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Hard-Core Potential and the Boundary-Condition Model for Nuclear Matter

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The usefulness of the boundary-condition model (BCM) for nuclear-matter calculations depends on the uniqueness of its pseudopotential. This uniqueness is examined in the present paper. A method is proposed for comparing different potentials. Explicit calculations of the average potential energy per particle and the energy gap for nuclear matter are carried out when a hard-core pseudopotential is introduced within the nuclear force range in addition to the pseudopotential of the original BCM. The influence of the hard core with various sizes on the results is discussed. It is found that the results are not sensitive to the hard core as long as the core radius is small, showing that the BCM pseudopotential is effectively quite unique. The fact that our results are affected greatly for larger core radius is linked to the importance of the effect of many-body-correlation.

I. INTRODUCTION

THE difficulties with the nuclear-matter problem are the uncertainty of the nucleon-nucleon interaction potential and the lack of a technique for comparing the different empirical nucleon-nucleon potentials. The construction of the different empirical nuclear potentials depends on two major criteria. Firstly, it must produce phase shifts which will match the experimental data for the nucleon-nucleon elastic scattering up to approximately 300 MeV. Secondly, it must give a binding energy comparable with the existing empirical binding-energy formula. Another test of the potential is obtained by applying it to the deuteron problem. However, because of the usual difficulty with many-body problems we are frequently unable to solve our problems without depending on perturbation methods even when we are given a well-defined empirical potential that will satisfy our criteria. It is well known that in the case of separable potentials[1] we can solve the many-body K-matrix equation without using perturbation methods; hence different forms of separable potentials[2] have been introduced. In particular, the recent boundary-condition model[3] (BCM) starts with the above-mentioned criteria in mind. However, the major difficulty with such a method is the nonuniqueness of the potential. In this paper a hard-core pseudopotential is introduced within the nuclear force range in addition to the pseudopotential[3] of the BCM. The purpose is twofold. Firstly, it is of interest to investigate the effect of a hard-core interaction itself on the properties of nuclear matter. Secondly, one can investigate the dependence of the calculated results for nuclear matter on the different forms of the BCM potential. A method for comparing separable potentials and other empirical potentials is also proposed.

In Sec. II we consider the K-matrix equation for the many-fermion system with a general short-range interaction, and compare it with the K-matrix equation with a separable potential which is assumed to give the same two-body phase shifts as those of the original short-range potential. It is found that we can construct a new integral equation which under certain conditions will allow us to use the usual Fredholm determinant method and which will converge uniformly. At the end of the section we construct a separable potential similar to the BCM potential, but include a hard-core pseudopotential within the range of the short-range nuclear force.

In Sec. III we solve the many-body singlet S-wave K-matrix equation with our model potential, including the hole-hole interaction. A set of graphs with $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{p}\rangle$ versus k is plotted. We find that for the case when our hard-core radius vanishes we obtain slightly different results from those given by Lomon and McMillan.[3] Since our graphs are different from those of Lomon and McMillan we expect that our ground-state energy will differ from theirs even without introducing the hard core.

In Sec. IV we have calculated the ${}^1S_0$ ground-state energy per nucleon and the energy gap, both as a function of the hard-core radius. The results are then plotted. It is interesting to observe that while the binding energy increases with the size of the repulsive hard core the energy gap reaches a minimum at $a \simeq 0.4$ F. Finally in Sec. V we give a discussion of our results as well as proposing some further studies on this problem.

II. THE K MATRIX

Recently, one of us (K. W. W.) has shown that the many-body K-matrix equation can be solved with a

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[1] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954). Y. Yamaguchi and Y. Yamaguchi, *ibid.* **95**, 1635 (1954).

[2] M. Bander, Phys. Rev. **139**, B322 (1965). More references can be found here.

[3] E. L. Lomon and M. McMillan, Ann. Phys. (N. Y.) **23**, 439 (1963).

finite sum of separable potentials[4]

$$V_s=\sum_{i=1}^{n}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_l^{(i)}(k')b_l^{(i)}(k)\times Y_{lm}(\mathbf{k}'|\mathbf{P})Y_{lm}^*(\mathbf{k}|\mathbf{P}), \quad (2.1)$$

where $2\mathbf{P}$ is the center-of-mass momentum.

The K-matrix equation for a fermion system with hole-hole interaction is given by

$$\langle k'|K|\mathbf{k},\mathbf{P}\rangle =\langle \mathbf{k}'|V|\mathbf{k}\rangle+(2\pi)^{-3}\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q \times G(\mathbf{P},\mathbf{k},\mathbf{q})\langle \mathbf{k}'|V|\mathbf{q}\rangle\langle \mathbf{q}|K|\mathbf{k},\mathbf{P}\rangle, \quad (2.2)$$

where

$$G(\mathbf{P},\mathbf{k},\mathbf{q}) =[E(\mathbf{P}+\mathbf{k})+E(\mathbf{P}-\mathbf{k})-E(\mathbf{P}+\mathbf{q})-E(\mathbf{P}-\mathbf{q})]^{-1}$$

and k_F is the Fermi momentum. Let us assume that both V and K_s are empirical nuclear potentials giving the same phase shifts up to 300 MeV. We have already mentioned that the K-matrix equation

$$\langle \mathbf{k}'|K_s|\mathbf{k},\mathbf{P}\rangle =\langle \mathbf{k}'|V_s|\mathbf{k}\rangle+(2\pi)^{-3}\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q \times G(\mathbf{P},\mathbf{k},\mathbf{q})\langle \mathbf{k}'|V_s|\mathbf{q}\rangle\langle \mathbf{q}|K_s|\mathbf{k},\mathbf{P}\rangle, \quad (2.3)$$

can be solved. Let us define $\langle \mathbf{k}'|\Delta K|\mathbf{k},\mathbf{P}\rangle$ as

$$\langle \mathbf{k}'|\Delta K|\mathbf{k},\mathbf{P}\rangle=\langle k'|K-K_s|\mathbf{k},\mathbf{P}\rangle, \quad (2.4)$$

then it follows from Eqs. (2.2) and (2.3) that the matrix ΔK satisfies the following integral equation:

$$\langle \mathbf{k}'|\Delta K|\mathbf{k},\mathbf{P}\rangle =\langle \mathbf{k}'|\Sigma|\mathbf{k},\mathbf{P}\rangle+(2\pi)^{-3}\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q \times G(\mathbf{P},\mathbf{k},\mathbf{q})\langle \mathbf{k}'|V|\mathbf{q}\rangle\langle \mathbf{q}|\Delta K|\mathbf{k},\mathbf{P}\rangle, \quad (2.5)$$

where

$$\langle k'|\Sigma|\mathbf{k},\mathbf{P}\rangle =\langle \mathbf{k}'|V-V_s|\mathbf{k}\rangle+(2\pi)^{-3}\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q \times G(\mathbf{P},\mathbf{k},\mathbf{q})\langle \mathbf{k}'|V-V_s|\mathbf{q}\rangle\langle \mathbf{q}|K_s|\mathbf{k},\mathbf{P}\rangle. \quad (2.6)$$

The Fredholm solution of (2.5) may be written as

$$\langle \mathbf{k}'|\Delta K|\mathbf{k},\mathbf{P}\rangle =\langle \mathbf{k}'|\Sigma|\mathbf{k},\mathbf{P}\rangle+(2\pi)^{-3}\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q \times\frac{N(\mathbf{k}',\mathbf{q},\mathbf{P})}{D(\mathbf{k}',\mathbf{P})}\langle \mathbf{q}|\Sigma|\mathbf{k},\mathbf{P}\rangle, \quad (2.7)$$

where $N(\mathbf{k}',\mathbf{q},\mathbf{P})$ and $D(\mathbf{k}',\mathbf{P})$ are the usual Fredholm minor determinant and the Fredholm determinant, respectively, given by the kernel $G(\mathbf{P},\mathbf{k},\mathbf{q})\langle \mathbf{k}'|V|\mathbf{q}\rangle$. N and D are expandable by uniformly convergent series, provided that the potential V satisfies the following conditions:

$$|V(r)|<M/r^2, \quad \int_{M'}^{\infty} r^2\,dr|V(r)|<\infty, \quad \int_0^{M''} dr\,r|V(r)|<\infty, \quad (2.8)$$

where M, M', M'' are finite positive numbers. Therefore, if, for a potential V, we can find a separable potential V_s such that $|\langle \mathbf{k}'|\Sigma|\mathbf{k},\mathbf{P}\rangle|<\epsilon$, then

$$|\langle \mathbf{k}'|\Delta K|\mathbf{k},\mathbf{P}\rangle|<\epsilon +\epsilon\frac{1}{(2\pi)^3}\left|\left[\int_{|\mathbf{P}\pm\mathbf{q}|>k_F}-\int_{|\mathbf{P}\pm\mathbf{q}|<k_F}\right]d^3q\frac{N(\mathbf{k}',\mathbf{q},\mathbf{P})}{D(\mathbf{k}',\mathbf{P})}\right|. \quad (2.9)$$

Physically we expect $|\Sigma|$ to be very small if both V and V_s give the same phase shifts up to 300 MeV, unless the extremely high-momentum region is of importance in the integral equation. The high-momentum component of the Fourier transform of the potential V corresponds to the extremely short-range part of the potential. In nuclear empirical potentials, such as the Breit[5] potential, one usually postulates a rather strong repulsive part at very short range. On the other hand, the separable potential derived from the BCM gives a free-particle solution inside the range of the nuclear force. Therefore it is desirable to investigate whether the presence of a hard-core inside the BCM potential will not change the calculated physical results of the system such as the binding energy and the energy gap. We modify the BCM potential[3]

$$V_{\mathrm{BCM}}=\lim_{\epsilon\to0,\epsilon>0}\frac{\hbar^2}{m}\sum_{l=0}^{\infty}\left[\frac{f_l}{r_{0l}}\delta(r-r_{0l}-\epsilon) -\delta(r-r_{0l}+\epsilon)\frac{\partial}{\partial r}\right]P_l, \quad (2.10)$$

where m is the mass of one nucleon and P_l is a projection operator, projecting out only the state with angular momentum l, by introducing a hard-core pseudopotential[6]

$$V_{\mathrm{Ps}}=\lim_{\epsilon\to0,\epsilon>0}\frac{\hbar^2}{ma}\delta(r-a)\left[\frac{\partial}{\partial r}r\right]_{r=a+\epsilon}, \quad (2.11)$$

where a is the radius of the hard core, taken to be less than r_{0l} for all l. While the addition of such potentials

[4] K. W. Wong, Nuovo Cimento 34, 591 (1964).

[5] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. 120, 2227 (1960).

[6] L. Liu and K. W. Wong, Phys. Rev. 132, 1349 (1963).

to the original interaction potential V_{BCM} does not change the scattering phase shifts and the bound states of the two-body problem, the many-body problem will be affected.

Our separable potential V_s is given by

$$V_s = V_{\mathrm{BCM}} + V_{\mathrm{P}s}. \tag{2.12}$$

We refer our readers to Refs. 3 and 6 for a more detailed discussion of V_{BCM} and $V_{\mathrm{P}s}$.

III. SOLUTION TO THE K-MATRIX EQUATION

In this section we shall solve for the K-matrix by assuming that the potential vanishes in all but the singlet S state. Inclusion of the triplet-spin S state introduces no particular difficulties but it is neglected here for simplicity. Following Lomon and McMillan, with the potential given by (2.12), one finds that the ${}^1S_0$-state contribution to the V matrix can be written as

$$4\pi\langle \mathbf{k}' | V_0 | \mathbf{k}\rangle = a_0^{(1)}(k')b_0^{(1)}(k) + a_0^{(2)}(k')b_0^{(2)}(k), \tag{3.1}$$

where

$$\begin{aligned} a_0^{(1)}(k') &= [(4\pi\hbar)^2/m] r_0 j_0(k'r_0), \\ b_0^{(1)}(k) &= f_0 j_0(kr_0^-) - kr_0 j_0'(kr_0^-), \\ a_0^{(2)}(k') &= [(4\pi\hbar)^2/m] a j_0(k'a), \\ b_0^{(2)}(k) &= j_0(ka^+) + ka j_0'(ka^+). \end{aligned} \tag{3.2}$$

Also, $r_0^\pm = r_0 \pm \epsilon$, $a^\pm = a \pm \epsilon$, where ϵ is an infinitesimally small positive number. In the right-hand side of (3.1), the first term is just the one considered by Lomon and McMillan,[3] the second term arises from the hard-core interaction.

Correspondingly, by expanding the K matrix into spherical harmonics and taking only the S-wave contribution, Eq. (2.3) becomes

$$4\pi\langle \mathbf{k}' | K_0 | \mathbf{k},\mathbf{P}\rangle = a_0^{(1)}(k') f_0^{(1)}(k,P) + a_0^{(2)}(k') f_0^{(2)}(k,P). \tag{3.3}$$

The f's in (3.3) satisfy the following equation:

$$\sum_{i=1,2} a_0^{(i)}(k') f_0^{(i)}(k,P) = \sum_{i=1,2} a_0^{(i)}(k') b_0^{(i)}(k) + \sum_{i,j=1,2} a_0^{(i)}(k') f_0^{(j)}(k,P)\Gamma^{ij}(k,P), \tag{3.4}$$

where

$$\Gamma^{ij}(k,P) \equiv \frac{1}{(2\pi)^3}\int_0^\infty dq\, q^2 G(P,k,q) \times C_0(q,P) b_0^{(i)}(q) a_0^{(j)}(q), \tag{3.5}$$

$$C_0(q,P) = A_0(q,P) - \lambda B_0(q,P), \tag{3.6a}$$

$$\begin{aligned} A_0(q,P) &= 1 \quad \text{if } q > k_F + P, \\ &= Q \quad \text{if } k_F + P > q > (k_F^2 - P^2)^{1/2}, \\ &= 0 \quad \text{if } (k_F^2 - P^2)^{1/2} > q, \end{aligned} \tag{3.6b}$$

$$\begin{aligned} B_0(q,P) &= 0 \quad \text{if } q > (k_F^2 - P^2)^{1/2}, \\ &= -Q \quad \text{if } (k_F^2 - P^2)^{1/2} > q > k_F - P, \\ &= 1 \quad \text{if } k_F - P > q, \\ Q &= (q^2 + P^2 - k_F^2)/2Pq. \end{aligned} \tag{3.6c}$$

The parameter λ is introduced in (3.6a) to keep track of the contributions from the hole-hole interaction.

Note that the energy denominator $G(\mathbf{P},\mathbf{k},\mathbf{q})$ of (2.2) is now assumed to be independent of angles and written as $G(P,k,q)$ in (3.5). Since Eq. (3.4) holds true for arbitrary values of k and k', the solution is

$$f_0^{(1)}(k,P) = \frac{b_0^{(2)}(k)\Gamma^{12}(k,P) + b_0^{(1)}(k)[1 - \Gamma^{22}(k,P)]}{D(k,P)},$$

$$f^{(2)}(k,P) = \frac{b_0^{(1)}(k)\Gamma^{21}(k,P) + b_0^{(2)}(k)[1 - \Gamma^{11}(k,P)]}{D(k,P)},$$

where

$$D(k,P) = [1 - \Gamma^{11}(k,P)][1 - \Gamma^{22}(k,P)] - \Gamma^{12}(k,P)\Gamma^{21}(k,P). \tag{3.7}$$

Substituting Eq. (3.7) into Eq. (3.3), one obtains

$$4\pi\langle \mathbf{k}' | K_0 | \mathbf{k},\mathbf{P}\rangle = N(k',k;P)/D(k,P), \tag{3.8}$$

where

$$N(k',k;P) = \sum_{i\neq j=1,2} a_0^{(i)}(k') \times \{ b_0^{(i)}(k)[1 - \Gamma^{jj}(k,P)] + b_0^{(j)}(k)\Gamma^{ij}(k,P)\}. \tag{3.9}$$

The two-body K matrix $\langle \mathbf{k}' | K_0 | \mathbf{k}\rangle$ can be obtained from Eq. (3.8) by putting $P=0$ and $\lambda=-1$. Thus, the hole-hole interactions tend to intensify the effect produced by the exclusion principle.[3]

To evaluate the integrals in Eq. (3.5) we shall use the effective-mass approximation so that

$$G(P,k,q) = [(\hbar^2/m^*)(k^2 - q^2)]^{-1},$$

where the effective mass m^* must be, in principle, determined self-consistently. The evaluation of the integrals are tedious but straightforward. The results are given below:

$$\Gamma^{11}(k,P) = -(2/\pi)(m^*/m)\{(f_0+1)I_1 + I_2 + \lambda[(f_0+1)I_3 + I_4]\}, \tag{3.10a}$$

$$\Gamma^{22}(k,P) = -(2/\pi)(m^*/m)(I_7 + \lambda I_{10}), \tag{3.10b}$$

$$\Gamma^{12}(k,P) = -(2/\pi)(m^*/m)\{(f_0+1)I_5 + \tfrac{1}{2}(I_6 - I_{11}) + \lambda(f_0+1)I_8 + \tfrac{1}{2}\lambda(I_9 - I_{12})\}, \tag{3.10c}$$

$$\Gamma^{21}(k,P) = -(1/\pi)(m^*/m) \times \{(I_6 + I_{11}) + \lambda(I_9 + I_{12})\}. \tag{3.10d}$$

The I integrals are given by

$$I_1=\frac{X}{4x}\left[\ln\left|\frac{\beta^2-x^2}{\alpha^2-x^2}\right|-2\left(\ln\frac{\beta}{\alpha}+\mathrm{Ci}2\alpha-\mathrm{Ci}2\beta\right)-(\varphi_\beta{}^+-\varphi_\alpha{}^+)\cos 2x-(\theta_\beta{}^--\theta_\alpha{}^-)\sin 2x\right]$$
$$+\frac{1}{4Pr_0}\left[\ln\frac{\beta}{\alpha}+\mathrm{Ci}2\alpha-\mathrm{Ci}2\beta\right]+\frac{1}{4x}\left[\ln\left|\frac{\beta+x}{\beta-x}\right|-\varphi_\beta{}^-\cos 2x+(\pi-\theta_\beta{}^+)\sin 2x\right], \quad (3.11a)$$

$$I_2=-\tfrac{1}{4}\pi+\tfrac{1}{4}X[-(\varphi_\beta{}^+-\varphi_\alpha{}^+)\sin 2x+(\theta_\beta{}^--\theta_\alpha{}^-)\cos 2x]$$
$$+(1/8Pr_0)[\cos 2\beta-\cos 2\alpha]-\tfrac{1}{4}[\varphi_\beta{}^-\sin 2x+(\pi-\theta_\beta{}^+)\cos 2x], \quad (3.11b)$$

$$I_3=\frac{X}{4x}\left[\ln\left|\frac{\alpha^2-x^2}{\gamma^2-x^2}\right|-2\left[\ln\frac{\alpha}{\gamma}+\mathrm{Ci}2\gamma-\mathrm{Ci}2\alpha\right]-(\varphi_\alpha{}^+-\varphi_\gamma{}^+)\cos 2x-(\theta_\alpha{}^--\theta_\gamma{}^-)\sin 2x\right]$$
$$+\frac{1}{4Pr_0}\left[\ln\frac{\alpha}{\gamma}+\mathrm{Ci}2\gamma-\mathrm{Ci}2\alpha\right]+\frac{1}{4x}\left[\ln\left|\frac{\gamma+x}{\gamma-x}\right|-\varphi_\gamma{}^-\cos 2x-\theta_\gamma{}^+\sin 2x\right], \quad (3.11c)$$

$$I_4=\tfrac{1}{4}X[-(\varphi_\alpha{}^+-\varphi_\gamma{}^+)\sin 2x+(\theta_\alpha{}^--\theta_\gamma{}^-)\cos 2x]+(1/8Pr_0)[\cos 2\alpha-\cos 2\gamma]-\tfrac{1}{4}[\varphi_\gamma{}^-\sin 2x-\theta_\gamma{}^+\cos 2x], \quad (3.11d)$$

$$I_5=(1/4x)\{[\Lambda_\beta{}^-+X(\Lambda_\beta{}^+-\Lambda_\alpha{}^+)]\cos\xi x$$
$$-[(\pi-\Omega_\beta{}^+)-X(\Omega_\beta{}^--\Omega_\alpha{}^-)]\sin\xi x+(x/Pr_0-2X)(\mathrm{Ci}\xi\beta-\mathrm{Ci}\xi\alpha)\}$$
$$-(1/4x)\{[K_\beta{}^-+X(K_\beta{}^+-K_\alpha{}^+)]\cos\eta x-[(\pi-\Sigma_\beta{}^+)-X(\Sigma_\beta{}^--\Sigma_\alpha{}^-)]\sin\eta x+(x/Pr_0-2X)(\mathrm{Ci}\eta\beta-\mathrm{Ci}\eta\alpha)\}. \quad (3.11e)$$

$$I_6=\tfrac{1}{2}[(\pi-\Omega_\beta{}^+)\cos\xi x+\Lambda_\beta{}^-\sin\xi x]$$
$$-\tfrac{1}{2}X[(\Lambda_\alpha{}^+-\Lambda_\beta{}^+)\sin\xi x-(\Omega_\alpha{}^--\Omega_\beta{}^-)\cos\xi x]+(1/2Pr_0\xi)(\cos\xi\alpha-\cos\xi\beta). \quad (3.11f)$$

$$I_7=\tfrac{1}{4}[(\pi-B_\beta{}^+)\cos 2\delta x+\Gamma_\beta{}^-\sin 2\delta x-\pi]+(1/8Pr_0\delta)(\cos 2\alpha\delta-\cos 2\beta\delta)$$
$$+\tfrac{1}{4}X[(B_\alpha{}^--B_\beta{}^-)\cos 2\delta x+(\Gamma_\beta{}^+-\Gamma_\alpha{}^+)\sin 2\delta x]. \quad (3.11g)$$

$$I_8=(1/4x)\{[\Omega_\gamma{}^+-X(\Omega_\gamma{}^--\Omega_\alpha{}^-)]\sin\xi x+[\Lambda_\gamma{}^--X(\Lambda_\gamma{}^+-\Lambda_\alpha{}^+)]\cos\xi x+(2X-x/Pr_0)(\mathrm{Ci}\xi\gamma-\mathrm{Ci}\xi\alpha)\}$$
$$-(1/4x)\{[\Sigma_\gamma{}^+-X(\Sigma_\gamma{}^--\Sigma_\alpha{}^-)]\sin\eta x+[K_\gamma{}^--X(K_\gamma{}^+-K_\alpha{}^+)]\cos\eta x+(2X-x/Pr_0)(\mathrm{Ci}\eta\gamma-\mathrm{Ci}\eta\alpha)\}. \quad (3.11h)$$

$$I_9=\tfrac{1}{2}\{[\Lambda_\gamma{}^-+X(\Lambda_\alpha{}^+-\Lambda_\gamma{}^+)]\sin\xi x-[\Omega_\gamma{}^++X(\Omega_\alpha{}^--\Omega_\gamma{}^-)]\cos\xi x+(1/Pr_0\xi)(\cos\xi\gamma-\cos\xi\alpha)\}. \quad (3.11i)$$

$$I_{10}=\tfrac{1}{4}\{[\Gamma_\gamma{}^-+X(\Gamma_\alpha{}^+-\Gamma_\gamma{}^+)]\sin 2\delta x-[B_\gamma{}^++X(B_\alpha{}^--B_\gamma{}^-)]\cos 2\delta x+(1/2Pr_0\delta)(\cos 2\delta\gamma-\cos 2\delta\alpha)\}. \quad (3.11j)$$

$$I_{11}=\tfrac{1}{2}[(\pi-\Sigma_\beta{}^+)\cos\eta x+K_\beta{}^-\sin\eta x]$$
$$-\tfrac{1}{2}X[(K_\alpha{}^+-K_\beta{}^+)\sin\eta x-(\Sigma_\alpha{}^--\Sigma_\beta{}^-)\cos\eta x]+(1/2Pr_0\eta)(\cos\eta\alpha-\cos\eta\beta), \quad (3.11k)$$

$$I_{12}=\tfrac{1}{2}\{[K_\gamma{}^-+X(K_\alpha{}^+-K_\gamma{}^+)]\sin\eta x-[\Sigma_\gamma{}^++X(\Sigma_\alpha{}^--\Sigma_\gamma{}^-)]\cos\eta x+(1/Pr_0\eta)(\cos\eta\gamma-\cos\eta\alpha)\}, \quad (3.11l)$$

where

$$x=kr_0,\qquad x_F=k_Fr_0,\qquad X=(x^2-\alpha^2)/2Pr_0x,$$
$$\alpha=r_0(k_F{}^2-P^2)^{1/2},\quad \beta=r_0(k_F+P),\quad \gamma=r_0(k_F-P),$$
$$\xi=(r_0-a)/r_0,\qquad \eta=(r_0+a)/r_0,\qquad \delta=a/r_0,$$

and where we have introduced the notation

$$\theta_a{}^\pm=\mathrm{Si}2(a+x)\pm\mathrm{Si}2(a-x),$$
$$\varphi_a{}^\pm=\mathrm{Ci}2(a+x)\pm\mathrm{Ci}2(a-x),$$
$$\Omega_a{}^\pm=\mathrm{Si}\xi(a+x)\pm\mathrm{Si}\xi(a-x),$$
$$\Lambda_a{}^\pm=\mathrm{Ci}\xi(a+x)\pm\mathrm{Ci}\xi(a-x),$$
$$\Sigma_a{}^\pm=\mathrm{Si}\eta(a+x)\pm\mathrm{Si}\eta(a-x),$$
$$K_a{}^\pm=\mathrm{Ci}\eta(a+x)\pm\mathrm{Ci}\eta(a-x),$$
$$B_a{}^\pm=\mathrm{Si}2\delta(a+x)\pm\mathrm{Si}2\delta(a-x),$$
$$\Gamma_a{}^\pm=\mathrm{Ci}2\delta(a+x)\pm\mathrm{Ci}2\delta(a-x),$$

with Si and Ci denoting the sine and cosine integrals as defined in Jahnke and Emde.[7]

To proceed with numerical calculations we have adopted with Lomon and McMillan that

$$f_0=-0.89,\quad r_0=0.95\ \mathrm{F},$$

which fit the YLAM $^1S_0$ phase shift of Breit *et al.*[5] for laboratory energies up to 160 MeV. Using $k_F\simeq 1.4(\mathrm{F})^{-1}$, we have plotted $\langle\mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ as a function of k for different values of the hard-core radius. In Fig. 1 we show $\langle\mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ with $m^*=m$ for $P=0$, $0.5k_F$, and $0.9k_F$ with the full hole-hole interaction included ($\lambda=1$) but in the absence of hard-core interaction. This plot should be the same as Fig. 4 of Lomon and McMillan.[3] How-

[7] E. Jahnke and F. Emde, *Table of Functions* (Dover Publications, New York, 1945), 4th ed.

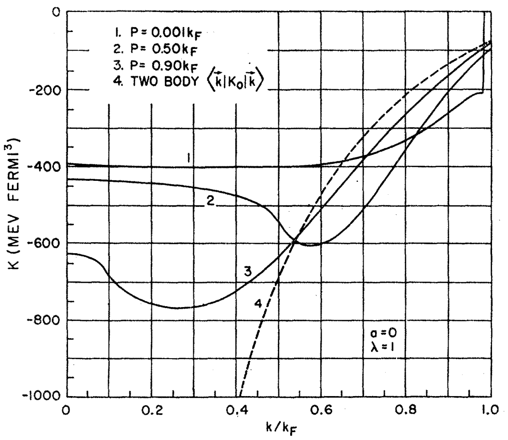

FIG. 1. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ as a function of k for various values of P for the case of no hard-core interaction ($a=0$) but when the hole-hole interactions are included ($\lambda=1$). Also included is the two-body $\langle \mathbf{k}|K_0|\mathbf{k}\rangle$ [Eq. (3.8) with $P=0$ and $\lambda=-1$].

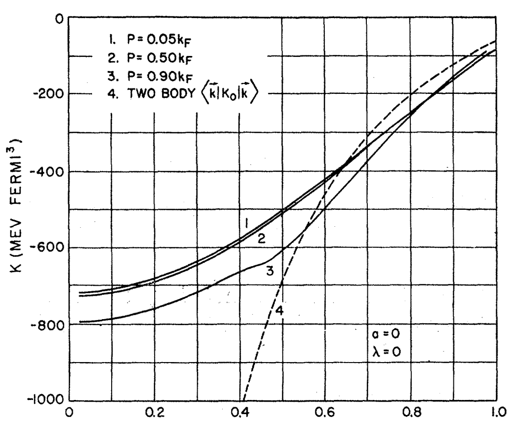

FIG. 2. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ in the absence of hard-core ($a=0$) and hole-hole interactions ($\lambda=0$). The two-body $\langle \mathbf{k}|K_0|\mathbf{k}\rangle$ is also included.

ever, probably due to some errors in the numerical computations, the curves for $P=0.5k_F$ and $P=0.9k_F$ look appreciably different in the two plots. Figures 2–4 show the variation of $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ as the hard-core radius is increased from zero to 0.8 F (we recall that $r_0=0.95$ F) when the hole-hole interactions are neglected. In Figs. 5 and 6 we show $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ when the hole-hole interactions are included. Discussions of these results will be deferred until later.

IV. AVERAGE ENERGY PER PARTICLE AND THE ENERGY GAP

Let us now consider the energy per particle, E/A:

$$\frac{E}{A}=\frac{3}{5}\frac{\hbar^2 k_F{}^2}{2m}+\frac{\mathrm{PE}}{A},$$

where PE/A denotes the average potential energy per particle. In the singlet S-wave approximation, it can be shown[3] that PE/A reduces to

$$\left(\frac{\mathrm{PE}}{A}\right)_0=\frac{9}{\pi^2 k_F{}^3}\int_0^{k_F} dP\,P^2\left[\int_0^{k_F-P} dk\,k^2\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle -\int_{k_F-P}^{(k_F{}^2-P^2)^{1/2}} dk\,k^2\langle \mathbf{k}|K_0|\mathbf{k},P\rangle\frac{k^2+P^2-k_F{}^2}{2Pk}\right]. \quad (4.1)$$

We have integrated this expression numerically both with and without the hole-hole interactions, assuming $m^*=m$. The results are plotted in Fig. 7 which shows the dependence of the potential energy on the size of the hard core. We observe that as the core becomes larger the absolute magnitude of the potential energy per particle increases. At $a=0.4$ F which is approximately the empirical value for the hard-core radius, the potential energy per particle reaches -19.5 MeV, exceeding the corresponding value at $a=0$ by more than 1

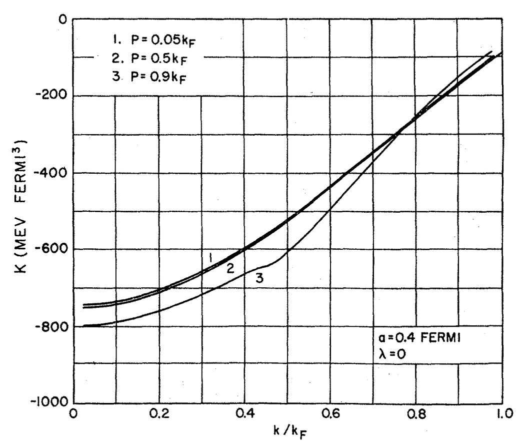

FIG. 3. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ in the absence of hole-hole interactions ($\lambda=0$) but when the hard-core radius $=0.4$ F.

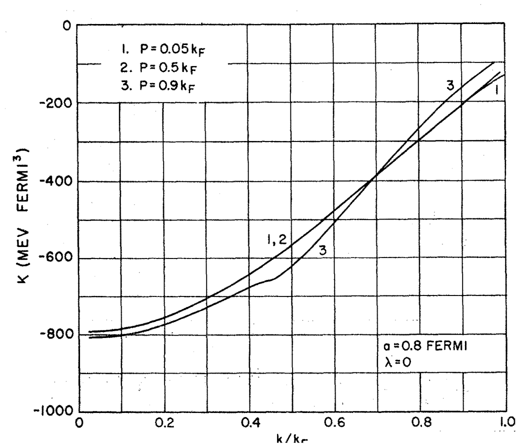

FIG. 4. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ when the hard-core radius $=0.8$ F, but with no hole-hole interaction.

MeV when hole-hole interactions are included. In the absence of hole-hole interactions, the corresponding increase of the absolute magnitude of the potential energy is somewhat less, being about 0.6 MeV. When a exceeds 0.6 F, the variation of the potential energy becomes more rapid, resulting in significant deviations from the corresponding values for $a=0$.

We have not attempted to calculate the effective mass m^* self-consistently. However, it has been shown[3] that m^* is about $0.9m$ when hole terms are included.

Next we consider the energy gap in nuclear matter. Analogous to superconductivity in metals, it has been suggested[8] that the ground state of nuclear matter is separated from the normal states by an energy gap. Emery[9] has proven that there exists a one-to-one correspondence between the energy gap and the singularity of the K matrix for an infinite fermion system when the

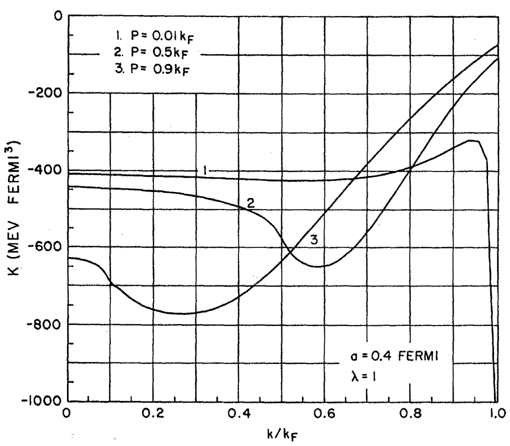

FIG. 5. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$ including hole-hole interactions and a hard-core of radius 0.4 F.

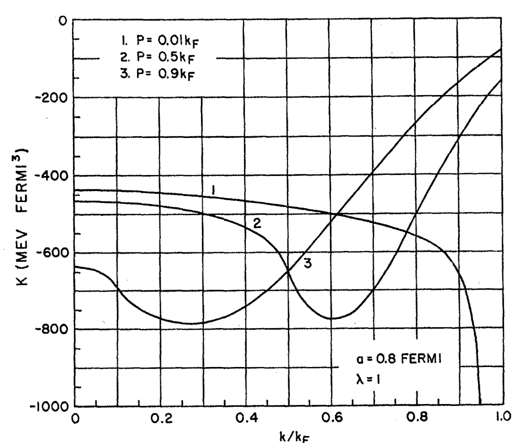

FIG. 6. The nuclear matter $\langle \mathbf{k}|K_0|\mathbf{k},\mathbf{P}\rangle$, including hole-hole interactions and a hard core of radius 0.8 F.

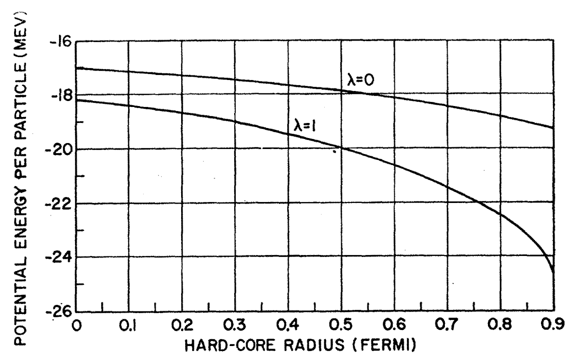

FIG. 7. The potential energy per particle as a function of the hard-core radius for both cases when the hole-hole interactions are included and excluded.

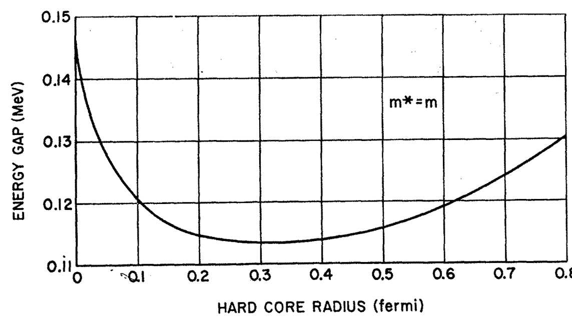

FIG. 8. The energy gap as a function of the hard-core radius when the hole-hole interactions are included.

hole-hole interactions are included, and that the size of the energy gap is related to the position of the pole of the K matrix by

$$\Delta=(2\hbar^2/m^*)(k_F^2-k^{*2}), \tag{4.2}$$

where k^* is given by $D(k^*,P)=0$ [see Eqs. (3.7) and (3.8)]. The existence of such a pole of the K matrix can already be discerned in Figs. 5 and 6. Figure 8 shows the dependence of the energy gap on the size of the hard core. We find that the energy gap is in the neighborhood of 0.13 MeV in agreement with the results of Emery and Sessler.[10] It is interesting to observe that the size of this gap has a minimum near $a=0.35$ F.

V. DISCUSSION AND CONCLUSIONS

In this section we shall discuss the results obtained in the two previous sections. From Figs. 1–8 we can make the following observations: (1) In the case of no hard-core potential (see Figs. 1 and 2) if one examines the many-body K matrices ($\lambda=1, \lambda=0$), using the two-body $\langle \mathbf{k}|K_0|\mathbf{k}\rangle$ as a reference, one finds that, for the same P, the many-body K matrices are modified much more at small k than at large k; and that for the same k, the many-body K matrices with large P are less affected than those with small P. (2) Due to the presence of

[8] A. Bohr, B. R. Mottelson, and D. Pines, Phys. Rev. **110**, 936 (1958).
[9] V. J. Emery, Nucl. Phys. **19**, 154 (1960).
[10] V. J. Emergy and A. M. Sessler, Phys. Rev. **119**, 248 (1960).

hard-core interaction ($a \neq 0$), the K matrices with large P are affected to a much lesser extent than those with small P as the hard-core radius a is increased from zero to 0.8 F (see Figs. 3–6). (3) As a increases, the K matrices become more negative or, in other words, the effective interaction becomes more attractive. This effect is more pronounced for $\lambda=1$ than $\lambda=0$. (4) The average potential energy per particle decreases as the core radius a increases. Again, the effect is more pronounced for $\lambda=1$ than for $\lambda=0$ (see Fig. 7). (5) The energy gap is rather insensitive to the variation of the hard-core radius as shown in Fig. 8. (6) The many-body K matrices and the average potential energy per particle start to show significant changes when a exceeds 0.5 F.

It is well known that the Brueckner K-matrix theory is essentially an "independent-pair" theory. The many-body correlation effect comes in only through the exclusion principle. The inclusion of the hole-hole interactions tends to "intensify" the effect produced by the exclusion principle as noted by Lomon and McMillan.[3] For two low-energy (small P and small k) nucleons the exclusion principle imposes a profound restriction on their interaction by making the low-energy states within the Fermi sphere unavailable as intermediate states in the K matrix. For high-energy nucleons (large P and k) the exclusion principle is not so restrictive since the two interacting nucleons have relatively less difficulty in getting out of the Fermi sphere. Therefore, for large P and k the many-body K matrix is very much like the two-body K matrix while for small P and k they are rather different as observed in (1) above. Introduction of the hard-core pseudopotential within the nuclear force range ($a<r_0$) does not change the two-body problem. The two-body K matrix remains the same, independent of the core radius a. Therefore, the effect of the hard core is felt only via the many-body correlation. Here again the exclusion principle comes into play while the hole-hole interactions tends to make it more prominent. This explains why for large P the hard core has hardly any influence on the K matrix as remarked in observation (2). The fact that at large k and small or moderate P the K matrix differs from the corresponding value at $a=0$ has to do, of course, with the nature of the hard core in modifying the correlations at short ranges. Observation (3) seems to be against our physical intuition at first in that the hard core effectively increases the attractive interaction instead of decreasing it. However, one must bear in mind that our hard-core potential was introduced in such a way that the phase shifts and the bound states of the two-body problem remain essentially unaltered. This implies that while we decrease the width of the attractive potential well by postulating a hard-core interaction at short distances, we must have effectively increased the depth of the attractive well at the same time. The resultant many-body K matrix is affected in a very complicated way via the many interfering terms of the hard core and the deepened attractive potential (such as $\langle q|$ hard core $|q\rangle\langle q|$ attractive potential $|q''\rangle$, $\langle q|$ hard core $|q'\rangle$ $\langle q'|$ hard core $|q''\rangle$, etc.) brought about by the many-body correlations. *A priori*, there is really no reason to expect that such a change of the potential will decrease the effective attractive interaction. In fact, one can see by studying numerically the various terms in Eqs. (3.7), (3.8), (3.9) that the effective attraction in the many-body system may gain or lose in strength, depending on the values of the parameters f_0, r_0, k_F, etc. Therefore, it should not be surprising if for the chosen values of the parameters the many-body K matrix becomes more negative with the increased depth of the attractive well. It also serves as an example to show that other intuitive feelings may not be valid. For example, the assumption that the reduction of the hard core to a soft core in the two-body interaction potential leads to a higher binding energy may not be correct if one keeps the same two-body scattering phase shifts. As a result the average potential energy per particle also decreases. However, in the neighborhood of $a=0.4$ F, we see that both the average potential energy and the size of the energy gap vary slowly as a function of the hard-core radius a.

In conclusion, we have demonstrated in this paper the effect of the hard-core potential on the properties of nuclear matter, namely, the average potential energy and the energy gap, calculated on the basis of the boundary-condition model. For reasonable values of the core radius, the results are not sensitive to the core radius, thus showing that the BCM pseudopotential is effectively rather unique for practical nuclear-matter calculations. However, the results vary greatly for $a>0.4$ F, showing the important effect of many-body correlation in this range. This is perhaps an indication that for better agreement with experiment, three- or more body correlation should be taken into account.[11]

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[11] H. A. Bethe, Phys. Rev. 138, B804 (1965).