wave function and does not disturb any of the assignments (3). Therefore we may, without loss of generality, set

$$\sigma=\rho_1. \tag{7}$$

The relations to be satisfied by an *antiunitary* time-

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[18] Notation: Latin indices run from 1 to 3. Summation over repeated indices is understood. δ and ϵ are the Kronecker and Levi-Civita symbols, respectively. Units such that $\hbar=c=1$ are assumed.

[19] The "spin" parts of the generators in (3c) and (3d) follow from the definitions of $D(0,s)$ and $D(s,0)$. Irreducible representations of the (homogeneous) Lorentz group are defined in terms of the operators $M_j=\frac{1}{2}(J_j+iK_j)$ and $N_j=\frac{1}{2}(J_j-iK_j)$ whose commutation relations $[M_i,N_j]=0$, $[M_i,M_j]=i\epsilon_{ijk}M_k$, $[N_i,N_j]=i\epsilon_{ijk}N_k$, deduced from Eqs. (1e), (1f), and (1g), characterize $\mathbf{M}$ and $\mathbf{N}$ as two independent "angular-momentum" vectors. $\mathbf{M}^2$ and $\mathbf{N}^2$ commute with all the M_i and N_i and are therefore represented, in any irreducible representation of these operators, by scalar multiples $m(m+1)$ and $n(n+1)$ of the $(2m+1)$- and $(2n+1)$-dimensional unit matrices. A given pair of (integral or half-integral) values m, n defines a particular irreducible representation $D(m,n)$; the representation space, being a direct product of those of $\mathbf{M}$ and $\mathbf{N}$, is $(2m+1)(2n+1)$-dimensional. In the special representation $D(s,0)$, the matrices N_i are evidently zero, while the matrices M_i are the spin-s representation of the angular-momentum operators. Denoting the latter by s_i we have, in $D(s,0)$, $\frac{1}{2}(\mathbf{J}-i\mathbf{K})=0$ and $\frac{1}{2}(\mathbf{J}+i\mathbf{K})=\mathbf{s}$, so that $\mathbf{J}=\mathbf{s}$ and $\mathbf{K}=-i\mathbf{s}$. Similarly, in $D(0,s)$, $\mathbf{J}=\mathbf{s}$ and $\mathbf{K}=+i\mathbf{s}$. On combining these we get the $\mathbf{S}$ and $i\boldsymbol{\lambda}$ of (3c) and (3d). Incidentally, we could take $\mathbf{J}=\mathbf{s}$ in $D(0,s)$ and $\mathbf{J}=\mathbf{s}'$ in $D(s,0)$ where $\mathbf{s}'$ is different from but equivalent to $\mathbf{s}$; but this would seem to be an avoidable and unnecessary complication.

[20] See, for instance, Foldy, Ref. 12.

[21] Though two successive space inversions return the system to its original state, the wave function need be reproduced only to within an arbitrary phase factor, since the correspondence between the state and the wave function is indeterminate to this extent. For similar reasons, the sign $\sim$ appears below in other relations between discrete operators.

reversal operator T are[20]

$$Tp_0=p_0T, \tag{8a}$$

$$T\mathbf{p}=-\mathbf{p}T, \tag{8b}$$

$$T\mathbf{J}=-\mathbf{J}T, \tag{8c}$$

$$T\mathbf{K}=\mathbf{K}T, \tag{8d}$$

$$TP\sim PT, \tag{8e}$$

$$T^2\sim 1. \tag{8f}$$

Equations (8a)–(8d) can be realized by defining

$$T\psi(\mathbf{x},t)=\tau\psi^*(\mathbf{x},-t), \tag{9}$$

where τ is a unitary matrix subject to the following requirements, obtained on introducing Eqs. (3) in (8):

$$T\mathbf{S}=-\mathbf{S}T, \quad \text{or} \quad \tau\mathbf{S}^*=-\mathbf{S}\tau, \tag{10}$$

$$T\boldsymbol{\lambda}=-\boldsymbol{\lambda}T, \quad \text{or} \quad \tau\boldsymbol{\lambda}^*=-\boldsymbol{\lambda}\tau. \tag{11}$$

These two equations imply that τ commutes with ρ_3, and therefore must be of the form

$$\tau=\begin{pmatrix}\tau' & 0\\ 0 & \tau''\end{pmatrix}. \tag{12}$$

It follows from (10) that

$$\tau'\mathbf{s}^*=-\mathbf{s}\tau', \quad \tau''\mathbf{s}^*=-\mathbf{s}\tau''. \tag{13}$$

However, it is known that the unitary transformation ζ_s which takes $\mathbf{s}$ over into $-\mathbf{s}^*$ is a unique one[22] (apart from an arbitrary phase factor), and satisfies

$$\zeta_s\zeta_s^*=(-1)^{2s}. \tag{14}$$

Thus, in view of (13), τ' and τ'' must be equal to ζ_s to within arbitrary phase factors. The condition (8e), which reduces to the requirement $\tau\sigma^*\sim\sigma\tau$ on the matrices σ and τ, restricts the *relative* phase between τ' and τ'' to ±1. Thus we finally have

$$\tau=\begin{pmatrix}\zeta_s & 0\\ 0 & \zeta_s e^{i\theta_t}\end{pmatrix}, \quad e^{i\theta_t}=\pm1, \tag{15}$$

apart from a possible over-all phase factor which we ignore. Note that this form automatically ensures (8f), by virtue of (14); in fact it leads to a sharpening of both (8e) and (8f):

$$\tau\sigma^*=e^{i\theta_t}\sigma\tau, \quad \text{or} \quad T\sigma=e^{i\theta_t}\sigma T, \tag{16a}$$

$$\tau\tau^*=(-1)^{2s}, \quad \text{or} \quad T^2=(-1)^{2s}. \tag{16b}$$

Consider now the antiunitary operation of charge conjugation. The requirements are

$$Cp_0=-p_0C, \tag{17a}$$

$$C\mathbf{p}=-\mathbf{p}C, \tag{17b}$$

$$C\mathbf{J}=-\mathbf{J}C, \tag{17c}$$

$$C\mathbf{K}=-\mathbf{K}C, \tag{17d}$$

$$CP\sim PC, \tag{17e}$$

$$CT\sim TC, \tag{17f}$$

$$C^2\sim 1. \tag{17g}$$

We can satisfy Eqs. (17a)–(17d) by defining C such that

$$C\psi(\mathbf{x},t)=\kappa\psi^*(\mathbf{x},t), \tag{18}$$

where κ is unitary, and

$$C\mathbf{S}=-\mathbf{S}C, \quad \text{or} \quad \kappa\mathbf{S}^*=-\mathbf{S}\kappa, \tag{19}$$

$$C\boldsymbol{\lambda}=\boldsymbol{\lambda}C, \quad \text{or} \quad \kappa\boldsymbol{\lambda}^*=\boldsymbol{\lambda}\kappa. \tag{20}$$

These two equations imply that κ anticommutes with ρ_3 and therefore must have the form

$$\kappa=\begin{pmatrix}0 & \kappa'\\ \kappa'' & 0\end{pmatrix}. \tag{21}$$

Unitarity of κ demands that of κ' and κ'', and then (19) requires these two matrices to be equal to ζ_s apart from phase factors. The relative phase factor between κ' and κ'' is restricted to $e^{i\theta_c}=\pm1$ by (17g), and (17e) and (17f) give no further restrictions. Thus, if as before we ignore an over-all phase,

$$\kappa=\begin{pmatrix}0 & \zeta_s\\ e^{i\theta_c}\zeta_s & 0\end{pmatrix}, \quad e^{i\theta_c}=\pm1. \tag{22}$$

The phase factors $e^{i\theta_t}$, $e^{i\theta_c}$ in (15) and (22) are uncorrelated. Again, with (22), the equalities (17e)–(17g) become sharper. We have

$$\kappa\sigma^*=e^{i\theta_c}\sigma\kappa, \quad \text{or} \quad C\sigma=e^{i\theta_c}\sigma C, \tag{23a}$$

$$\kappa\tau^*=e^{i\theta_t}\tau\kappa^*, \quad \text{or} \quad CT=e^{i\theta_t}TC, \tag{23b}$$

$$\kappa\kappa^*=e^{i\theta_c}(-1)^{2s}, \quad \text{or} \quad C^2=e^{i\theta_c}(-1)^{2s}. \tag{23c}$$

III. THE RELATIVISTICALLY INVARIANT WAVE EQUATION

Having determined the operators representing the Poincaré generators and the discrete operations, we now proceed to construct a Hamiltonian operator which has the same transformation properties as $-p_0$ so that the invariance of the equation

$$-p_0\psi\equiv i\partial\psi/\partial t=H\psi \tag{24}$$

would be ensured. The operators available to us, out of which H can be built up, are $\mathbf{p}$, $\mathbf{S}$ or $\boldsymbol{\lambda}$, and the Pauli

[22] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959), Appendix C.

matrices ρ_i. The final expression for H must be such that replacement of p_0 by $-H$ in Eqs. (1), (4), (8), and (17) leaves the commutation relations unchanged. Translation invariance, Eq. (1a), forbids the appearance of $\mathbf{x}$ in H. For invariance under rotations, Eq. (1c), H must be a scalar operator; it cannot be a pseudoscalar, according to (4a). Now, it is easy to see that all scalars that can be constructed from the above operators can be written as a linear combination of terms $(\boldsymbol{\lambda}\cdot\mathbf{p})^l$ and $\sigma(\boldsymbol{\lambda}\cdot\mathbf{p})^l$, $(l=0, 1, 2, \cdots, 2s)$. Hence H must be a linear combination of this form. Incidentally, since $P_0^2-P_1^2-P_2^2-P_3^2=m^2$ for a particle of definite mass m, our Hamiltonian operator must satisfy the condition

$$H^2=p^2+m^2\equiv E^2,\quad p\equiv|\mathbf{p}|,\quad E=(p^2+m^2)^{1/2}. \tag{25}$$

The imposition of this condition is facilitated by expressing H in terms of projection operators to various eigenvalues of $(\boldsymbol{\lambda}\cdot\mathbf{p})$ rather than directly in powers of $(\boldsymbol{\lambda}\cdot\mathbf{p})$. The eigenvalues of $\lambda_p\equiv(\boldsymbol{\lambda}\cdot\mathbf{p})/p$ are of course $-s, -s+1, \cdots, s-1, s$, each value occurring twice. The projection operator to an eigenvalue ν of λ_p is

$$\Lambda_\nu=\prod_\mu{}'\frac{\lambda_p-\mu}{\nu-\mu},\quad (\nu,\mu=-s, -s+1, \cdots, s), \tag{26}$$

where the prime on the product sign indicates that $\mu=\nu$ is to be excluded. Evidently,

$$\Lambda_\mu\Lambda_\nu=\delta_{\mu\nu}\Lambda_\nu. \tag{27}$$

We shall see below that λ_p changes sign under charge conjugation; thus Λ_ν, which contains both odd and even powers of λ_p, has no simple transformation property under this operation. It is convenient therefore to split Λ_ν into "odd" and "even" parts (which contain, respectively, only odd powers and only even powers of λ_p). This is accomplished by defining

$$B_\nu=\Lambda_\nu+\Lambda_{-\nu}, \tag{28a}$$

$$C_\nu=\Lambda_\nu-\Lambda_{-\nu}, \tag{28b}$$

except that when $\nu=0$ (which can be the case only for integral spin),

$$B_0=\Lambda_0. \tag{28c}$$

The B_ν are even in λ_p and the C_ν are odd. This is evident from the explicit forms of these operators:

When s is an integer:

$$B_0=\Lambda_0=\prod_{\mu=1}^{s}\frac{\lambda_p^2-\mu^2}{-\mu^2}, \tag{29a}$$

$$B_\nu=B_{-\nu}=\frac{\lambda_p^2}{\nu^2}\prod_{\mu=1}^{s}{}'\frac{\lambda_p^2-\mu^2}{\nu^2-\mu^2}, \tag{29b}$$

$$C_\nu=-C_{-\nu}=\frac{\lambda_p}{\nu}\prod_{\mu=1}^{s}{}'\frac{\lambda_p^2-\mu^2}{\nu^2-\mu^2}. \tag{29c}$$

When s is a half-odd integer:

$$B_\nu=B_{-\nu}=\prod_{\mu=\frac{1}{2}}^{s}{}'\frac{\lambda_p^2-\mu^2}{\nu^2-\mu^2}, \tag{29d}$$

$$C_\nu=-C_{-\nu}=\frac{\lambda_p}{\nu}\prod_{\mu=\frac{1}{2}}^{s}{}'\frac{\lambda_p^2-\mu^2}{\nu^2-\mu^2}. \tag{29e}$$

It follows from (27) that in all cases

$$B_\mu B_\nu=C_\mu C_\nu=B_\mu\delta_{\mu\nu}, \tag{30a}$$

$$B_\mu C_\nu=C_\mu\delta_{\mu\nu}. \tag{30b}$$

Since any power of λ_p can be expressed in terms of the Λ_ν and hence in terms of the B_ν and C_ν, our earlier statement regarding the form of H is equivalent to taking

$$H=\sum_{\nu\geqslant 0}b_\nu B_\nu+\sum_{\nu>0}c_\nu C_\nu+\sigma\Big[\sum_{\nu\geqslant 0}b_\nu' B_\nu+\sum_{\nu>0}c_\nu' C_\nu\Big], \tag{31}$$

with the coefficients b_ν, c_ν, b_ν', c_ν' undetermined as yet. It can be easily seen with the aid of (30) that Eq. (25) demands, for any given ν, either

$$b_\nu^2=E^2\quad\text{and}\quad c_\nu=b_\nu'=c_\nu'=0 \tag{32a}$$

or

$$b_\nu=0\quad\text{and}\quad c_\nu^2+b_\nu'^2-c_\nu'^2=E^2. \tag{32b}$$

The choice of (32a) or (32b) may be made independently for the various values of ν. The further restrictions that arise from invariance under time reversal and charge conjugation will now be considered. For invariance under T, according to (8a), T must commute with H. Now, by (8b) and (11), T commutes with $(\boldsymbol{\lambda}\cdot\mathbf{p})$ and hence with the B_ν and C_ν. But T may commute or anticommute with σ according to (16a). Recalling that T is an antiunitary operation, we conclude that T will commute with H, Eq. (31), provided b_ν, c_ν, b_ν', c_ν' are all real in case $T\sigma=\sigma T$, while if $T\sigma=-\sigma T$ we must have b_ν, c_ν real and b_ν' and c_ν' pure imaginary.

In the case of charge conjugation, invariance of the wave Eq. (24) requires $CH=-HC$, according to (17a). From (17b) and (20) we find that C anticommutes with $(\boldsymbol{\lambda}\cdot\mathbf{p})$, so that $CB_\nu=B_\nu C$ and $CC_\nu=-C_\nu C$. Combining this with the fact that C may commute or anticommute with σ according to (23a), and recalling that C is antilinear, we conclude that the wave equation will be invariant under C, provided b_ν, b_ν' are imaginary and c_ν, c_ν' real in case $C\sigma=\sigma C$, while if $C\sigma=-\sigma C$, invariance requires b_ν, c_ν' to be imaginary and b_ν', c_ν to be real.

The above conditions for invariance under C and T are listed in Table I. It is evident from the table that invariance under *both* C and T requires the b_ν to vanish; this rules out the possibility of choosing the coefficients according to (32a)—instead, b_ν', c_ν, and c_ν' must satisfy (32b) for every ν. Further, either the set of coefficients b_ν' or the set c_ν' must vanish, since the b_ν' can be nonvanishing only when either T or C (but not both)

TABLE I. Conditions on H, Eq. (31), for invariance of the wave equation under T and C.

Symmetry operation[a]	b_ν	c_ν	b_ν'	c_ν'
T (+)	R[b]	R	R	R
T (−)	R	R	I	I
C (+)	I	R	I	R
C (−)	I	R	R	I
T and C	0	R	R if T (+), C (−) I if T (−), C (+) 0 otherwise	R if T (+), C (+) I if T (−), C (−) 0 otherwise

[a] A + or − sign following T or C indicates that the operator is taken to commute (+) or anticommute (−) with σ.
[b] Notation: R = Real, I = Imaginary.

commutes with σ, while if the c_ν' are to be nonzero, both C and T must commute with σ or both must anticommute.

We have now exhausted all the conditions on H arising from invariance of (24) under the Poincaré group and the discrete operations, with one exception: invariance under boosts, which, according to (1i), requires $[H,\mathbf{K}]=i\mathbf{P}$, or

$$[H,(t\mathbf{p}-\mathbf{x}H+i\boldsymbol{\lambda})]=i\mathbf{p}. \tag{33}$$

Explicit evaluation of the left-hand member[23] in (33) is difficult in the general case, but this can be avoided, as we shall see in the Appendix. At the moment we wish merely to observe that the application of (33) would not help to settle in favor of any particular one of the possibilities left open in Table I. It is simplest to see this in the spin-$\frac{1}{2}$ case. In this case $\boldsymbol{\lambda}$ and σ can be interpreted in terms of the conventional Dirac matrices as $\frac{1}{2}\boldsymbol{\alpha}$ and β. The operators $B_{1/2}$ and $C_{1/2}$ are 1 and $(\boldsymbol{\alpha}\cdot\mathbf{p}/p)$, respectively. If we now choose, *a priori*, the coefficient $b_{1/2}'$ to be nonzero and $c_{1/2}'$ to be zero, the form of the Hamiltonian is $c_{1/2}(\boldsymbol{\alpha}\cdot\mathbf{p}/p)+\beta b_{1/2}'$. The application of (33) and (32b) then leads to the actual values of the coefficients, and the final result is the familiar Dirac Hamiltonian $\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m$. On the other hand, if we started by choosing $b_{1/2}'$ to be zero and $c_{1/2}'$ to be nonzero, the same procedure would lead to a new Hamiltonian,

$$H=(E^2/p^2)(\boldsymbol{\alpha}\cdot\mathbf{p})+(Em/p^2)\beta(\boldsymbol{\alpha}\cdot\mathbf{p}), \tag{34}$$

which also satisfies all the invariance conditions. Thus it is clear that relativistic invariance alone does not determine the wave Eq. (24) uniquely.

There is, however, another condition of a physical nature which any reasonable Hamiltonian should satisfy and which is not satisfied by (34), nor indeed by any Hamiltonian of the form

$$\sum c_\nu C_\nu+\sigma\sum c_\nu' C_\nu. \tag{35}$$

It is that if we make $\mathbf{p}\to 0$ (passage to the rest system) H should tend to a well-defined limit.[24] It is clear that λ_p is not well-defined in this limit since it would depend on the direction of the unit vector $(\mathbf{p}/p)$; and the C_ν, being odd in λ_p, share this property. In view of this, it is easy to convince oneself that there is no choice of the coefficients c_ν and c_ν' in (35) which would lead to the desired limiting property for H. The only alternative left is to take H to be of the form

$$H=\sum c_\nu C_\nu+\sigma\sum b_\nu' B_\nu. \tag{36}$$

Here again the coefficients c_ν must vanish as $\mathbf{p}\to 0$, and in the same limit, all powers of λ_p higher than the zeroth in the second term in (36) must cancel out, leaving just a multiple of the identity matrix. But from the definition of the B_ν in terms of projection operators, it follows that the identity operator is $1=\sum B_\nu$. The conclusion is, therefore, that all the b_ν' must tend to the same value when $\mathbf{p}\to 0$; and by virtue of (32b) and the fact that in this limit $E\to m$ and all the $c_\nu\to 0$ (as already noted), the common limit of all the b_ν' must be m. The Hamiltonian in the rest system is thus uniquely determined as $H_0=\rho_1 m$.

It must be pointed out here that our requirement regarding the existence of the rest-system limit has accomplished two things: first to establish that the coefficients c_ν' must vanish, and secondly to show that the b_ν' (in the rest system) must be *real*. It follows immediately from Table I that the commutation properties of T and C with σ must be

$$T\sigma=+\sigma T, \tag{37}$$

$$C\sigma=-\sigma C. \tag{38}$$

These results, in combination with (16a) and (23a), eliminate the ambiguity in the matrices τ and κ as given by (15) and (22). We now have

$$\tau=\begin{pmatrix}\zeta_s & 0\\ 0 & \zeta_s\end{pmatrix} \tag{39}$$

and

$$\kappa=\begin{pmatrix}0 & \zeta_s\\ -\zeta_s & 0\end{pmatrix}. \tag{40}$$

One need hardly stress that (37) and (38) ensure that the coefficients b_ν' are always real (not only in the rest system). Consequently the Hamiltonian is Hermitian in the ordinary sense—a nice property to have though all we need for conservation of the invariant scalar product $\int\psi^\dagger M\psi d^3x$ is the condition $H^\dagger M=MH$.

To return to the explicit form of the Hamiltonian, we still have to determine the values of the coefficients c_ν and b_ν' in (36). For any given spin s, this can be done by imposing on (36) the condition (33) for invariance under

[23] Note that in writing out the expression for $\mathbf{K}$ in (33), we have replaced the p_0 which occurs in the definition (3d) by $-H$. This is permissible here because p_0 commutes with H and is equivalent to $-H$ when operating on any solution of Eq. (24).

[24] The same condition appears in the guise of a "regularity property" of the wave function at the origin of momentum space, in the investigation of localized states of relativistic particles by T. D. Newton and E. P. Wigner, Rev. Mod. Phys. **21**, 400 (1949).

boosts.[25] The result in the spin-$\frac{1}{2}$ case, for instance, is the Dirac Hamiltonian, while for spin 1, we obtain

$$H=\frac{2E^2}{2E^2-m^2}(\boldsymbol{\lambda}\cdot\mathbf{p})+\sigma\left[E-\frac{2E}{2E^2-m^2}(\boldsymbol{\lambda}\cdot\mathbf{p})^2\right]. \quad (41)$$

These coincide with the Hamiltonians derived in the corresponding cases by Weaver, Hammer, and Good,[14,26] Such agreement is, in fact, to be expected for any spin since their starting point is the assumption that the rest-system Hamiltonian is $H_0=\beta m=\sigma m$ (which we have shown to be the only one consistent with covariance and regularity conditions), and they utilize an integrated form of (3d) to pass to an arbitrary reference frame. The general expression for the Hamiltonian of a particle of arbitrary spin is

$$H=\sum_\nu \frac{(E+p)^{4\nu}-m^{4\nu}}{(E+p)^{4\nu}+m^{4\nu}}EC_\nu+\sigma\sum_\nu \frac{2Em^{2\nu}(E+p)^{2\nu}}{(E+p)^{4\nu}+m^{4\nu}}B_\nu. \quad (42)$$

The derivation of this expression is given in the Appendix.

It may be recalled that we had explicitly excluded spinless particles from the treatment given above. When $s=0$, the parity operation P can be defined on the single-component wave function $\psi^{(0,0)}$, but it is necessary to adjoin another component, also transforming like $\psi^{(0,0)}$, in order to accommodate the two antilinear operations T and C, which must commute and anticommute, respectively, with the Hamiltonian. The form of the wave function then fits into the general type $\psi^{(m,n)}\oplus\psi^{(n,m)}$ that we had considered earlier, and it is easy to see that the Hamiltonian also can quite generally be taken in the form (36), which now reduces to $H=\rho_1 E$ since $\mathbf{S}$ and $\boldsymbol{\lambda}$ are now zero. The last mentioned fact releases the parity operator σ from the necessity to anticommute with ρ_3; it may therefore coincide with ρ_1 as before or may just be the unit operator, as has already been noted by WHG. A similar ambiguity exists in the choice of C and T, and what is more, there seems to be no reason to preclude independent choice of the possible alternatives in P, C, and T.

A final remark regarding the representation of the Poincaré group provided by (3): The operators $\mathbf{P}$, $\mathbf{J}$ are clearly Hermitian and so is the Hamiltonian H, (36), to which $-P_0$ is equivalent when operating directly on a wave function; but in order that $\mathbf{K}$ be Hermitian we must have

$$-\mathbf{x}H+i\boldsymbol{\lambda}=-H\mathbf{x}-i\boldsymbol{\lambda}$$

or

$$-i[\mathbf{x},H]=2\boldsymbol{\lambda}.$$

It is clear from the form (36) of H that it is only in the spin-$\frac{1}{2}$ case that this requirement can be met. This substantiates our statement in Sec. I that except for $s=\frac{1}{2}$ the representation (3) is not unitary with respect to the scalar product $\int\psi^\dagger\psi d^3x$.

ACKNOWLEDGMENT

It is a pleasure to thank Professor D. L. Falkoff for his hospitality at Brandeis University.

APPENDIX

To determine the coefficients of c_ν and b_ν' in (36) by requiring invariance under boosts, we make use of the alternative form of the boost condition (33) given in Ref. 25, namely

$$[H,\{-i[\mathbf{x},H]-2\boldsymbol{\lambda}\}]=0. \quad (A1)$$

It turns out to be sufficient for our purposes, and very much simpler, to consider the weaker condition obtained by scalar multiplication of (A1) on the left by $\mathbf{p}$, namely

$$[H,-i\mathbf{p}\cdot[\mathbf{x},H]]=2[H,\boldsymbol{\lambda}\cdot\mathbf{p}]. \quad (A2)$$

In applying (A2), we shall find it convenient to make use of two different ways of writing H:

$$H=\sum_{\nu>0}c_\nu C_\nu+\sigma\sum_{\nu\geqslant 0}b_\nu' B_\nu \quad (36)$$

or

$$H=\sum_{l\text{ odd}}f_l(\boldsymbol{\lambda}\cdot\mathbf{p})^l+\sigma\sum_{l\text{ even}}g_l(\boldsymbol{\lambda}\cdot\mathbf{p})^l, \quad (A3)$$

where l ranges from 0 to $2s$, the f_l being nonzero only for odd values of l and the g_l for even values. The relation between the coefficients in the above two equations can be easily obtained from the spectral representation of $(\boldsymbol{\lambda}\cdot\mathbf{p})$:

$$(\lambda_p)^l=(\boldsymbol{\lambda}\cdot\mathbf{p}/p)^l=\sum_{\nu=-s}^{s}\nu^l\Lambda_\nu \quad (A4)$$

so that

$$(\boldsymbol{\lambda}\cdot\mathbf{p})^l=\sum_{\nu=-s}^{s}(\nu p)^l\Lambda_\nu=\sum_{\nu\geqslant 0}(\nu p)^l B_\nu,\quad (l\text{ even}), \quad (A5a)$$

$$=\sum_{\nu>0}(\nu p)^l C_\nu,\quad (l\text{ odd}). \quad (A5b)$$

Introducing (A5) in (A3) and comparing with (36), we find that

$$c_\nu=\sum_l(\nu p)^l f_l,\quad b_\nu'=\sum_l(\nu p)^l g_l. \quad (A6)$$

Now, using the fact that $-i[\mathbf{x},H]$ is just the gradient of

[25] By writing $i\mathbf{p}=\frac{1}{2}[\mathbf{x},E^2]=\frac{1}{2}(\mathbf{x}H^2-H^2\mathbf{x})$, we can reduce (33) to the form $[H,\{-i[\mathbf{x},H]-2\boldsymbol{\lambda}\}]=0$, which shows that H does not commute with $-i[\mathbf{x},H]$. The obvious interpretation is that the particle (whatever be its spin) exhibits "Zitterbewegung." It is possible, however, to argue that it is not $\mathbf{x}$ that is observable, but some "mean position" as defined by Foldy and Wouthuysen [Phys. Rev. **78**, 29 (1950)] for the spin-$\frac{1}{2}$ case. This "mean" or "observable" position leads to a velocity which is a constant of the motion. For further considerations on this point in relation to the Dirac electron, and to particles of spins 0 and 1 as described by the Sakata-Taketani form of the Kemmer equation [N. Kemmer, Proc. Roy. Soc. (London) **A173**, 91 (1939), S. Sakata and M. Taketani, Proc. Phys.-Math. Soc. Japan **22**, 757 (1940)], see P. M. Mathews and A. Sankaranarayanan, Progr. Theoret. Phys. (Kyoto) **26**, 499 (1961); **27**, 1063 (1962), and **32**, 159 (1964). A discussion of observables of particles of arbitrary spin in the context of the present formalism will be published elsewhere.

[26] Our matrices $\boldsymbol{\lambda}$ and ρ_1 (to which we have equated σ) are, in the notation of WHG, $s\boldsymbol{\alpha}$ and β, respectively.

H in $\mathbf{p}$ space, it is easy to verify that

$$-i\mathbf{p}\cdot[\mathbf{x},H]=\sum((df_l/dp)p+f_l l)(\boldsymbol{\lambda}\cdot\mathbf{p})^l +\sigma\sum((dg_l/dp)p+g_l l)(\boldsymbol{\lambda}\cdot\mathbf{p})^l. \quad (A7)$$

By using (A5) and (A6) we can reduce this to

$$-i\mathbf{p}\cdot[\mathbf{x},H]=\sum p(dc_\nu/dp)C_\nu+\sigma\sum p(db_\nu'/dp)B_\nu. \quad (A8)$$

Substituting this in the left-hand side of (A2) and using the representation (A5b) for $(\boldsymbol{\lambda}\cdot\mathbf{p})$ on the right-hand side, we obtain, after evaluating the commutators,

$$(dc_\nu/dp)b_\nu'-c_\nu(db_\nu'/dp)=2\nu b_\nu'. \quad (A9)$$

But we already have the relation

$$c_\nu^2+b_\nu'^2=E^2 \quad (A10)$$

from which it follows that

$$c_\nu dc_\nu/dp+b_\nu' db_\nu'/dp=p. \quad (A11)$$

Solution of the simultaneous Eqs. (A9) and (A11) for the derivatives of c_ν and b_ν' yields

$$\frac{dc_\nu}{dp}=\frac{c_\nu p+2\nu b_\nu'^2}{E^2}=\frac{c_\nu p+2\nu(E^2-c_\nu^2)}{E^2}, \quad (A12)$$

$$\frac{db_\nu'}{dp}=\frac{-2\nu c_\nu b_\nu'+pb_\nu'}{E^2}. \quad (A13)$$

The substitution $c_\nu=Eh_\nu$ in (A12) leads to a simple equation for h_ν. Solution of this equation, with the initial condition $h_\nu=mc_\nu=0$ at $p=0$, yields the result

$$c_\nu=E\frac{(E+p)^{4\nu}-m^{4\nu}}{(E+p)^{4\nu}+m^{4\nu}}. \quad (A14)$$

The coefficient b_ν' is then obtained from (A10):

$$b_\nu'=\frac{2Em^{2\nu}(E+p)^{2\nu}}{(E+p)^{4\nu}+m^{4\nu}}. \quad (A15)$$

The expression (42) for H follows on introducing (A14) and (A15) in (36).

It is a striking characteristic of the coefficients as given by (A14) and (A15) that they are *independent* of the spin of the particle. Thus, for example, the *form* of the Hamiltonian for a particle with integral spin s would differ from that of a particle with spin $(s-1)$ only in having the extra terms c_sC_s and $\sigma b_s'B_s$, the values of the coefficients $c_1, c_2, \cdots, c_{s-1}$ and $b_0', b_1', \cdots, b_{s-1}'$ being the same in both cases. But it must be kept in mind that the matrices C_ν, B_ν do not remain unchanged from one spin to another, their dimensionality, $2(2s+1)$, being determined by the spin.

PHYSICAL REVIEW VOLUME 143, NUMBER 4 MARCH 1966

Invariant Scalar Product and Observables in a Relativistic Theory of Particles of Arbitrary Spin

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(Received 27 September 1965)

In a recent paper a relativistically covariant Schrödinger equation was derived for particles of arbitrary spin s, locally covariant wave functions without redundant components being used to describe states of a particle. Here we determine the invariant scalar product with respect to which the representation of Poincaré transformations on these wave functions is unitary. It is shown that the conventional position and spin operators, not being Hermitian with respect to this scalar product, cannot be observables. New operators which can represent these observables are constructed with the aid of a generalized Foldy-Wouthuysen transformation which is determined explicitly for arbitrary spin.

I. INTRODUCTION

It has been shown in a recent paper[1] that there exists a unique relativistically invariant Schrödinger equation which describes a free particle of arbitrary spin s and nonzero mass m. The requirements on which the derivation was based are the following: (i) The wave function ψ must transform locally under the operations of the Poincaré group as well as under the discrete operations of space and time inversions and charge conjugation. (ii) Its behavior under rotations must be such as to ensure, without the aid of any supplementary conditions, that the particle has a unique spin s. These conditions determine the operators which represent the generators of the Poincaré group on the states of the particle. An operator H was then constructed such that (iii) the Schrödinger equation $i\partial\psi/\partial t=H\psi$ is invariant with respect to the operations mentioned under (i),

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[1] P. M. Mathews, this issue, Phys. Rev. 143, 978 (1966). The notation of this paper will be followed in the present work.