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Remarks on the Forward Peak at High Energies*

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In the framework of the axiomatic field theory, it is proved that s-u crossing-symmetric elastic-scattering amplitude of any two stable particles has forward peak at high energies, unless $|\mathrm{Re}f(s,1)/\mathrm{Im}f(s,1)| \to \infty$ or $|f(s,1)| < C(\ln s)^{-1}$. If one exchange amplitude dominates the others, this result also holds for any elastic-scattering amplitude without s-u symmetry. It turns out that the dispersion relation is not essential and the analytic property recently found by Bros, Epstein, and Glaser is sufficient for the proof. In the proof, the properties of the Herglotz function are extensively used.

ONE of the better established experimental facts in high-energy elastic scattering is the existence of the forward peak. If one assumes the high-energy behavior $\sigma_{\mathrm{total}} \to \mathrm{const} \neq 0$ for an s-u crossing-symmetric amplitude $f(s, \cos\theta)$ it is obvious that there should be a forward peak, since this behavior implies[1] $|\mathrm{Re}f(s, 1)/\mathrm{Im}f(s,1)| \to 0$ and $\mathrm{Im}f(s, \cos\theta)$ has its maximum at $\theta=0$ because of the unitarity $\mathrm{Im}f_l(s) \geqslant 0$. Suppose we drop this assumption on the high-energy behavior of σ_{total}; then no longer is the existence of a forward peak self-evident, for $\mathrm{Re}f(s,1)$ may now be comparable to $\mathrm{Im}f(s,1)$, while we do not know where $\mathrm{Re}f(s, \cos\theta)$ takes its maximum value. Some time ago, however, Aramaki[2] showed the existence of a forward peak in a general framework where the Mandelstam representation is assumed. From his result it is clear that the forward peak is a general feature which follows from analyticity and unitarity, and not a consequence of a specific high-energy behavior of the scattering amplitude or dominance of the imaginary part.

In this paper, we shall show that under general analytic properties which have recently been proved in the Lehmann-Symanzik-Zimmermann (LSZ) formalism by Bros, Epstein, and Glaser,[3] there exists a forward peak at high energies for any s-u-symmetric scattering amplitudes, provided that the following two additional conditions:

$$\text{(i)} \quad |\mathrm{Re}f(s,1)/\mathrm{Im}f(s,1)| < \infty\,,$$

and

$$\text{(ii)} \quad |f(s,1)| > C(\ln s)^{-1}$$

are satisfied by the scattering amplitude at high energies. Condition (i) is an extremely reasonable physical assumption, for at high energies many inelastic channels become open. As to (ii), it is a rather weak restriction on the high-energy behavior, and it can be replaced by an even weaker assumption, i.e., the inequality (12).

Let us consider the elastic scattering $A+B \to A+B$, where A and B are any stable particles and at least one of them is self-conjugate. Introducing the symmetric variable $z=(s-m_A^2-m_B^2)^2$, the analyticity obtained in Ref. 3 allows us to write the forward scattering amplitude as follows:

$$f(s,1) \equiv F(z,1) = A + Bz + \frac{1}{2\pi i}\int_\Gamma \frac{F(z',1)}{z'-z}dz' + \frac{z^2}{\pi}\int_{x_0}^{\infty} \frac{\mathrm{Im}F(z',1)}{z'^2(z'-z)}dz', \quad (1)$$

where Γ denotes a sufficiently large but finite circle which contains the region in which $F(z,1)$ may be singular. [In the derivation of (1), without losing generality we assume that the subtraction point $z=0$ lies outside of Γ. If this is not the case and, say, the real point $z=\bar{x}$ lies outside of Γ and the physical cut, then

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[1] Y. S. Jin and S. W. MacDowell, Phys. Rev. **138**, B1279 (1965).
[2] S. Aramaki, Progr. Theoret. Phys. (Kyoto) **30**, 265 (1963).
[3] J. Bros, H. Epstein, and V. Glaser, Comm. Math. Phys. (to be published).

make the subtraction at this point and use a new variable $\zeta = z - \bar{x}$.]

While forward dispersion relations have been proved only for π-N, π-K, π-Λ, π-Σ, π-Ξ, K-K, and π-π processes, representation (1) holds for the elastic-scattering amplitude of any stable particles without restrictions on their masses, e.g., N-N, K-N, Λ-N Λ-Λ, Λ-Σ, Σ-Σ, etc. In those processes for which dispersion relations have been proved, the finite contour integral in (1) is absent and x_0 corresponds to the least massive intermediate state. The convergence of the second integral in (1) is warranted by the Greenberg-Low bound,[4] i.e., $|F(z,1)| < C|z|\ln^2|z|$. In the derivation of the Greenberg-Low bound, however, the analyticity in the Lehmann ellipse was essential and no analyticity at all in the s plane was used. Thus, the Greenberg-Low bound holds even if one cannot prove the dispersion relation in s, provided that the amplitude is analytic in the Lehmann ellipse. It is easy to see that any scattering amplitude of two stable particles is analytic in a corresponding Lehmann ellipse in the $\cos\theta$ plane whose precise form is determined by the mass-spectrum condition.[5]

Recently, however, it was shown by Khuri and Kinoshita[6] as well as by MacDowell and the author[1] that under assumption (i), i.e., at high energies, the real part of the scattering amplitude does not dominate the imaginary part; one subtraction in (1) can be removed to give

$$F(z,1) = A + \frac{1}{2\pi i}\int_\Gamma \frac{F(z',1)}{z'-z}dz' + \frac{z}{\pi}\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{z'(z'-z)}dz'. \quad (2)$$

Now define

$$G(z) \equiv F(z,1) - \frac{1}{2\pi i}\int_\Gamma \frac{F(z',1)}{z'-z}dz', \quad (3)$$

then since $\mathrm{Im}G(x) = \mathrm{Im}F(x,1) \geqslant 0$ for $x > x_0$, we get

$$G(z) = A + \frac{z}{\pi}\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{z'(z'-z)}dz', \quad (4)$$

and $G(z)$ is clearly a Herglotz function, i.e., $\mathrm{Im}G(z)/\mathrm{Im}z > 0$. We now consider the following two cases (see, for instance, Ref. 11):

(a) $G(x) \leqslant 0$ for a certain $\bar{x} < x_0$.
In this case (4) can be written as

$$G(z) = \bar{A} + \frac{z-\bar{x}}{\pi}\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{(z'-\bar{x})(z'-z)}dz', \quad \text{with} \quad \bar{A} \leqslant 0,$$

and consequently $G(z)/(z-\bar{x})$ is also a Herglotz function. Hence $-(z-\bar{x})/G(z)$ is again a Herglotz function and we obtain[7]

$$\int_{x_0}^\infty \frac{\mathrm{Im}\{(z'-\bar{x})/G(z')\}dz'}{z'^2} < \infty,$$

and thus

$$\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{z'|G(z')|^2}dz' < \infty. \quad (5)$$

(b) $G(x) > 0$ for any $x < x_0$.
In this case we have

$$G(-\infty) = A - \frac{1}{\bar{x}}\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{z'}dz' > 0,$$

and hence it follows that

$$\frac{1}{\bar{x}}\int_{x_0}^\infty \frac{\mathrm{Im}F(z',1)}{z'}dz' < A. \quad (6)$$

But under assumption (ii) the left-hand side of (6) evidently diverges. Thus from (i) and (ii) the inequality (5) necessarily follows. Since $G(z) - F(z,1) = O(z^{-1})$ by definition (3), and $|F(z,1)| > C/\ln|z|$ by (ii), (5) implies

$$\int_{s_0}^\infty \frac{\mathrm{Im}f(s,1)}{s|f(s,1)|^2}ds < \infty, \quad (7)$$

and hence

$$\liminf_{s\to\infty} \frac{\mathrm{Im}f(s,1)}{|f(s,1)|^2} = 0. \quad (8)$$

By using the unitarity condition

$$\mathrm{Im}f(s,1) \geqslant \frac{k}{\sqrt{s}}\int_{-1}^1 |f(s,\cos\theta)|^2 d\cos\theta, \quad (9)$$

we obtain

$$\liminf_{s\to\infty} \int_{-1}^1 \left|\frac{f(s,\cos\theta)}{f(s,1)}\right|^2 d\cos\theta = 0. \quad (10)$$

Now, if $\lim_{s\to\infty}|f(s,\cos\theta)/f(s,1)|$ exists, then the Fatou theorem[8] says

$$\liminf_{s\to\infty} \int_{-1}^1 \left|\frac{f(s,\cos\theta)}{f(s,1)}\right|^2 d\cos\theta > \int_{-1}^1 \lim_{s\to\infty}\left|\frac{f(s,\cos\theta)}{f(s,1)}\right|^2 d\cos\theta. \quad (11)$$

[4] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).
[5] H. Lehmann, Nuovo Cimento **10**, 579 (1958).
[6] N. N. Khuri and T. Kinoshita, Phys. Rev. **137**, B720 (1965); **140**, B706 (1965).
[7] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments* (American Mathematical Society, New York, 1943), p. 23.
[8] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), p. 389.

Therefore,

$$\lim_{s\to\infty}\left|\frac{f(s,\cos\theta)}{f(s,1)}\right|^2=0 \tag{12}$$

almost everywhere on the interval $-1\leqslant\cos\theta\leqslant 1$. This proves the existence of the forward peak; however, the possible existence of other peaks is not excluded.

In Ref. 2 it was claimed that if one uses a wider analyticity domain in the $\cos\theta$ plane implied by the Mandelstam representation,[9] then Eq. (12) is valid *everywhere* on $-1+\epsilon\leqslant\cos\theta\leqslant 1-\epsilon$. It is easy to see that this is not true, unless one assumes *uniform convergence* in the relevant region of $\cos\theta$. Take as an example,

$$f(s,\cos\theta)=Cs^{1-(\cos\theta-1)^2\cos^2\theta}.$$

This function is analytic in the $\cos\theta$ plane, and

$$\lim_{s\to\infty}|f(s,\cos\theta)/f(s,1)|=0$$

everywhere except at $\theta=0$ and $\theta=\frac{1}{2}\pi$. Consequently, there exist peaks at $\theta=\frac{1}{2}\pi$ as well as at $\theta=0$. If, however, one assumes such *uniform convergence* at all, the size of the analyticity domain is inessential. The continuity of $f(s,\cos\theta)$ in $\cos\theta$ is then sufficient to exclude a nonforward (or nonbackward) peak. Without such an additional assumption, the best we can expect is

$$\lim_{s\to 0}\left|\frac{f(s,\cos\theta)}{f(s,1)}\right|=0$$

uniformly in a set of measure greater than $m(E)-\delta$ *where* δ *is any arbitrarily small positive number and* $E=\{\cos\theta: -1\leqslant\cos\theta\leqslant 1\}$ (Egoroff's theorem).[10]

Having proved the existence of a forward peak, let us briefly discuss its relation to the low-energy behavior of the scattering amplitude. If $G(x_0)<0$, then evidently we have case (a) and assumption (ii) is redundant. In the case $G(x_0)>0$, if

$$G(x_0)<\frac{1}{\pi}\int_{x_0}^{\infty}\frac{\operatorname{Im}F(z',1)}{z'-x_0}dz', \tag{13}$$

the alternative case (a) again holds and there exists the forward peak. Suppose we have a dispersion relation; then the finite contour integral in (2) drops out. Hence, *there exists always a forward peak,* unless

$$a>\frac{1}{8\pi^2}\int_{(m_A+m_B)^2}^{\infty}ds\times\frac{(s-m_A{}^2-m_B{}^2)\sigma_{\text{tot}}(s)}{[s-(m_A+m_B)^2]^{1/2}[s-(m_A-m_B)^2]^{1/2}}, \tag{14}$$

where $a\equiv f(s_0,1)\equiv G(x_0)$ is the scattering length and $\sigma_{\text{tot}}(s)=(4\pi/k\sqrt{s})\operatorname{Im}f(s,1)$. However, as has been discussed by Martin and the author[11] as well as by Sugawara,[12] the inequality (14) will be violated as soon as there exists a pronounced resonance in the low-energy region, irrespective of the high-energy behavior. If $a<0$, Eq. (14) imposes no restriction at all, and there always exists a forward peak.

Although our discussion so far applies only to an s-u-symmetric amplitude, e.g., $f^{(1)}(s,\cos\theta)\equiv f_{p\pi^+}(s,\cos\theta)+f_{p\pi^-}(s,\cos\theta)$, we can prove the existence of the high-energy forward peak for any elastic amplitude with an additional assumption that:

(iii) *At high energies, one exchange amplitude dominates the others.* For, on this assumption together with (i), it has been shown[13] that the dominant amplitude corresponds to the quantum number of the vacuum. Hence, for instance, the symmetric amplitude $f_{p\pi^+}+f_{p\pi^-}$ is dominant over $f_{p\pi^+}-f_{p\pi^-}$, and as the former has a peak at $\theta=0$, each of $f_{p\pi^+}$ and $f_{p\pi^-}$ also has a forward peak.

Summarizing our discussions, the conclusion is that the elastic-scattering amplitude of any two stable particles has a forward peak at high energies unless one of the physically extremely reasonable assumptions (i), (ii), and (iii) fails. For the proof, we used the properties that

(a) $f(s,\cos\theta)$ is analytic in the Lehmann ellipse,

(b) $f(s,1)$ is analytic in the cut s plane minus an arbitrarily large but finite complex domain,

(c) $|f(s,\cos\theta)|<C|s|^n$ in the Lehmann ellipse,

and unitarity. It should be pointed out that in the discussion of the high-energy asymptotic behavior the dispersion relation in the s plane is not essential. For instance, in the proof of the Pomeranchuk theorem, it is sufficient to assume (b), as recently discussed by Martin.[14]

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[9] If one assumes the analyticity in the t plane implied by the Mandelstam representation, then the bound of Kinoshita, Loeffel, and Martin [Phys. Rev. Letters **10**, 460 (1963)], viz.,

$$|f(s,\cos\theta)|<C(\ln s)^{3/2}\quad\text{for}\quad\theta\neq 0,\pi,$$

holds. Hence, if one replaces the assymption (i) by $|f(s,1)|\geqslant C(\ln s)^{3/2+\epsilon}$, it follows that (11) is valid *everywhere* on the interval $-1+\epsilon\leqslant\cos\theta\leqslant 1-\epsilon$.

[10] See Ref. 6, p. 339.

[11] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[12] M. Sugawara, Phys. Rev. Letters **14**, 336 (1965).

[13] See, for instance, L. Van Hove, in *Theoretical Problems in Strong Interactions at High Energies*, (CERN, Geneva, 1964), p. 36 and references quoted therein.

[14] A. Martin, Nuovo Cimento **39**, 704 (1965).