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General Nondynamical Formalism for Reactions with Particles of Arbitrary Spin: Composite Reactions and Partial-Wave Expansions*

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The mathematical techniques developed in a previous paper for basic constituent reactions are now extended to composite reactions involving four particles of arbitrary spins and parities. The nondynamical description of the various ways to couple the spins is considered in detail. The coupling together of the initial spins, on the one hand, and of the final spins, on the other, permits one to develop the partial-wave expansion of the resulting nondynamical form factors. The Appendices include many specific examples of commonly occurring reactions.

I. INTRODUCTION

IN this paper we shall consider the various nondynamical ways to analyze in spin space a "composite" reaction

$$s_1+s_2 \rightarrow s_1'+s_2' \tag{1.1}$$

of particles with arbitrary values of the spins s_1, s_2, s_1', s_2'. This will not only give us insight into the underlying spin space structure of composite reactions, but will enable us to develop logically the partial wave expansions of the form factors occurring in such reactions.

One technique presented in previous papers[1–4] (we shall use essentially the same notation as in Ref. 1) is that of factorization:

$$s_1+0 \rightarrow s_1'+0\,, \tag{1.2}$$

$$0+s_2 \rightarrow 0+s_2'\,, \tag{1.3}$$

where fermions always appear pairwise in the constituents. Each constituent reaction [Eqs. (1.2) and (1.3)] is a "basic" reaction of the form

$$0+s \rightarrow 0+s'\,, \tag{1.4}$$

which can be readily analyzed for arbitrary values of s and s'.[1] The basic constituent for bosons is even simpler and includes only one particle with nonzero spin. The M matrix for these general basic reactions can be written in spin space as a sum of products of energy- and angular-dependent form factors a_J (whose energy and angular dependence we shall delete for clarity), and spin-momentum rotation scalars $T_{[J]}(p^r):S_{[J]}(s',s)$ (J integer). Thus we have

$$(s'|M|s)=\sum_{J,r} a_J{}^r T_{[J]}(p^r):S_{[J]}(s',s)\,, \tag{1.5}$$

where $J=|s'-s|,\cdots(s'+s)$. The $T_{[J]}(p^r)$ is an irreducible momentum tensor of rank J, while p^r is some set of basis momentum vectors (see Appendix 1), and $S_{[J]}(s',s)$ is the rank J, irreducible, rectangular, spin matrix tensor. The notation $(s'|M|s)$ will mean that M is a matrix in a $(2s'+1)\times(2s+1)$ dimensional spin space with matrix elements $(s'|M|s)_{\Lambda'\Lambda}=\langle s'\Lambda'|M|s\Lambda\rangle$, Λ' and Λ being the spin projections of s' and s along any $\hat{z}$ axis in the rest frames of each particle. All physical observables of this basic reaction Eq. (1.4) can then be expressed as bilinear combinations of these form factors a_J multiplied by "9-j symbols" which depend[1] on the relevant spins and the nonphysical angular momentum type quantity J.

The M matrix (and observables) for the composite reaction can then be factorized into two direct product spin spaces,

$$(s_1's_2'|M|s_1s_2)=(s_1'|M|s_1)\otimes(s_2'|M|s_2) \tag{1.6}$$

or more explicitly as

$$(s_1's_2'|M|s_1s_2)=\sum_{J_1J_2}\sum_{r_1r_2} C_{J_1J_2}{}^{r_1r_2}(T_{[J_1]}(p^{r_1}):S_{[J_1]}(s_1',s_1))\otimes(T_{[J_2]}(p^{r_2}):S_{[J_2]}(s_2',s_2))\,. \tag{1.7}$$

This "factorized" expansion is the natural one with which to calculate the composite reaction observables as bilinear combinations of the $C_{J_1J_2}$'s.[1] However the partial wave expansion of these $C_{J_1J_2}$ form factors is indeed difficult to find because angular momentum expansions demand that the *total* angular momentum be conserved. That is, we first must recouple[5] s_1 with s_2 to form s, s_1' with s_2' to form s' and then couple the initial orbital angular momentum l with s to form j_I, and the final orbital angular momentum l' with s' to form j_F and finally make $j_I=j_F$. Moreover, if two fermions appear on one side of Eq. (1.1) and two bosons on the other, then the factorization technique breaks down unless we first use a "reshuffling theorem"[6] to "cross" one fermion and one boson in spin space.

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[1] P. Csonka, M. Moravcsik, and M. Scadron, Ann. Phys. (N. Y.) (to be published).

[2] P. Csonka, M. Moravcsik, and M. Scadron, Phys. Letters 15, 353 (1965).

[3] P. Csonka, M. Moravcsik, and M. Scadron, Phys. Rev. Letters 14, 861 (1965).

[4] P. Csonka, M. Moravcsik, and M. Scadron, Nuovo Cimento (to be published).

[5] We shall henceforth use the word "couple" to mean initial to final spin space couplings and the word "recouple" to mean initial to initial and final to final spin-space couplings.

[6] See, for example, R. Stora, University of Maryland, Technical Report No. 250, 1962 (unpublished).

In Sec. II we will consider an "irreducible" spin-space expansion of Eq. (1.1) in analogy to Eq. (1.5) by combining the two spin-momentum scalars in Eq. (1.7) into one "compound" spin-momentum scalar. We can then express the "factorized" form factors $C_{J_1J_2}$ in terms of the new "irreducible" form factors $A_{JJ_1J_2}$. In Sec. III we make a Clebsch-Gordan expansion of the composite amplitude to "recouple" the initial and final spins together and write the irreducible form factors $A_{JJ_1J_2}$ in terms of now new "recoupled" form factors $a_J(s',s)$. Then in Secs. IV and V we proceed with the analysis of partial-wave expansions of the form factors $a_J(s',s)$. Examples are worked out in the Appendices for all values of $J\leq 3$, where $|s-s'|\leq J\leq s+s'$.

II. COMPOUND SPIN MATRIX TENSORS

Given the spin matrix tensors $S_{[J_1]}(s_1',s_1)$ and $S_{[J_2]}(s_2',s_2)$, we can form a compound spin matrix tensor $S_{[J;J_1J_2]}(s_1',s_1;s_2's_2)$ in much the same way as combining two momentum tensors $T_{[J_1]}$ and $T_{[J_2]}$ into a compound momentum tensor $T_{[J;J_1J_2]}$ (see Ref. 1). Formally, we define such objects with Cartesian indices by their spherical counterparts

$$\Omega_{J;J_1J_2}{}^M(s_1',s_1;s_2's_2)=\sum_{M_1M_2}\langle J_1J_2M_1M_2|JM\rangle \Omega_{J_1M_1}(s_1',s_1)\Omega_{J_2M_2}(s_2',s_2)\,, \quad (2.1)$$

with

$$(\Omega_{JM}(s',s))_{\Lambda'\Lambda}\equiv(-)^{s-\Lambda}\langle s's\Lambda'-\Lambda|JM\rangle\,. \quad (2.2)$$

Just as[1]

$$(S_{[J]}(s',s))_{\mu_1\cdots\mu_J}=(\beta_J(s',s))^{1/2}\times\sum_M\langle 1\cdots 1\mu_1\cdots\mu_J|JM\rangle\Omega_{JM}(s',s)\,, \quad (2.3)$$

with $\mu_1\cdots\mu_J$ being spherical vector indices, and $\langle 1\cdots 1\mu_1\cdots\mu_J|JM\rangle$ being the parallel coupling coefficient discussed in Ref. 1, and

$$\beta_J(s',s)\equiv\frac{(s'+J-s)!(s+J-s')!(s'+s+J+1)!}{2^J(2J+1)!(s'+s-J)!}\,, \quad (2.4)$$

so now

$$[S_{[J;J_1J_2]}(s_1',s_1;s_2',s_2)]_{\mu_1\cdots\mu_J}=[\beta_{J_1}(s_1',s_1)\beta_{J_2}(s_2',s_2)]^{1/2}\times\sum_M\langle 1\cdots 1\mu_1\cdots\mu_J|JM\rangle\Omega_{J;J_1J_2}{}^M(s_1',s_1;s_2',s_2)\,. \quad (2.5)$$

Note that $S_{[J;J_1J_2]}$ and $\Omega_{J;J_1J_2}$ are matrices in the direct product spin spaces (s_1',s_1) and (s_2',s_2).

In analogy with momentum tensors,[1] Cartesian spin matrix tensors can be found directly by contracting over the appropriate number of indices of $S_{[J_1]}$ and $S_{[J_2]}$ as

$$S_{[J,J_1J_2]}(s_1',s_1;s_2's_2)=\alpha_{J_1J_2J}[S_{[J_1]}(s_1',s_1)\otimes S_{[J_2]}(s_2',s_2)]_{[J]}\,, \quad (2.6)$$

with

$$\alpha_{J_1J_2J}=(-)^{(J_1+J_2-J)/2}\left[\frac{(2J+1)!(2J_1)!(2J_2)!}{(J_1+J_2+J+1)!(J+J_1-J_2)!(J+J_2-J_1)!(J_1+J_2-J)!}\right]^{1/2} \quad (2.7)$$

and $\alpha\to\alpha/\sqrt{2}$ for J_1+J_2+J odd.

$$[S_{[J_1]}(s_1',s_1)\otimes S_{[J_2]}(s_2',s_2)]_{[J]}$$

is the tensor-traceless, symmetric tensor of rank J formed from $S_{[J_1]}$ and $S_{[J_2]}$ by the process of symmetrization and contraction. For example, if $J_1=J_2=1$, then

$$S_{[0;11]}=-\frac{1}{\sqrt{3}}\mathbf{S}_{[1]}(s_1',s_1)\cdot\mathbf{S}_{[1]}(s_2',s_2)\,, \quad (2.8)$$

$$\mathbf{S}_{[1;11]}=\frac{i}{\sqrt{2}}\mathbf{S}_{[1]}(s_1',s_1)\times\mathbf{S}_{[1]}(s_2',s_2)\,, \quad (2.9)$$

$$(S_{[2;11]})_{ij}=\tfrac{1}{2}[S_i(s_1',s_1)S_j(s_2',s_2)+S_j(s_1',s_1)S_i(s_2',s_2)]-\tfrac{1}{3}\mathbf{S}_{[1]}(s_1',s_1)\cdot\mathbf{S}_{[1]}(s_2',s_2)\delta_{ij}\,. \quad (2.10)$$

In general, the trace properties of such spin-matrix tensors are

$$\mathrm{Tr}\,\Omega_{J;J_1J_2}{}^M\Omega_{J';J_1'J_2'}{}^{\dagger M'}=\delta_{J_1'J_1}\delta_{J_2'J_2}\delta_{J'J}\delta_{M'M} \quad (2.11)$$

and

$$\mathrm{Tr}\,S_{[J;J_1J_2]}(s_1's_1;s_2's_2)S_{[J';J_1'J_2']}(s_1's_1;s_2's_2)=\beta_{J_1}(s_1',s_1)\beta_{J_2}(s_2',s_2)\delta_{J_1'J_1}\delta_{J_2'J_2}\delta_{J'J}\mathcal{P}_J\,, \quad (2.12)$$

$\mathcal{P}_J$ being the projection operator explained in Ref. 1.

So now we can make an "irreducible" spin-space expansion of the composite reaction $s_1+s_2\to s_1'+s_2'$ by first coupling s_1 and s_1' together and s_2 and s_2' together to form J_1 and J_2, respectively, and then coupling J_1 and J_2 together to form J:

$$(s_1's_2'|M|s_1s_2)=\sum_{J,J_1J_2}\sum_r A_{JJ_1J_2}{}^rT_{[J]}(p^r):S_{[J;J_1J_2]}(s_1',s_1;s_2',s_2)\,. \quad (2.13)$$

Since Eqs. (1.7) and (2.13) are identical we can express the $C_{JJ_1J_2}$ form factors in terms of the $A_{JJ_1J_2}$ form factors by decomposing the compound spin matrix tensors, Eq. (2.6). Examples of this procedure are worked out in Appendix 2 for the composite reactions $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$, $\frac{1}{2}+0\to\frac{1}{2}+1$, $\frac{1}{2}+1\to\frac{1}{2}+1$, and $\frac{1}{2}+0\to\frac{1}{2}+2$.

In passing, note that the irreducible spin space expansion, Eq. (2.13), also can be used instead of the reshuffling discussed in Sec. I, provided we let J_1 and J_2 take on noninteger values (but J still remains integral). The recoupling technique discussed in the next section can also be used when fermions are in one state and bosons in the other.

III. RECOUPLING

A third way to treat composite reactions $s_1+s_2\rightarrow s_1'+s_2'$ is, instead of coupling s_1 with s_1' and s_2 with s_2', to recouple s_1 with s_2 and s_1' with s_2' to form s and s', respectively.

$$(s_1's_2'|M|s_1s_2)=\sum_{s',s}(s_1's_2'|s')(s'|M|s)(s|s_1s_2)\,, \qquad (3.1)$$

where $(s|s_1s_2)$ and $(s_1's_2'|s')$ are Clebsch-Gordan coefficients in spin space. More explicitly we can write

$$\langle s_1's_2'\Lambda_1'\Lambda_2'|M|s_1s_2\Lambda_1\Lambda_2\rangle=\sum_{s',s,\Lambda',\Lambda}\langle s_1's_2'\Lambda_1'\Lambda_2'|s'\Lambda'\rangle \times\langle s'\Lambda'|M|s\Lambda\rangle\langle s\Lambda|s_1s_2\Lambda_1\Lambda_2\rangle\,. \qquad (3.2)$$

Now the composite reaction has been reduced to a *sum* of basic reactions $0+s\rightarrow 0+s'$, where $s=s_1+s_2,\cdots,|s_1-s_2|$ and $s'=s_1'+s_2',\cdots,|s_1'-s_2'|$. Given the fact that each intermediate $(s'|M|s)$ can be expanded as

$$(s'|M|s)=\sum_{J,r}a_J{}^r(s',s)T_{[J]}(p^r):S_{[J]}(s',s)\,, \qquad (3.3)$$

we can express $(s_1's_2'|M|s_1s_2)$ as sums over various sets of basic form factors $a_J{}^r(s',s)$. This can be achieved with the help of the recoupling identity

$$\sum_{\Lambda'\Lambda}(s_1's_2'|s'\Lambda'\rangle[S_{[J]}(s',s)]_{\Lambda'\Lambda}\langle s\Lambda|s_1s_2) =\sum_{J_1,J_2}\xi_{JJ_1J_2}(s',s)S_{[J;J_1J_2]}(s_1's_1;s_2's_2)\,, \qquad (3.4)$$

obtained by Racah algebra, which gives for the recoupling coefficient $\xi_{JJ_1J_2}(s',s)$ a 9-j symbol,

$$\xi_{JJ_1J_2}(s',s)=\left[\frac{\beta_J(s',s)(2s'+1)(2s+1)(2J_1+1)(2J_2+1)}{\beta_{J_1}(s_1',s_1)\beta_{J_2}(s_2',s_2)}\right]^{1/2}\begin{Bmatrix}J_1 & J_2 & J\\ s_1' & s_2' & s'\\ s_1 & s_2 & s\end{Bmatrix}. \qquad (3.5)$$

Putting Eqs. (3.3) and (3.4) into Eq. (3.1) and comparing to the composite amplitude irreducible expansion Eq. (2.13), we obtain the relation between the irreducible form factors $(A_{JJ_1J_2}{}^r)$ and the recoupled form factors $(a_J{}^r(s',s))$,

$$A_{JJ_1J_2}{}^r=\sum_{s',s}\xi_{JJ_1J_2}(s',s)a_J{}^r(s',s)\,. \qquad (3.6)$$

In Appendix II, these relations are worked out for the composite reactions $\frac{1}{2}+\frac{1}{2}\rightarrow\frac{1}{2}+\frac{1}{2}$, $\frac{1}{2}+0\rightarrow\frac{1}{2}+1$, $\frac{1}{2}+1\rightarrow\frac{1}{2}+1$ and $\frac{1}{2}+0\rightarrow\frac{1}{2}+2$.

As an aside we consider the relation of the composite reaction observables

$$L_{I_1I_2;F_1F_2}(s_1's_1;s_2's_2)=\mathrm{Tr}\{(s_1's_2'|M|s_1s_2)S_{I_1}(s_1)S_{I_2}(s_2)(s_1s_2|M^\dagger|s_1's_2')S_{F_1}(s_1')S_{F_2}(s_2')\} \qquad (3.7)$$

to observables of basic reactions

$$L_{IF}(s',s)=\mathrm{Tr}\{(s'|M|s)S_I(s)(s|M^\dagger|s')S_F(s')\}\,, \qquad (3.8)$$

where $S_I(s)\equiv T_{[J_I]}:S_{[J_I]}(s)$. In Refs. 1 and 2 it is shown that the factorized form of the amplitude Eqs. (1.6) and (1.7), gives $L_{I_1I_2;F_1F_2}(s_1's_1;s_2's_2)$ as a sum of *products* of the constituent observables or "pseudo" observables. From Eq. (3.1) we also see that $L_{I_1I_2;F_1F_2}(s_1's_1;s_2's_2)$ can be expressed as a *sum* of basic observables and "pseudo" observables.

IV. PROJECTION OPERATORS

Consider for the moment the recoupling identity Eq. (3.4) with $J=0$, i.e., $S_{[0]}(s',s)=I(s)\delta_{s's}$. The left-hand side of (3.4) is then $\sum_\Lambda(s_1's_2'|s\Lambda\rangle\langle s\Lambda|s_1s_2)$ which, as is well known, is just the projection operator $\mathcal{P}_s(s_1's_1;s_2's_2)$ in the direct product spin space $(s_1's_1)$, $(s_2's_2)$, i.e., $\mathcal{P}^2=\mathcal{P}$. Setting $J=0$ (and letting $J_1\rightarrow J$) in the recoupling coefficient Eq. (3.5) yields

$$\mathcal{P}_s(s_1's_1;s_2's_2)\equiv\sum_\Lambda(s_1's_2'|s\Lambda\rangle\langle s\Lambda|s_1s_2) =(-)^{s+s_1+s_2'}(2s+1)\sum_J[\beta_J(s_1',s_1)\beta_J(s_2',s_2)]^{-1/2}\begin{Bmatrix}s_1' & s_2' & s\\ s_2 & s_1 & J\end{Bmatrix}S_{[J]}(s_1's_1):S_{[J]}(s_2's_2)\,. \qquad (4.1)$$

This formula can also be used for isotopic spin projection operators or even for spin-orbital angular momentum projection operators. Consider the latter possibility with $s_1\rightarrow l$, $s_2\rightarrow s$, $s_1'\rightarrow l'$, $s_2'\rightarrow s'$, $s\rightarrow j$. We then have, in terms of Cartesian or spherical rectangular spin matrix tensors,

$$\mathcal{P}_j(l'l;s's)=\sum_{m_j}|l's'jm_j\rangle\langle lsjm_j|$$

$$=(-)^{j+l+s'}(2j+1)\sum_J[\beta_J(l',l)\beta_J(s',s)]^{-1/2}\begin{Bmatrix} l' & l & J \\ s & s' & j \end{Bmatrix}L_{[J]}(l',l):S_{,J]}(s',s) \quad (4.2)$$

$$=(-)^{j+l+s'}(2j+1)\sum_J\begin{Bmatrix} l' & l & J \\ s & s' & j \end{Bmatrix}\sum_M(-)^M\Omega_{JM}(l',l)\Omega_{J,-M}(s',s), \quad (4.3)$$

where $L_{[J]}(l',l)$ and $\Omega_{JM}(l',l)$ are the irreducible rectangular orbital angular momentum tensor operators of rank J.

At this point we wish to stress that the operators $L_{[J]}(l',l)$ and $S_{[J]}(s',s)$ are in general *rectangular* (for $l'\neq l$ or $s'\neq s$). Such rectangular projection operators have not been used before, as the properties of rectangular matrix tensors have been thought to be undetermined. Only *square* matrix tensors have previously been considered, being constructed with the help of eigenvalues $l(l+1)$ and $s(s+1)$ (which do not exist for rectangular operators), where

$$\langle s\Lambda'|\mathbf{S}^2|s\Lambda\rangle=s(s+1)\delta_{\Lambda'\Lambda} \quad (4.4)$$

and

$$\langle lm'|\mathbf{L}^2|lm\rangle=l(l+1)\delta_{m'm}. \quad (4.5)$$

Recall the (square) projections operators for $0+\frac{1}{2}\to 0+\frac{1}{2}$ scattering,[7] $\mathcal{P}_\pm$ for $j=l\pm\frac{1}{2}$:

$$\mathcal{P}_+(l;\tfrac{1}{2})=(l+1+\mathbf{L}\cdot\boldsymbol{\sigma})/(2l+1), \quad (4.6)$$

$$\mathcal{P}_-(l;\tfrac{1}{2})=(l-\mathbf{L}\cdot\boldsymbol{\sigma})/(2l+1). \quad (4.7)$$

These too can be obtained from the general rectangular formula (4.2) using $J=0$, 1 and

$$\begin{Bmatrix} l & l & 0 \\ \frac{1}{2} & \frac{1}{2} & j \end{Bmatrix}=(-)^{j+l+\frac{1}{2}}\frac{1}{[2(2l+1)]^{1/2}}, \quad (4.8)$$

$$\begin{Bmatrix} l & l & 1 \\ \frac{1}{2} & \frac{1}{2} & j \end{Bmatrix}=(-)^{j+l+\frac{1}{2}}\frac{[j(j+1)-l(l+1)-\frac{3}{4}]}{[6(l)(l+1)(2l+1)]^{1/2}}, \quad (4.9)$$

$$S_{[0]}(\tfrac{1}{2})=I,\quad L_{[0]}(l)=I,$$
$$\mathbf{S}_{[1]}(\tfrac{1}{2})=\tfrac{1}{2}\boldsymbol{\sigma},\quad \mathbf{L}_{[1]}(l)\to -i\mathbf{q}\times\nabla_q, \quad (4.10)$$

so that

$$\mathcal{P}_j(l;\tfrac{1}{2})=\frac{2j+1}{2l+1}\left\{\tfrac{1}{2}I+\frac{[j(j+1)-l(l+1)-\frac{3}{4}]}{2l(l+1)}\mathbf{L}\cdot\boldsymbol{\sigma}\right\} \quad (4.11)$$

and using $j=l\pm\frac{1}{2}$ gives $\mathcal{P}_\pm$, Eqs. (4.6) and (4.7).

Note that the space in which the $L_{[J]}$ tensor acts is in momentum space, as

$$\langle\hat{q}'|\mathbf{L}|\hat{q}\rangle\to i\hat{q}\times\hat{q}'\frac{d}{d\cos\theta},\quad \cos\theta=\hat{q}'\cdot\hat{q}. \quad (4.12)$$

Herein lies one advantage of the rectangular formula Eq. (4.3) even for the special "square" case ($l'=l$, $s'=s$). Even though we cannot explicitly construct rectangular gradient tensor operators, we can compute their effect when taken between momentum states. Using the notation $\langle\hat{q}|lm\rangle=Y_l^m(\hat{q})$, consider the spherical rectangular angular momentum tensor operator $\Omega_{JM}(l',l)$ between momentum states:

$$\langle\hat{q}'|\Omega_{JM}(l',l)|\hat{q}\rangle=\sum_{m',m}\langle\hat{q}'|l'm'\rangle \times(\Omega_{J,M}(l',l))_{m'm}\langle lm|\hat{q}\rangle. \quad (4.13)$$

But

$$(\Omega_{JM}(l',l))_{m'm}=(-)^{l-m}\langle l'lm'-m|JM\rangle \quad (4.14)$$

implies

$$\langle\hat{q}'|\Omega_{JM}(l',l)|\hat{q}\rangle=(-)^lY_{J;l',l}^M(\hat{q}',\hat{q}), \quad (4.15)$$

where $Y_{J;l',l}^M$ is a compound spherical harmonic. Our final result is then

$$\langle\hat{q}'|\mathcal{P}_j(l'l;s's)|\hat{q}\rangle=(-)^{j+s'}(2j+1)\sum_J\begin{Bmatrix} l' & l & J \\ s & s' & j \end{Bmatrix} \times\sum_M(-)^MY_{J;l'l}(\hat{q}',\hat{q})\Omega_{J,-M}(s',s). \quad (4.16)$$

V. PARTIAL-WAVE EXPANSIONS

Consider the basic reaction $0+s\to 0+s'$ either as the physical reaction of interest or as an intermediate recoupled reaction of $s_1+s_2\to s_1'+s_2'$ with $\mathbf{s}_1+\mathbf{s}_2=\mathbf{s}$, $\mathbf{s}_1'+\mathbf{s}_2'=\mathbf{s}'$. Our goal is to expand the form factors $a_J^r(\theta)$ of

$$(s'|M|s)\to(\hat{q}'s'|M|\hat{q}s) =\sum_{J,r}a_J^r(\theta)T_{[J]}(p^r):S_{[J]}(s',s) \quad (5.1)$$

into partial waves, where $\hat{q}'$ is the final and $\hat{q}$ the initial c.m. momenta. Such a partial-wave expansion corresponds to first coupling the spins s, s' with the orbital angular momenta l, l' to form the total angular momenta[8] $\mathbf{j}=\mathbf{l}+\mathbf{s}=\mathbf{l}'+\mathbf{s}'$. Note that l, l', and j can take on a countably infinite set of values. Contrast this with the spin-space expansion Eq. (1.5), where the nonphysical angular momentum is $\mathbf{J}=\mathbf{s}-\mathbf{s}'=\mathbf{l}'-\mathbf{l}$ and

[7] See, for example, J. J. Sakurai, in *Lectures in Theoretical Physics, Brandeis Summer Institute, 1961* (W. A. Benjamin and Company, New York, 1962), Vol. 1.

[8] We remark that this angular momentum decomposition, as well as the spin-space expansions, is also valid relativistically when the spin directions Λ, Λ', etc., are measured in the instantaneous rest frames of each (massive) particle.

takes on a finite set of integral values, $|s-s'|\leq J\leq s+s'$. Recall the parity-conserving positive intrinsic parity expansion[7] for $0+\frac{1}{2}\rightarrow 0+\frac{1}{2}$;

$$(\tfrac{1}{2}|M|\tfrac{1}{2})=a_0I+a_1T(\hat{m}):S_{[1]}(\tfrac{1}{2},\tfrac{1}{2}),\quad (5.2)$$
$$[\text{with } T(m):S_{[1]}(\tfrac{1}{2},\tfrac{1}{2})=\tfrac{1}{2}\hat{m}\cdot\boldsymbol{\sigma}],$$

and

$$4\pi a_0(q,\theta)=\sum_{l=0}^{\infty}[(l+1)M_{l+}(q)+lM_{l-}(q)]P_l^0(\cos\theta),\quad (5.3)$$

$$4\pi a_1(q,\theta)=2i\sum_{l=0}^{\infty}[M_{l+}(q)-M_{l-}(q)]P_l^1(\cos\theta),\quad (5.4)$$

where $M_{l\pm}(q)=e^{i\delta_{l\pm}(q)}\sin\delta_{l\pm}(q)$, and $l\pm$ means $j=l\pm\frac{1}{2}$. We generalize these results for arbitrary values of s and s' by using the projection operator of the last section. First, we may write

$$(\hat{q}'s'|M|\hat{q}s)_{\Lambda'\Lambda}=\sum\langle\hat{q}'|l'm'\rangle\langle l's'm'\Lambda'|j'm_j'\rangle \times\langle l's'j'm_j'|M|lsjm_j\rangle\langle jm_j|lsm\Lambda\rangle\langle lm|\hat{q}\rangle.\quad (5.5)$$

We assume parity is conserved ($M\rightarrow M^{\pm}$) along with the total angular momentum. Recall that $M^{\pm}$ means that the product of all the intrinsic parities is ± 1, and therefore only $l'+l$ states which are $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ can contribute, which we denote by $\delta\left[l'+l^{\text{even}}_{\text{odd}}\right]$. Then the conservation theorems state

$$l's'j'm_j'|M^{\pm}|lsjm_j\rangle =\delta_{j'j}\delta_{m_j'm_j}\delta\begin{pmatrix}l'+l\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\end{pmatrix}M_{l',l}^{j}(s',s)\quad (5.6)$$

by the Wigner-Eckart theorem.[9] $M_{l',l}^{j}(s',s)$ is the reduced matrix element, or physically, the matrix generalization of the partial-wave amplitude $e^{i\delta_l}\sin\delta_l$. That is, the application of unitarity on the partial-wave S matrix for fixed j, $S^j=1+2iM^j$, gives $M_{l',l}^{j}(s',s)$ in terms of phase shifts and mixing parameters. This step will not be discussed here in detail, since it refers to a specific way of describing the *dynamics* of the reaction, which is not the purpose of this paper. For an example of a simple specific case, see Ref. 10. Now putting Eq. (5.6) into Eq. (5.5) yields

$$(\hat{q}'s'|M^{\pm}|\hat{q}s) =\sum_{l,l',j}\langle\hat{q}'|\mathcal{P}_j(l'l;s's)|\hat{q}\rangle M_{l'l}^{j}(s',s)\delta\begin{pmatrix}l'+l\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\end{pmatrix},\quad (5.7)$$

and using (4.16)

$$(\hat{q}'s'|M^{\pm}|\hat{q}s)=\sum_{J,l'l}\sum_{M}(-)^M Y_{J;l'l}{}^{M}(\hat{q}',\hat{q})\Omega_{J,-M}(s',s) \times\sum_j(-)^{j+s'}(2j+1)\begin{Bmatrix}l' & l & J\\ s & s' & j\end{Bmatrix} \times M_{l'l}{}^{j}(s',s)\delta\begin{pmatrix}l'+l\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\end{pmatrix}.\quad (5.8)$$

Now we compare (5.8) with (5.1), but in order to do so we must choose a basis set of momentum tensors $T_{[J]}(p^r)$. To keep things quite general we relegate Appendix I to the choice of such a basis, and here just formally extract from the $Y_{J;l'l}(\hat{q}',\hat{q})$ a general momentum tensor basis. We shall find it convenient to choose the coefficients of such a basis by first letting $\hat{q}'\rightarrow\hat{e}_3$, for then

$$Y_{J;l'l}{}^{M}(\hat{q}',\hat{q}) =\sum_{m',m}\langle l'lm'm|JM\rangle Y_{l'}{}^{m'}(\hat{q}')Y_l{}^{m}(\hat{q})\rightarrow\frac{(-i)^M}{4\pi} \times\left[(2l'+1)(2l+1)\frac{(l-M)!}{(l+M)!}\right]^{1/2} \times\langle l'l0M|JM\rangle P_l{}^{M}(\hat{q}'\cdot\hat{q})\quad (5.9)$$

[where $P_l{}^{M}(\cos\theta)$ is the associated Legendre function] and then setting $J=M$;

$$Y_{J;l'l}{}^{J}(\hat{q}',\hat{q}) =(-i)^J\frac{(-)^{l'}}{4\pi}\left[\frac{(2l'+1)(2l+1)}{2^J\beta_J(l',l)}\right]^{1/2}P_l{}^{J}(\hat{q}'\cdot\hat{q}).\quad (5.10)$$

Hence, we may write

$$Y_{J,l'l}{}^{M}(\hat{q}',\hat{q})=(-i)^J\frac{(-)^{l'}}{4\pi}\left[\frac{(2l'+1)(2l+1)}{2^J\beta_J(l',l)}\right]^{1/2} \times\sum_r f_{J;l'l}{}^{r}(\theta)T_J{}^{M}(p^r),\quad (5.11)$$

where[1]

$$T_J{}^{M}(p^r)=\sum_{\mu_1\cdots\mu_J}\langle 1\cdots 1\mu_1\cdots\mu_J|JM\rangle p_{r_1}{}^{\mu_1}\cdots p_{r_J}{}^{\mu_J}\quad (5.12)$$

is the spherical tensor counterpart to $T_{[J]}(p^r)$, and $f_{J;l'l}{}^{r}(\theta)$ is the "angular function" which we shall explicitly find in Appendix IV.

Finally, putting (5.11) into (5.8) and comparing with (5.1) yields the partial-wave expansions

$$4\pi a_J{}^{r}(\theta)=(-i)^J\sum_{l'l}\left[\frac{(2l'+1)(2l+1)}{2^J\beta_J(l',l)\beta_J(s',s)}\right]^{1/2}f_{J;l'l}{}^{r}(\theta) \times\sum_j(-)^{j+l'+s'}(2j+1)\begin{Bmatrix}l' & l & J\\ s & s' & j\end{Bmatrix} \times M_{l',l}{}^{j}(s',s)\delta\begin{pmatrix}l'+l\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\end{pmatrix}.\quad (5.13)$$

[9] See, for example, A. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

[10] H. Stapp, T. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

In Appendix III we shall work out the partial-wave expansions for such basic reactions as $0+\frac{1}{2}\to 0+\frac{1}{2}$, $0+\frac{1}{2}\to 0+\frac{3}{2}$, $0+\frac{1}{2}\to 0+\frac{5}{2}$, $0+\frac{3}{2}\to 0+\frac{3}{2}$, $0+s\to 0+0$, $0+0\to 0+s'$, $0+1\to 0+1$, and $0+1\to 0+2$.

For composite reactions $s_1+s_2\to s_1'+s_2'$ we need only consider $a_J{}^r(\theta)$ of Eq. (5.13) as $a_J{}^r(\theta;s',s)$ and then use Appendix II which expresses the composite form factors $C_{J_1J_2}$ in terms of the basic form factors $a_J(s',s)$ in order to obtain the partial-wave expansions of $C_{J_1J_2}$.

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APPENDIX I: CHOICE OF THE TENSOR BASIS

There are many possible choices for basis sets of momenta. In Ref. 1 we displayed the merits of using orthogonal unit vectors defined from q' and q. For partial-wave expansions this is almost a necessity because it transfers all the angular dependence in observables (bilinear combinations of the form factors) into the form factors themselves. There are even then at least three such choices for orthogonal momentum bases.

$$(1)\quad \hat{l}=\frac{\mathbf{q}-\mathbf{q}'}{|\mathbf{q}-\mathbf{q}'|},\quad \hat{m}=\frac{\mathbf{q}\times\mathbf{q}'}{|\mathbf{q}\times\mathbf{q}'|},\quad \hat{n}\equiv\frac{\mathbf{q}+\mathbf{q}'}{|\mathbf{q}+\mathbf{q}'|}.$$

These are only orthogonal for equal mass, elastic scattering.

$$(2)\quad \hat{l}\equiv\frac{\mathbf{q}-\mathbf{q}'}{|\mathbf{q}-\mathbf{q}'|},\quad \hat{m}\equiv\frac{\mathbf{q}\times\mathbf{q}'}{|\mathbf{q}\times\mathbf{q}'|},\quad \hat{n}\equiv\hat{l}\times\hat{m}.$$

Now these vectors are always orthogonal, but their dependence on θ ($\cos\theta=\hat{q}'\cdot\hat{q}$) becomes complicated for all but equal-mass elastic scattering. This choice is used in Ref. 1.

$$(3)\quad \hat{l}\equiv\frac{\hat{q}-\hat{q}'}{|\hat{q}-\hat{q}'|}=\frac{1}{2\sin(\theta/2)}(\hat{q}-\hat{q}'),$$

$$\hat{m}\equiv\frac{\hat{q}\times\hat{q}'}{|\hat{q}\times\hat{q}'|}=\frac{1}{\sin\theta}(\hat{q}\times\hat{q}'),$$

$$\hat{n}\equiv\hat{l}\times\hat{m}=\frac{\hat{q}+\hat{q}'}{|\hat{q}+\hat{q}'|}=\frac{1}{2\cos(\theta/2)}(\hat{q}+\hat{q}').$$

That is, we use all unit vectors $\hat{q}$, $\hat{q}'$. This is a new choice. Now the angular dependence is the *same* for elastic and inelastic scattering. Instead, all of the effects of inelastic scattering are now embedded in the complex part of the energy dependent phase shift

$$2iM_{V\,l}{}^{j}(s'q',sq)=e^{2i\,\mathrm{Re}\,\delta_l(q',q)}e^{-2\,\mathrm{Im}\,\delta_l(q',q)}-1.$$

Now we write

$$(s'|M|s)=\sum_{J=|s'-s|}^{s'+s}(s'|M_J|s),$$

and separating the positive and negative intrinsic parity amplitudes M^+ and M^- we use the property of our orthogonal basis $[(\hat{l},\hat{m},\hat{n})\to(-\hat{l},\hat{m},-\hat{n})]$ under space inversion to write

$$(s'|M_{J=0}{}^+|s)=a_0(s',s)I(s)\delta_{s's},$$

$$(s'|M_{J=1}{}^+|s)=a_1{}^1(s',s)T(\hat{m}):S_{[1]}(s',s),$$

$$(s'|M_{J=1}{}^-|s)=[a_1{}^2(s',s)T(\hat{l})+a_1{}^3(s',s)T(\hat{n})]:S_{[1]}(s',s),$$

$$(s'|M_{J=2}{}^+|s)=[a_2{}^1(s',s)T(\hat{l}\hat{l})+a_2{}^2(s',s)T(\hat{n}\hat{n})+a_2{}^3(s',s)T(\hat{l}\hat{n})]:S_{[2]}(s',s),$$

$$(s'|M_{J=2}{}^-|s)=[a_2{}^4(s',s)T(\hat{l}\hat{m})+a_2{}^5(s',s)T(\hat{n}\hat{m})]:S_{[2]}(s',s),$$

$$(s'|M_{J=3}{}^+|s)=[a_3{}^1(s',s)T(\hat{l}\hat{l}\hat{m})+a_3{}^2(s',s)T(\hat{n}\hat{n}\hat{m})+a_3{}^3(s',s)T(\hat{l}\hat{n}\hat{m})]:S_{[3]}(s',s),\ \text{etc.}$$

So our choice for the basis set $T_{[J]}(p^r)$ will be 1, $T(\hat{m})$, $T(\hat{l}\hat{l})$, $T(\hat{l}\hat{n})$, $T(\hat{l}\hat{l}\hat{m})$, $T(\hat{n}\hat{n}\hat{m})$, $T(\hat{l}\hat{n}\hat{m})$, etc., for M^+ amplitudes and $T(\hat{l})$, $T(\hat{n})$, $T(\hat{l}\hat{m})$, $T(\hat{l}\hat{n})$, etc., for M^- amplitudes.

APPENDIX II: COMPOSITE REACTION FORM FACTORS

Using the tensor basis defined in Appendix I, we list the relations between the factorized ($C_{J_1J_2}$), irreducible ($A_{JJ_1J_2}$) and recoupled ($a_J(s',s)$) form factors for M^+ (parity conserved with the product of the intrinsic parities equal to $+1$) for the composite reactions $\frac{1}{2}+\frac{1}{2}\to\frac{1}{2}+\frac{1}{2}$, $\frac{1}{2}+0\to\frac{1}{2}+1$, $\frac{1}{2}+1\to\frac{1}{2}+1$, $\frac{1}{2}+0\to\frac{1}{2}+2$, $0+1\to 0+1$, and $0+1\to 0+2$. (Note that we abbreviate $C_{J_1J_2}$ by C_i and $A_{JJ_1J_2}$ by A_J for clarity.) Similarly, analogous relations can be worked out for M^-.

$$\tfrac{1}{2}+\tfrac{1}{2}\to\tfrac{1}{2}+\tfrac{1}{2}$$

We factorize into $\frac{1}{2}+0\to\frac{1}{2}+0$ and $0+\frac{1}{2}\to 0+\frac{1}{2}$ with $S_{[1]}(s_1'=\frac{1}{2},s_1=\frac{1}{2})=\frac{1}{2}\boldsymbol{\sigma}\equiv S$, $S_{[1]}(s_2'=\frac{1}{2},s_2=\frac{1}{2})\equiv S'$:

$$\begin{aligned}(\tfrac{1}{2}\,\tfrac{1}{2}|M^+|\tfrac{1}{2}\,\tfrac{1}{2})&=C_1+C_2\hat{m}\cdot\mathbf{S}+C_3\hat{m}\cdot\mathbf{S}'+C_4\hat{m}\cdot\mathbf{S}\hat{m}\cdot\mathbf{S}'+C_5\hat{l}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'+C_6\hat{n}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_7\hat{l}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_8\hat{n}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'\\&=A_0{}^1S_{[0,00]}+A_0{}^2S_{[0,11]}+A_1{}^1T(\hat{m}):S_{[1,10]}+A_1{}^2T(\hat{m}):S_{[1,01]}+A_1{}^3T(\hat{m}):S_{[1,11]}\\&\qquad+[A_2{}^1T(\hat{l}\hat{l})+A_2{}^2T(\hat{n}\hat{n})+A_2{}^3T(\hat{l}\hat{n})]:S_{[2,11]}\\&=(\tfrac{1}{2}\,\tfrac{1}{2}|0)\{a_0(0,0)\}(0|\tfrac{1}{2}\,\tfrac{1}{2})+(\tfrac{1}{2}\,\tfrac{1}{2}|0)\{a_1(0,1)T(\hat{m}):S_{[1]}(0,1)\}(1|\tfrac{1}{2}\,\tfrac{1}{2})\\&\quad+(\tfrac{1}{2}\,\tfrac{1}{2}|1)\{a_1(1,0)T(\hat{m}):S_{[1]}(1,0)\}(0|\tfrac{1}{2}\,\tfrac{1}{2})+(\tfrac{1}{2}\,\tfrac{1}{2}|1)\{a_0(1,1)+a_1(1,1)T(\hat{m}):S_{[1]}(1,1)\\&\qquad+[a_2{}^1(1,1)T(\hat{l}\hat{l})+a_2{}^2(1,1)T(\hat{n}\hat{n})+a_2{}^3(1,1)T(\hat{l}\hat{n})]:S_{[2]}(1,1)\}(1|\tfrac{1}{2}\,\tfrac{1}{2}).\end{aligned}$$

Now using $\mathbf{S}\cdot\mathbf{S}'=\hat{l}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'+\hat{m}\cdot\mathbf{S}\hat{m}\cdot\mathbf{S}'+\hat{n}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'$ and $\hat{m}\cdot\mathbf{S}\times\mathbf{S}'=\hat{n}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'-\hat{l}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'$ we find

$$\begin{aligned}C_1&=A_0{}^1, & C_5&=-(1/\sqrt{3})A_0{}^2+\tfrac{2}{3}A_2{}^1-\tfrac{1}{3}A_2{}^2,\\C_2&=A_1{}^1, & C_6&=-(1/\sqrt{3})A_0{}^2-\tfrac{1}{3}A_2{}^1+\tfrac{2}{3}A_2{}^2,\\C_3&=A_1{}^2, & C_7&=-(i/\sqrt{2})A_1{}^3+\tfrac{1}{2}A_2{}^3,\\C_4&=-(1/\sqrt{3})A_0{}^2-\tfrac{1}{3}A_2{}^1-\tfrac{1}{3}A_2{}^2, & C_8&=(i/\sqrt{2})A_1{}^3+\tfrac{1}{2}A_2{}^3.\end{aligned}$$

Next, Eqs. (3.5) and (3.6) yield

$$\begin{aligned}A_0{}^1&=\tfrac{1}{4}a_0(0,0)+\tfrac{3}{4}a_0(1,1), & A_1{}^2&=\tfrac{1}{2}a_1(0,1)-\tfrac{1}{2}a_1(1,0)+a_1(1,1),\\A_0{}^2&=\sqrt{3}a_0(0,0)-\sqrt{3}a_0(1,1), & A_1{}^3&=\sqrt{2}a_1(0,1)+\sqrt{2}a_1(1,0),\\A_1{}^1&=-\tfrac{1}{2}a_1(0,1)+\tfrac{1}{2}a_1(1,0)+a_1(1,1), & A_2{}^r&=2a_2{}^r(1,1)\quad r=1,2,3,\end{aligned}$$

so that

$$\begin{aligned}C_1&=\tfrac{1}{4}a_0(0,0)+\tfrac{3}{4}a_0(1,1), & C_5&=-a_0(0,0)+a_0(1,1)+\tfrac{4}{3}a_2{}^1(1,1)-\tfrac{2}{3}a_2{}^2(1,1),\\C_2&=-\tfrac{1}{2}a_1(0,1)+\tfrac{1}{2}a_1(1,0)+a_1(1,1), & C_6&=-a_0(0,0)+a_0(1,1)-\tfrac{2}{3}a_2{}^1(1,1)+\tfrac{4}{3}a_2{}^2(1,1),\\C_3&=\tfrac{1}{2}a_1(0,1)-\tfrac{1}{2}a_1(1,0)+a_1(1,1), & C_7&=-ia_1(0,1)-ia_1(1,0)+a_2{}^3(1,1),\\C_4&=-a_0(0,0)+a_0(1,1)-\tfrac{2}{3}a_2{}^1(1,1)-\tfrac{2}{3}a_2{}^2(1,1), & C_8&=+ia_1(0,1)+ia_1(1,0)+a_2{}^3(1,1).\end{aligned}$$

Note that for elastic scattering, time reversal invariance implies $\hat{l}\to+\hat{l}$, $\hat{m}\to-\hat{m}$, $\hat{n}\to-\hat{n}$, $S\to-S$ so that $C_7=C_8=0$, $A_1{}^3=A_2{}^3=0$ and $a_1(0,1)=-a_1(1,0)$, $a_2{}^3(1,1)=0$.

$$\tfrac{1}{2}+0\to\tfrac{1}{2}+1$$

We factorize into $\frac{1}{2}+0\to\frac{1}{2}+0$ and $0+0\to 0+1$ with $S_{[1]}(\frac{1}{2},\frac{1}{2})\equiv S$, $S_{[1]}(1,0)\equiv S'$:

$$\begin{aligned}(\tfrac{1}{2}1|M^+|\tfrac{1}{2}0)&=C_1\hat{m}\cdot\mathbf{S}'+C_2\hat{m}\cdot\mathbf{S}\hat{m}\cdot\mathbf{S}'+C_3\hat{l}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'+C_4\hat{n}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_5\hat{l}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_6\hat{n}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'\\&=A_0S_{[0,11]}+A_1{}^1T(\hat{m}):S_{[1,01]}+A_1{}^2T(\hat{m}):S_{[1,11]}+[A_2{}^1T(\hat{l}\hat{l})+A_2{}^2T(\hat{n}\hat{n})+A_2{}^3T(\hat{l}\hat{n})]:S_{[2,11]}\\&=(\tfrac{1}{2}1|\tfrac{1}{2})\{a_0(\tfrac{1}{2},\tfrac{1}{2})+a_1(\tfrac{1}{2},\tfrac{1}{2})T(\hat{m}):S_{[1]}(\tfrac{1}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}0)+(\tfrac{1}{2}1|\tfrac{3}{2})\{a_1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{m}):S_{[1]}(\tfrac{3}{2},\tfrac{1}{2})\\&\qquad+[a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{l})+a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}T(\hat{n}\hat{n})+a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{3}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}0),\end{aligned}$$

with

$$\begin{aligned}C_1&=A_1{}^1, & C_4&=-(1/\sqrt{3})A_0{}^1-\tfrac{1}{3}A_2{}^1+\tfrac{2}{3}A_2{}^2,\\C_2&=-(1/\sqrt{3})A_0{}^1-\tfrac{1}{3}A_2{}^1-\tfrac{1}{3}A_2{}^2, & C_5&=-(i/\sqrt{2})A_1{}^2+\tfrac{1}{2}A_2{}^3,\\C_3&=-(1/\sqrt{3})A_0{}^1+\tfrac{2}{3}A_2{}^1-\tfrac{1}{3}A_2{}^2, & C_6&=(i/\sqrt{2})A_1{}^2+\tfrac{1}{2}A_2{}^3,\end{aligned}$$

and

$$\begin{aligned}A_0&=-2a_0(\tfrac{1}{2},\tfrac{1}{2}), & A_1{}^2&=(\sqrt{2}/\sqrt{3})a_1(\tfrac{1}{2},\tfrac{1}{2})-(\sqrt{2}/\sqrt{3})a_1(\tfrac{3}{2},\tfrac{1}{2}),\\A_1{}^1&=(1/2\sqrt{3})a_1(\tfrac{1}{2},\tfrac{1}{2})+(2/\sqrt{3})a_1(\tfrac{3}{2},\tfrac{1}{2}), & A_2{}^r&=(1/\sqrt{3})a_2{}^r(\tfrac{3}{2},\tfrac{1}{2}),\quad r=1,2,3,\end{aligned}$$

so that

$$\begin{aligned}\sqrt{3}C_1&=\tfrac{1}{2}a_1(\tfrac{1}{2},\tfrac{1}{2})+2a_1(\tfrac{3}{2},\tfrac{1}{2}), & \sqrt{3}C_4&=2a_0(\tfrac{1}{2},\tfrac{1}{2})-\tfrac{1}{3}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{2}{3}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}),\\\sqrt{3}C_2&=2a_0(\tfrac{1}{2},\tfrac{1}{2})-\tfrac{1}{3}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{1}{3}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}), & \sqrt{3}C_5&=-ia_1(\tfrac{1}{2},\tfrac{1}{2})+2ia_1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{1}{2}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2}),\\\sqrt{3}C_3&=2a_0(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{2}{3}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{1}{3}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}), & \sqrt{3}C_6&=ia_1(\tfrac{1}{2},\tfrac{1}{2})-2ia_1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{1}{2}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2}).\end{aligned}$$

$$\tfrac{1}{2}+1 \to \tfrac{1}{2}+1$$

We factorize into $\tfrac{1}{2}+0 \to \tfrac{1}{2}+0$ and $0+1 \to 0+1$ with $S_{[1]}(\tfrac{1}{2},\tfrac{1}{2}) \equiv S$, $S_{[1]}(1,1) \equiv S'$

$$\begin{aligned}
(\tfrac{1}{2}1|M^+|\tfrac{1}{2}) = {} & C_1 + C_2\hat{m}\cdot\mathbf{S} + C_3\hat{m}\cdot\mathbf{S}' + C_4\hat{m}\cdot\mathbf{S}\hat{m}\cdot\mathbf{S}' + C_5T(\hat{l}\hat{l}):S_{[2]}' + C_6T(\hat{n}\hat{n}):S_{[2]}' + C_7T(\hat{l}\hat{n})\cdot S_{[2]}' + C_8\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{l}):S_{[2]}' \\
& + C_9\hat{m}\cdot\mathbf{S}T(\hat{n}\hat{n}):S_{[2]}' + C_{10}\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{n}):S_{[2]}' + C_{11}\hat{l}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}' + C_{12}\hat{n}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}' + C_{13}\hat{l}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}' + C_{14}\hat{n}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}' \\
& + C_{15}\hat{l}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}' + C_{16}\hat{n}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}' + C_{17}\hat{l}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}' + C_{18}\hat{n}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}' \\
= {} & A_0{}^1S_{[0,00]} + A_0{}^2S_{[0,11]} + A_1{}^1T(\hat{m}):S_{[1,10]} + A_1{}^2T(\hat{m}):S_{[1,01]} + A_1{}^3T(\hat{m}):S_{[1,11]} + A_1{}^4T(\hat{m}):S_{[1,12]} \\
& + [A_2{}^1T(\hat{l}\hat{l}) + A_2{}^2T(\hat{n}\hat{n}) + A_2{}^3T(\hat{l}\hat{n})]:S_{[2,11]} + [A_2{}^4T(\hat{l}\hat{l}) + A_2{}^5T(\hat{n}\hat{n}) + A_2{}^6T(\hat{l}\hat{n})]:S_{[2,02]} \\
& + [A_2{}^7T(\hat{l}\hat{l}) + A_2{}^8T(\hat{n}\hat{n}) + A_2{}^9T(\hat{l}\hat{n})]:S_{[2,12]} + [A_3{}^1T(\hat{l}\hat{l}\hat{m}) + A_3{}^2T(\hat{n}\hat{n}\hat{m}) + A_3{}^3T(\hat{l}\hat{n}\hat{m})]:S_{[3,12]} \\
= {} & (\tfrac{1}{2}1|\tfrac{1}{2})\{a_0(\tfrac{1}{2},\tfrac{1}{2}) + a_1(\tfrac{1}{2},\tfrac{1}{2})T(\hat{m}):S_{[1]}(\tfrac{1}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}1) \\
& + (\tfrac{1}{2}1|\tfrac{1}{2})\{a_1(\tfrac{1}{2},\tfrac{3}{2})T(\hat{m}):S_{[1]}(\tfrac{1}{2},\tfrac{3}{2}) + [a_2{}^1(\tfrac{1}{2},\tfrac{3}{2})T(\hat{l}\hat{l}) + a_2{}^2(\tfrac{1}{2},\tfrac{3}{2})T(\hat{n}\hat{n}) + a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{1}{2},\tfrac{3}{2})\}(\tfrac{3}{2}|\tfrac{1}{2}1) \\
& + (\tfrac{1}{2}1|\tfrac{3}{2})\{a_1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{m}):S_{[1]}(\tfrac{3}{2},\tfrac{1}{2}) + [a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{l}) + a_2{}^2(\tfrac{3}{2},\tfrac{1}{2})T(\hat{n}\hat{n}) + a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{3}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}1) \\
& + (\tfrac{1}{2}1|\tfrac{3}{2})\{a_0(\tfrac{3}{2},\tfrac{3}{2}) + a_1(\tfrac{3}{2},\tfrac{3}{2})T(\hat{m}):S_{[1]}(\tfrac{3}{2},\tfrac{3}{2}) + [a_2{}^1(\tfrac{3}{2},\tfrac{3}{2})T(\hat{l}\hat{l}) + a_2{}^2(\tfrac{3}{2},\tfrac{3}{2})T(\hat{n}\hat{n}) + a_2{}^3(\tfrac{3}{2},\tfrac{3}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{3}{2},\tfrac{3}{2}) \\
& + [a_3{}^1(\tfrac{3}{2},\tfrac{3}{2})T(\hat{l}\hat{l}\hat{m}) + a_3{}^2(\tfrac{3}{2},\tfrac{3}{2})T(\hat{n}\hat{n}\hat{m}) + a_3{}^3(\tfrac{3}{2},\tfrac{3}{2})T(\hat{l}\hat{n}\hat{n})]:S_{[3]}(\tfrac{3}{2},\tfrac{3}{2})\}(\tfrac{3}{2}|\tfrac{1}{2}1),
\end{aligned}$$

with

$$\begin{aligned}
&C_1 = A_0{}^1, && C_8 = (\sqrt{3}/\sqrt{5})A_1{}^4 - (i/2)(\sqrt{2}/\sqrt{3})A_2{}^9 + (7/15)A_3{}^1 + (2/15)A_3{}^2, \\
&C_2 = A_1{}^1, && C_9 = (\sqrt{3}/\sqrt{5})A_1{}^4 + (i/2)(\sqrt{2}/\sqrt{3})A_2{}^9 + (2/15)A_3{}^1 + (7/15)A_3{}^2, \\
&C_3 = A_1{}^2, && C_{10} = i(\sqrt{2}/\sqrt{3})A_2{}^7 - i(\sqrt{2}/\sqrt{3})A_2{}^8 + \tfrac{1}{3}A_3{}^3, \\
&C_4 = (-1/\sqrt{3})A_0{}^2 - \tfrac{1}{3}A_2{}^1 - \tfrac{1}{3}A_2{}^2, && C_{11} = -(1/\sqrt{3})A_0{}^2 + \tfrac{2}{3}A_2{}^1 - \tfrac{1}{3}A_2{}^2, \\
&C_5 = A_2{}^4, && C_{12} = -(1/\sqrt{3})A_0{}^2 - \tfrac{1}{3}A_2{}^1 + \tfrac{2}{3}A_2{}^2, \\
&C_6 = A_2{}^5, && C_{13} = -(i/\sqrt{2})A_1{}^3 + \tfrac{1}{2}A_2{}^3, \\
&C_7 = A_2{}^6, && C_{14} = (i/\sqrt{2})A_1{}^3 + \tfrac{1}{2}A_2{}^3,
\end{aligned}$$

$$\begin{aligned}
&C_{15} = -(\sqrt{3}/\sqrt{5})A_1{}^4 + (i/2)(\sqrt{2}/\sqrt{3})A_2{}^9 + (8/15)A_3{}^1 - (2/15)A_3{}^2, \\
&C_{16} = -(\sqrt{3}/\sqrt{5})A_1{}^4 - (i/2)(\sqrt{2}/\sqrt{3})A_2{}^9 - (2/15)A_3{}^1 + (8/15)A_3{}^2, \\
&C_{17} = i(\sqrt{2}/\sqrt{3})A_2{}^8 + \tfrac{1}{3}A_3{}^3, \\
&C_{18} = -i(\sqrt{2}/\sqrt{3})A_2{}^7 + \tfrac{1}{3}A_3{}^3,
\end{aligned}$$

and

$$\begin{aligned}
&A_0{}^1 = \tfrac{1}{3}a_0(\tfrac{1}{2},\tfrac{1}{2}) + \tfrac{2}{3}a_0(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_0{}^2 = \tfrac{2}{3}\sqrt{3}a_0(\tfrac{1}{2},\tfrac{1}{2}) - \tfrac{2}{3}\sqrt{3}a_0(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_1{}^1 = -\tfrac{1}{9}a_1(\tfrac{1}{2},\tfrac{1}{2}) - (8/9)a_1(\tfrac{1}{2},\tfrac{3}{2}) + (8/9)a_1(\tfrac{3}{2},\tfrac{1}{2}) + (10/9)a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_1{}^2 = \tfrac{1}{6}a_1(\tfrac{1}{2},\tfrac{1}{2}) + \tfrac{1}{3}a_1(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{3}a_1(\tfrac{3}{2},\tfrac{1}{2}) + \tfrac{5}{6}a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_1{}^3 = 0a_1(\tfrac{1}{2},\tfrac{1}{2}) + \sqrt{2}a_1(\tfrac{1}{2},\tfrac{3}{2}) + \sqrt{2}a_1(\tfrac{3}{2},\tfrac{1}{2}) + 0a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_1{}^4 = (2/9)(\sqrt{15})a_1(\tfrac{1}{2},\tfrac{1}{2}) - (2/9)(\sqrt{15})a_1(\tfrac{1}{2},\tfrac{3}{2}) + (2/9)(\sqrt{15})a_1(\tfrac{3}{2},\tfrac{1}{2}) - (2/9)(\sqrt{15})a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&A_2{}^r = -\tfrac{1}{2}a_2{}^r(\tfrac{1}{2},\tfrac{3}{2}) + \tfrac{1}{2}a_2{}^r(\tfrac{3}{2},\tfrac{1}{2}) + 2a_2{}^r(\tfrac{3}{2},\tfrac{3}{2}), \quad r=1,2,3, \\
&A_2{}^{\bar{r}} = \tfrac{1}{2}a_2{}^r(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{2}a_2{}^r(\tfrac{3}{2},\tfrac{1}{2}) + a_2{}^r(\tfrac{3}{2},\tfrac{3}{2}), \quad \bar{r}=4,5,6, \quad r=1,2,3, \\
&A_2{}^{\bar{r}} = (\sqrt{3}/\sqrt{2})a_2{}^r(\tfrac{1}{2},\tfrac{3}{2}) + (\sqrt{3}/\sqrt{2})a_2{}^r(\tfrac{3}{2},\tfrac{1}{2}) + 0a_2{}^r(\tfrac{3}{2},\tfrac{3}{2}), \quad \bar{r}=7,8,9, \quad r=1,2,3, \\
&A_3{}^r = 3a_3{}^r(\tfrac{3}{2},\tfrac{3}{2}), \quad r=1,2,3,
\end{aligned}$$

so that

$$\begin{aligned}
&C_1 = \tfrac{1}{3}a_0(\tfrac{1}{2},\tfrac{1}{2}) + \tfrac{2}{3}a_0(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_2 = -\tfrac{1}{9}a_1(\tfrac{1}{2},\tfrac{1}{2}) - (8/9)a_1(\tfrac{1}{2},\tfrac{3}{2}) + (8/9)a_1(\tfrac{3}{2},\tfrac{1}{2}) + (10/9)a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_3 = \tfrac{1}{6}a_1(\tfrac{1}{2},\tfrac{1}{2}) + \tfrac{1}{3}a_1(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{3}a_1(\tfrac{3}{2},\tfrac{1}{2}) + \tfrac{5}{6}a_1(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_4 = -\tfrac{2}{3}a_0(\tfrac{1}{2},\tfrac{1}{2}) + \tfrac{2}{3}a_0(\tfrac{3}{2},\tfrac{3}{2}) + \tfrac{1}{6}a_2{}^1(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{6}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2}) - \tfrac{2}{3}a_2{}^1(\tfrac{3}{2},\tfrac{3}{2}) + \tfrac{1}{6}a_2{}^2(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{6}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}) - \tfrac{2}{3}a_2{}^2(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_5 = \tfrac{1}{2}a_2{}^1(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{2}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2}) + a_2{}^1(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_6 = \tfrac{1}{2}a_2{}^2(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{2}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}) + a_2{}^2(\tfrac{3}{2},\tfrac{3}{2}), \\
&C_7 = \tfrac{1}{2}a_2{}^3(\tfrac{1}{2},\tfrac{3}{2}) - \tfrac{1}{2}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2}) + a_2{}^3(\tfrac{3}{2},\tfrac{3}{2}),
\end{aligned}$$

$$C_8=\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{1}{2})-\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{3}{2})-(i/2)a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})-(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+(7/5)a_3{}^1(\tfrac{3}{2},\tfrac{3}{2})+\tfrac{2}{5}a_3{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_9=\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{1}{2})-\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{3}{2})+(i/2)a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})+(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{2}{5}a_3{}^1(\tfrac{3}{2},\tfrac{3}{2})+(7/5)a_3{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{10}=ia_2{}^1(\tfrac{1}{2},\tfrac{3}{2})+ia_2{}^1(\tfrac{3}{2},\tfrac{1}{2})-ia_2{}^2(\tfrac{1}{2},\tfrac{3}{2})-ia_2{}^2(\tfrac{3}{2},\tfrac{1}{2})+a_3{}^3(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{11}=-\tfrac{2}{3}a_0(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{2}{3}a_0(\tfrac{3}{2},\tfrac{3}{2})-\tfrac{1}{3}a_2{}^1(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{1}{3}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{4}{3}a_2{}^1(\tfrac{3}{2},\tfrac{3}{2})+\tfrac{1}{6}a_2{}^2(\tfrac{1}{2},\tfrac{3}{2})-\tfrac{1}{6}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{2}{3}a_2{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{12}=-\tfrac{2}{3}a_0(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{2}{3}a_0(\tfrac{3}{2},\tfrac{3}{2})+\tfrac{1}{6}a_2{}^1(\tfrac{1}{2},\tfrac{3}{2})-\tfrac{1}{6}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{2}{3}a_2{}^1(\tfrac{3}{2},\tfrac{3}{2})-\tfrac{1}{3}a_2{}^2(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{1}{3}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{4}{3}a_2{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{13}=-ia_1(\tfrac{1}{2},\tfrac{3}{2})-ia_1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{1}{4}a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{1}{4}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+a_2{}^3(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{14}=ia_1(\tfrac{1}{2},\tfrac{3}{2})+ia_1(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{1}{4}a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})+\tfrac{1}{4}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+a_2{}^3(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{15}=-\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{3}{2})-\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{3}{2})+(i/2)a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})+(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+(8/5)a_3{}^1(\tfrac{3}{2},\tfrac{3}{2})-\tfrac{2}{5}a_3{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{16}=-\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{1}{2})+\tfrac{2}{3}a_1(\tfrac{1}{2},\tfrac{3}{2})-\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{1}{2})+\tfrac{2}{3}a_1(\tfrac{3}{2},\tfrac{3}{2})-(i/2)a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})-(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})-\tfrac{2}{5}a_3{}^1(\tfrac{3}{2},\tfrac{3}{2})+(8/5)a_3{}^2(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{17}=ia_2{}^2(\tfrac{1}{2},\tfrac{3}{2})+ia_2{}^2(\tfrac{3}{2},\tfrac{1}{2})+a_3{}^3(\tfrac{3}{2},\tfrac{3}{2}),$$

$$C_{18}=-ia_2{}^1(\tfrac{1}{2},\tfrac{3}{2})-ia_2{}^1(\tfrac{3}{2},\tfrac{1}{2})+a_3{}^3(\tfrac{3}{2},\tfrac{3}{2}).$$

For elastic scattering, time-reversal invariance implies

$$C_7=C_{10}=C_{13}=C_{14}=C_{17}=C_{18}=0,\quad A_1{}^3=A_2{}^3=A_2{}^6=A_2{}^7=A_2{}^8=A_3{}^3=0,$$

and

$$a_1(\tfrac{1}{2},\tfrac{3}{2})=-a_1(\tfrac{3}{2},\tfrac{1}{2}),\quad a_2{}^3(\tfrac{1}{2},\tfrac{3}{2})=a_2{}^3(\tfrac{3}{2},\tfrac{1}{2}),\quad a_2{}^3(\tfrac{3}{2},\tfrac{3}{2})=a_3{}^3(\tfrac{3}{2},\tfrac{3}{2})=0,\quad a_2{}^1(\tfrac{1}{2},\tfrac{3}{2})=-a_2{}^1(\tfrac{3}{2},\tfrac{1}{2}),\quad a_2{}^2(\tfrac{1}{2},\tfrac{3}{2})=-a_2{}^2(\tfrac{3}{2},\tfrac{1}{2}).$$

$\tfrac{1}{2}+0\rightarrow\tfrac{1}{2}+2$

We factorize into $\tfrac{1}{2}+0\rightarrow\tfrac{1}{2}+0$ and $0+0\rightarrow 0+2$ with $S_{[1]}(\tfrac{1}{2},\tfrac{1}{2})\equiv S$, $S_{[2]}(2,0)\equiv S_{[2]}'$:

$$\begin{aligned}(\tfrac{1}{2}2|M^+|\tfrac{1}{2}0)&=C_1T(\hat{l}\hat{l}):S_{[2]}'+C_2T(\hat{n}\hat{n}):S_{[2]}'+C_3T(\hat{l}\hat{n}):S_{[2]}'+C_4\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{l}):S_{[2]}'+C_5\hat{m}\cdot\mathbf{S}T(\hat{n}\hat{n}):S_{[2]}'\\&\quad+C_6\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{n}):S_{[2]}'+C_7\hat{l}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}'+C_8\hat{n}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}'+C_9\hat{l}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}'+C_{10}\hat{n}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}'\\&=A_1T(\hat{m}):S_{[1,12]}+[A_2{}^1T(\hat{l}\hat{l})+A_2{}^2T(\hat{n}\hat{n})+A_2{}^3T(\hat{l}\hat{n})]:S_{[2,02]}\\&\quad+[A_2{}^4T(\hat{l}\hat{l})+A_2{}^5T(\hat{n}\hat{n})+A_2{}^6T(\hat{l}\hat{n})]:S_{[2,12]}+[A_3{}^1T(\hat{l}\hat{l}\hat{m})+A_3{}^2T(\hat{n}\hat{n}\hat{m})+A_3{}^3T(\hat{l}\hat{n}\hat{m})]:S_{[3,12]}\\&=(\tfrac{1}{2}2|\tfrac{3}{2})\{a_1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{m}):S_{[1]}(\tfrac{3}{2},\tfrac{1}{2})+[a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{l})+a_2{}^2(\tfrac{3}{2},\tfrac{1}{2})T(\hat{n}\hat{n})+a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{3}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}0)\\&\quad+(\tfrac{1}{2}2|\tfrac{5}{2})\{[a_2{}^1(\tfrac{5}{2},\tfrac{1}{2})T(\hat{l}\hat{l})+a_2{}^2(\tfrac{5}{2},\tfrac{1}{2})T(\hat{n}\hat{n})+a_2{}^3(\tfrac{5}{2},\tfrac{1}{2})T(\hat{l}\hat{n})]:S_{[2]}(\tfrac{5}{2},\tfrac{1}{2})\\&\quad+[a_3{}^1(\tfrac{5}{2},\tfrac{1}{2})T(\hat{l}\hat{l}\hat{m})+a_3{}^2(\tfrac{5}{2},\tfrac{1}{2})T(\hat{n}\hat{n}\hat{m})+a_3{}^3(\tfrac{5}{2},\tfrac{1}{2})T(\hat{l}\hat{n}\hat{m})]:S_{[3]}(\tfrac{5}{2},\tfrac{1}{2})\}(\tfrac{1}{2}|\tfrac{1}{2}0),\end{aligned}$$

with

$$C_1=A_2{}^1,$$
$$C_2=A_2{}^2,$$
$$C_3=A_2{}^3,$$
$$C_4=(\sqrt{3}/\sqrt{5})A_1-(i/2)(\sqrt{2}/\sqrt{3})A_2{}^6+(7/15)A_3{}^1+(2/15)A_3{}^2,$$
$$C_5=(\sqrt{3}/\sqrt{5})A_1+(i/2)(\sqrt{2}/\sqrt{3})A_2{}^6+(2/15)A_3{}^1+(7/15)A_3{}^2,$$
$$C_6=i(\sqrt{2}/\sqrt{3})A_2{}^4-i(\sqrt{2}/\sqrt{3})A_2{}^5+\tfrac{1}{3}A_3{}^3,$$
$$C_7=-(\sqrt{3}/\sqrt{5})A_1+(i/2)(\sqrt{2}/\sqrt{3})A_2{}^6+(8/15)A_3{}^1-(2/15)A_3{}^2,$$
$$C_8=-(\sqrt{3}/\sqrt{5})A_1-(i/2)(\sqrt{2}/\sqrt{3})A_2{}^6-(2/15)A_3{}^1+(8/15)A_3{}^2,$$
$$C_9=i(\sqrt{2}/\sqrt{3})A_2{}^5+\tfrac{1}{3}A_3{}^3,$$
$$C_{10}=-i(\sqrt{2}/\sqrt{3})A_2{}^4+\tfrac{1}{3}A_3{}^3,$$

and

$$A_1=-(2/\sqrt{3})a_1(\tfrac{3}{2},\tfrac{1}{2}),$$
$$A_2{}^r=(1/2\sqrt{5})a_2{}^r(\tfrac{3}{2},\tfrac{1}{2})+(3/\sqrt{5})a_2{}^r(\tfrac{5}{2},\tfrac{1}{2}),\quad r=1,2,3,$$
$$A_2{}^{\bar{r}}=(\sqrt{3}/\sqrt{10})a_2{}^r(\tfrac{3}{2},\tfrac{1}{2})-(4\sqrt{3}/\sqrt{10})a_2{}^r(\tfrac{5}{2},\tfrac{1}{2}),\quad \bar{r}=4,5,6,\quad r=1,2,3,$$
$$A_3{}^r=(\sqrt{5})a_3{}^r(\tfrac{5}{2},\tfrac{1}{2}),\quad r=1,2,3,$$

so that

$$(\sqrt{5})C_1=\tfrac{1}{2}a_2{}^1(\tfrac{3}{2},\tfrac{1}{2})+3a_2{}^1(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_2=\tfrac{1}{2}a_2{}^2(\tfrac{3}{2},\tfrac{1}{2})+3a_2{}^2(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_3=\tfrac{1}{2}a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+3a_2{}^3(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_4=-2a_1(\tfrac{3}{2},\tfrac{1}{2})-(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+2ia_2{}^3(\tfrac{5}{2},\tfrac{1}{2})+(7/3)a_3{}^1(\tfrac{5}{2},\tfrac{1}{2})+\tfrac{2}{3}a_3{}^2(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_5=-2a_1(\tfrac{3}{2},\tfrac{1}{2})+(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})-2ia_2{}^3(\tfrac{5}{2},\tfrac{1}{2})+\tfrac{2}{3}a_3{}^1(\tfrac{5}{2},\tfrac{1}{2})+(7/3)a_3{}^2(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_6=ia_2{}^1(\tfrac{3}{2},\tfrac{1}{2})-4ia_2{}^1(\tfrac{5}{2},\tfrac{1}{2})-ia_2{}^2(\tfrac{3}{2},\tfrac{1}{2})+4ia_2{}^2(\tfrac{5}{2},\tfrac{1}{2})+(5/3)a_3{}^3(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_7=-2a_1(\tfrac{3}{2},\tfrac{1}{2})+(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})-2ia_2{}^3(\tfrac{5}{2},\tfrac{1}{2})+(8/3)a_3{}^1(\tfrac{5}{2},\tfrac{1}{2})-\tfrac{2}{3}a_3{}^2(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_8=-2a_1(\tfrac{3}{2},\tfrac{1}{2})-(i/2)a_2{}^3(\tfrac{3}{2},\tfrac{1}{2})+2ia_2{}^3(\tfrac{5}{2},\tfrac{1}{2})-\tfrac{2}{3}a_3{}^2(\tfrac{5}{2},\tfrac{1}{2})+(8/3)a_3{}^1(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_9=ia_2{}^2(\tfrac{3}{2},\tfrac{1}{2})-4ia_2{}^2(\tfrac{5}{2},\tfrac{1}{2})+(5/3)a_3{}^3(\tfrac{5}{2},\tfrac{1}{2}),$$
$$(\sqrt{5})C_{10}=-ia_2{}^1(\tfrac{3}{2},\tfrac{1}{2})+4ia_2{}^1(\tfrac{5}{2},\tfrac{1}{2})+(5/3)a_3{}^3(\tfrac{5}{2},\tfrac{1}{2}).$$

For such boson reactions as $0+s\to 0+s'$, which may be taken as either basic or composite (with constituents $0+0\to 0+s'$ and $0+s\to 0+0$), it is clear that the composite form factors (C_i) are most convenient for the calculation of observables, whereas the recoupled, basic form factors ($a_J(s',s)$) are most convenient for partial-wave expansions. Moreover, for the compound spin matrix tensors, Eqs. (2.1)–(2.5), we have

$$(\Omega_{J_1M_1}(s',0))_{\Lambda',0}=\delta_{s',J_1}\delta_{\Lambda',M_1},\quad (\Omega_{J_2M_2}(0,s))_{0,\Lambda}=(-)^{s-\Lambda}\delta_{s,J_2}\delta_{\Lambda,-M_2},$$

so that

$$(\Omega_{J;J_1J_2}{}^M(s',0;0,s))_{\Lambda'\Lambda}=(-)^{s-\Lambda}\langle s'{s}\Lambda',-\Lambda|JM\rangle\delta_{s',J_1}\delta_{s,J_2}$$
$$=(\Omega_{JM}(s',s))_{\Lambda',\Lambda}\delta_{s',J_1}\delta_{s,J_2},$$

and

$$S_{[J;J_1J_2]}(s',0;0,s)=\left(\frac{\beta_{s'}(s',0)\beta_s(0,s)}{\beta_J(s',s)}\right)^{1/2}S_{[J]}(s',s)\delta_{s',J_1}\delta_{s,J_2}.$$

$0+1\to 0+1$

We factorize into $0+0\to 0+1$ and $0+1\to 0+0$ with $S_{[1]}(1,0)\equiv S$, $S_{[1]}(0,1)\equiv S'$

$$(01|M^+|01)=C_1\hat{m}\cdot\mathbf{S}\hat{m}\cdot\mathbf{S}'+C_2\hat{l}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'+C_3\hat{n}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_4\hat{l}\cdot\mathbf{S}\hat{n}\cdot\mathbf{S}'+C_5\hat{n}\cdot\mathbf{S}\hat{l}\cdot\mathbf{S}'$$
$$=A_0S_{[0,11]}+A_1T(\hat{m}):S_{[1,11]}+[A_2{}^1T(\hat{l}\hat{l})+A_2{}^2T(\hat{n}\hat{n})+A_2{}^3T(\hat{l}\hat{n})]:S_{[2,11]}$$
$$=a_0(1,1)+a_1(1,1)T(\hat{m}):S_{[1]}(1,1)+[a_2{}^1(1,1)T(\hat{l}\hat{l})+a_2{}^2(1,1)T(\hat{n}\hat{n})+a_2{}^3(1,1)T(\hat{l}\hat{n})]:S_{[2]}(1,1),$$

with $A_0=\sqrt{3}a_0(1,1)$, $A_1=\sqrt{2}a_1(1,1)$ and $A_2{}^r=a_2{}^r(1,1)$ $r=1$, 2, 3 which gives

$$C_1=-a_0(1,1)-\tfrac{1}{3}a_2{}^1(1,1)-\tfrac{1}{3}a_2{}^2(1,1),$$
$$C_2=-a_0(1,1)+\tfrac{2}{3}a_2{}^1(1,1)-\tfrac{1}{3}a_2{}^2(1,1),$$
$$C_3=-a_0(1,1)-\tfrac{1}{3}a_2{}^1(1,1)+\tfrac{2}{3}a_2{}^2(1,1),$$
$$C_4=-ia_1(1,1)+\tfrac{1}{2}a_2{}^3(1,1),$$
$$C_5=ia_1(1,1)+\tfrac{1}{2}a_2{}^3(1,1).$$

For elastic scattering, time-reversal invariance implies $S\to -S'$, and hence $C_4=-C_5$ and $a_2{}^3(1,1)=0$.

$0+1\to 0+2$

We factorize into $0+0\to 0+2$ and $0+1\to 0+0$ with $S_{[1]}(0,1)\equiv S$ and $S_{[2]}(2,0)\equiv S_{[2]}'$:

$$(02|M^+|01)=C_1\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{l}):S_{[2]}'+C_2\hat{m}\cdot\mathbf{S}T(\hat{n}\hat{n}):S_{[2]}'+C_3\hat{m}\cdot\mathbf{S}T(\hat{l}\hat{n}):S_{[2]}'+C_4\hat{l}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}'$$
$$+C_5\hat{n}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}'+C_6\hat{l}\cdot\mathbf{S}T(\hat{n}\hat{m}):S_{[2]}'+C_7\hat{n}\cdot\mathbf{S}T(\hat{l}\hat{m}):S_{[2]}'$$
$$=A_1T(\hat{m}):S_{[1,21]}+[A_2{}^1T(\hat{l}\hat{l})+A_2{}^2T(\hat{n}\hat{n})+A_2{}^3T(\hat{l}\hat{n})]:S_{[2,21]}$$
$$+[A_3{}^1T(\hat{l}\hat{l}\hat{m})+A_3{}^2T(\hat{n}\hat{n}\hat{m})+A_3{}^3T(\hat{l}\hat{n}\hat{m})]:S_{[3,21]}$$
$$=a_1(2,1)T(\hat{m}):S_{[1]}(2,1)+[a_2{}^1(2,1)T(\hat{l}\hat{l})+a_2{}^2(2,1)T(\hat{n}\hat{n})+a_2{}^3(2,1)T(\hat{l}\hat{n})]:S_{[2]}(2,1)$$
$$+[a_3{}^1(2,1)T(\hat{l}\hat{l}\hat{m})+a_3{}^2(2,1)T(\hat{n}\hat{n}\hat{m})+a_3{}^3(2,1)T(\hat{l}\hat{n}\hat{m})]:S_{[3]}(2,1),$$

with $A_1=(\sqrt{5}/\sqrt{3})a_1(2,1)$, $A_2{}^r=(\sqrt{3}/\sqrt{2})a_2{}^r(2,1)$, $A_3{}^r=a_3{}^r(2,1)$ $r=1, 2, 3$ which gives

$$C_1=a_1(2,1)+(i/2)a_2{}^3(2,1)+(7/15)a_3{}^1(2,1)+(2/15)a_3{}^2(2,1)\,,$$
$$C_2=a_1(2,1)-(i/2)a_2{}^3(2,1)+(2/15)a_3{}^1(2,1)+(7/15)a_3{}^2(2,1)\,,$$
$$C_3=-ia_2{}^1(2,1)+ia_2{}^2(2,1)+\tfrac{1}{3}a_3{}^3(2,1)\,,$$
$$C_4=-a_1(2,1)-(i/2)a_2{}^3(2,1)+(8/15)a_3{}^1(2,1)-(2/15)a_3{}^2(2,1)\,,$$
$$C_5=-a_1(2,1)+(i/2)a_2{}^3(2,1)-(2/15)a_3{}^1(2,1)+(8/15)a_3{}^2(2,1)\,,$$
$$C_6=-ia_2{}^2(2,1)+\tfrac{1}{3}a_3{}^3(2,1)\,,$$
$$C_7=ia_2{}^1(2,1)+\tfrac{1}{3}a_3{}^3(2,1)\,.$$

APPENDIX III: PARTIAL-WAVE EXPANSIONS OF $a_J(s',s)$

Now we work out Eq. (5.13) for the basic reactions $0+\frac{1}{2}\to 0+\frac{1}{2}$, $0+\frac{1}{2}\to 0+\frac{3}{2}$, $0+\frac{1}{2}\to 0+\frac{5}{2}$, $0+\frac{3}{2}\to 0+\frac{3}{2}$, $0+s\to 0+0$, $0+0\to 0+s'$, $0+1\to 0+1$ and $0+1\to 0+2$. Since all of the angular dependence of $a_J(\theta;s',s)$ is embedded in the angular functions $f_{J;l',l}{}^r(\theta)$ these expansions are valid for any choice of the momentum tensor basis. For our choice of tensor basis (Appendix I) the $f_{J;l',l}{}^r$'s (again we omit the angular dependence) are worked out in Appendix IV. Recall the definition of $M_{l',l}{}^j(s',s)$,

$$M_{l',l}{}^j(s',s)=\langle l's'jm_j|M|lsjm_j\rangle\,.$$

Consider first the positive-intrinsic-parity amplitudes M^+ where $l'+l$ is even:

$$0+\tfrac{1}{2}\xrightarrow{+}0+\tfrac{1}{2}$$

$$J=0:\quad 4\pi a_0(\tfrac{1}{2},\tfrac{1}{2})=\sum_l f_{0;l,l}[(l+1)M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})+lM_{l,l}{}^{l-\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})]\,.$$

$$J=1:\quad 4\pi a_1(\tfrac{1}{2},\tfrac{1}{2})=-i\sqrt{2}\sum_l f_{1;l,l}[M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})-M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})]\,.$$

$$0+\tfrac{1}{2}\xrightarrow{+}0+\tfrac{3}{2}$$

$$J=1:\quad 4\pi a_1(\tfrac{3}{2},\tfrac{1}{2})=-\frac{i}{2}\sum_l f_{1;l,l}\left[\left(\frac{2l+3}{2l}\right)^{1/2}M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})+\left(\frac{2l-1}{2l+2}\right)^{1/2}M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\right]\,.$$

$$J=2:\quad 4\pi a_2{}^r(\tfrac{3}{2},\tfrac{1}{2})=-\sqrt{2}\sum_l\left\{f_{2;l,l}{}^r\left[\left(\frac{1}{(2l)(2l+3)}\right)^{1/2}M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})-\left(\frac{1}{(2l-1)(2l+2)}\right)^{1/2}M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\right]\right.$$
$$\left.+f_{2;l+2,l}\left(\frac{1}{3(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})-f_{2;l-2,l}\left(\frac{1}{3(2l-2)(2l-1)}\right)^{1/2}M_{l-2,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\right\}\,.$$

$$0+\tfrac{1}{2}\xrightarrow{+}0+\tfrac{5}{2}$$

$$J=2:\quad 4\pi a_2{}^r(\tfrac{5}{2},\tfrac{1}{2})=-\tfrac{1}{6}\sqrt{3}\sum_l\left\{f_{2;l,l}{}^r\left[\left(\frac{2l+4}{(2l-1)(2l)(2l+3)}\right)^{1/2}M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})+\left(\frac{2l-2}{(2l-1)(2l+2)(2l+3)}\right)^{1/2}M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]\right.$$
$$+\tfrac{1}{6}\sqrt{2}f_{2;l+2,l}{}^r\left[\left(\frac{2l+6}{(2l+1)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right.$$
$$\left.+2l\left(\frac{5}{(2l+1)(2l+2)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]$$
$$+\tfrac{1}{6}\sqrt{2}f_{2;l-2,l}{}^r\left[(2l+2)\left(\frac{5}{(2l-2)(2l-1)(2l)(2l+1)}\right)^{1/2}M_{l-2,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right.$$
$$\left.\left.+\left(\frac{2l-4}{(2l-2)(2l-1)(2l+1)}\right)^{1/2}M_{l-2,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]\right\}\,.$$

$$J=3:\quad 4\pi a_3{}^r(\tfrac{5}{2},\tfrac{1}{2})=i\frac{\sqrt{2}}{\sqrt{3}}\sum_l\left\{f_{3;l,l}{}^r\left[\left(\frac{1}{(2l-1)(2l)(2l+3)(2l+4)}\right)^{1/2}M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})-\left(\frac{1}{(2l-2)(2l-1)(2l+2)(2l+3)}\right)^{1/2}M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]\right.$$
$$+\frac{1}{\sqrt{2}}f_{3;l+2,l}{}^r\left[\left(\frac{1}{(2l+1)(2l+3)(2l+4)(2l+6)}\right)^{1/2}M_{l+2,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})-\left(\frac{1}{5(2l+1)(2l+2)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]$$
$$\left.+\frac{1}{\sqrt{2}}f_{3;l-2,l}{}^r\left[\left(\frac{1}{5(2l-2)(2l-1)(2l)(2l+1)}\right)^{1/2}M_{l-2,l}{}^{l+\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})-\left(\frac{1}{(2l-4)(2l-2)(2l-1)(2l+1)}\right)^{1/2}M_{l-2,l}{}^{l-\frac{1}{2}}(\tfrac{5}{2},\tfrac{1}{2})\right]\right\}.$$

$$0+\tfrac{3}{2}\xrightarrow{+}0+\tfrac{3}{2}$$

$$J=0:\quad 4\pi a_0(\tfrac{3}{2},\tfrac{3}{2})=\tfrac{1}{2}\sum_l f_{0;l,l}[(l+2)M_{l,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})+(l+1)M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})+lM_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})+(l-1)M_{l,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})].$$

$$J=1:\quad 4\pi a_1(\tfrac{3}{2},\tfrac{3}{2})=-i\frac{3}{10}\sqrt{2}\sum_l f_{1;l,l}\left[\left(\frac{2l+4}{2l+2}\right)M_{l,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})+\left(\frac{2l-6}{3(2l)}\right)M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{2l+8}{3(2l+2)}\right)M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{2l-2}{2l}\right)M_{l,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right].$$

$$J=2:\quad 4\pi a_2{}^r(\tfrac{3}{2},\tfrac{3}{2})=-\tfrac{1}{2}\sum_l\left\{f_{2;l,l}{}^r\left[\left(\frac{2l+4}{(2l+2)(2l+3)}\right)M_{l,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{2l+6}{(2l)(2l+3)}\right)M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{2l-4}{(2l-1)(2l+2)}\right)M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})+\left(\frac{2l-2}{(2l-1)(2l)}\right)M_{l,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]\right.$$
$$+\frac{1}{\sqrt{3}}f_{2;l+2,l}{}^r\left[\frac{1}{(2l+3)}\left(\frac{2l+6}{2l+2}\right)^{1/2}M_{l+2,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})+\frac{1}{(2l+3)}\left(\frac{2l}{2l+4}\right)^{1/2}M_{l+2,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]$$
$$\left.+\frac{1}{\sqrt{3}}f_{2;l-2,l}{}^r\left[\frac{1}{(2l-1)}\left(\frac{2l+2}{2l-2}\right)^{1/2}M_{l-2,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})+\frac{1}{(2l-1)}\left(\frac{2l-4}{2l}\right)^{1/2}M_{l-2,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]\right\}.$$

$$J=3:\quad 4\pi a_3{}^r(\tfrac{3}{2},\tfrac{3}{2})=\frac{i}{3}\sqrt{2}\sum_l\left\{f_{3;l,l}{}^r\left[\left(\frac{1}{(2l+2)(2l+3)}\right)M_{l,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{3}{(2l)(2l+3)}\right)M_{l,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})+\left(\frac{3}{(2l-1)(2l+2)}\right)M_{l,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\left(\frac{1}{(2l-1)(2l)}\right)M_{l,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]\right.$$
$$+\sqrt{3}f_{3;l+2,l}{}^r\left[\frac{1}{(2l+3)}\left(\frac{1}{(2l+2)(2l+6)}\right)^{1/2}M_{l+2,l}{}^{l+\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\frac{1}{(2l+3)}\left(\frac{1}{(2l)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]$$
$$\left.+\sqrt{3}f_{3;l-2,l}{}^r\left[\frac{1}{(2l-1)}\left(\frac{1}{(2l-2)(2l+2)}\right)^{1/2}M_{l-2,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{3}{2})-\frac{1}{(2l-1)}\left(\frac{1}{(2l-4)(2l)}\right)^{1/2}M_{l-2,l}{}^{l-\frac{3}{2}}(\tfrac{3}{2},\tfrac{3}{2})\right]\right\}.$$

$$0+0\xrightarrow{\pm}0+0$$

$$J=0:\quad 4\pi a_0(0,0)=\sum_l f_{0;l,l}(2l+1)M_{l,l}{}^l(0,0)\,.$$

$$0+1\xrightarrow{\pm}0+0$$

$$J=1:\quad 4\pi a_1(0,1)=i\sqrt{2}\sum_l f_{1;l,l}(2l+1)\left(\frac{1}{(2l)(2l+2)}\right)^{1/2}M_{l,l}{}^l(0,1)\,.$$

$$0+0\xrightarrow{\pm}0+1$$

$$J=1:\quad 4\pi a_1(1,0)=-i\sqrt{2}\sum_l f_{1;l,l}(2l+1)\left(\frac{1}{(2l)(2l+2)}\right)^{1/2}M_{l,l}{}^l(1,0)\,.$$

$$0+2\xrightarrow{\pm}0+0$$

$$\begin{aligned}J=2:\quad 4\pi a_2{}^r(0,2)=-\sum_l\bigg\{&f_{2;l,l}{}^r(2l+1)\left(\frac{1}{(2l-1)(2l)(2l+2)(2l+3)}\right)^{1/2}M_{l,l}{}^l(0,2)\\&+\frac{1}{\sqrt{6}}f_{2;l+2,l}{}^r\left(\frac{2l+5}{(2l+2)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l+2}(0,2)\\&+\frac{1}{\sqrt{6}}f_{2;l-2,l}\left(\frac{2l-3}{(2l-2)(2l-1)(2l)}\right)^{1/2}M_{l-2,l}{}^{l-2}(0,2)\bigg\}\,.\end{aligned}$$

$$0+0\xrightarrow{\pm}0+2$$

$$\begin{aligned}J=2:\quad 4\pi a_2{}^r(2,0)=-\sum_l\bigg\{&f_{2;l,l}{}^r(2l+1)\left(\frac{1}{(2l-1)(2l)(2l+2)(2l+3)}\right)^{1/2}M_{l,l}{}^l(2,0)\\&+\frac{1}{\sqrt{6}}f_{2;l+2,l}{}^r\left(\frac{2l+5}{(2l+2)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^l(2,0)\\&+\frac{1}{\sqrt{6}}f_{2;l-2,l}{}^r\left(\frac{2l-3}{(2l-2)(2l-1)(2l)}\right)^{1/2}M_{l-2,l}{}^l(2,0)\bigg\}\,.\end{aligned}$$

$$0+1\xrightarrow{\pm}0+1$$

$$J=0:\quad 4\pi a_0(1,1)=\tfrac{1}{3}\sum_l f_{0;l,l}[(2l+3)M_{l,l}{}^{l+1}(1,1)+(2l+1)M_{l,l}{}^l(1,1)+(2l-1)M_{l,l}{}^{l-1}(1,1)]\,.$$

$$J=1:\quad 4\pi a_1(1,1)=-\frac{i}{\sqrt{2}}\sum_l f_{1;l,l}\left[\left(\frac{2l+3}{2l+2}\right)M_{l,l}{}^{l+1}(1,1)-\left(\frac{2(2l+1)}{(2l)(2l+2)}\right)M_{l,l}{}^l(1,1)-\left(\frac{2l-1}{2l}\right)M_{l,l}{}^{l-1}(1,1)\right]\,.$$

$$\begin{aligned}J=2:\quad 4\pi a_2{}^r(1,1)=-\sum_l\bigg\{&f_{2;l,l}{}^r\left[\left(\frac{1}{2l+2}\right)M_{l,l}{}^{l+1}(1,1)-\left(\frac{2(2l+1)}{(2l)(2l+2)}\right)M_{l,l}{}^l(1,1)+\left(\frac{1}{2l}\right)M_{l,l}{}^{l-1}(1,1)\right]\\&+f_{2;l+2,l}{}^r\left(\frac{1}{(2l+2)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l+1}(1,1)+f_{2;l-2,l}{}^r\left(\frac{1}{(2l-2)(2l)}\right)^{1/2}M_{l-2,l}{}^{l-1}(1,1)\bigg\}\,.\end{aligned}$$

$$0+1\xrightarrow{+}0+2$$

$$J=1:\quad 4\pi a_1(2,1)=-i\frac{\sqrt{3}}{5\sqrt{2}}\sum_l f_{1;l,l}\Bigg[\left(\frac{2l+3}{2l+2}\right)\left(\frac{2l+4}{2l}\right)^{1/2}M_{l,l}{}^{l+1}(2,1)$$
$$+\frac{2}{\sqrt{3}}\left(\frac{2l+1}{(2l)(2l+2)}\right)((2l-1)(2l+3))^{1/2}M_{l,l}{}^{l}(2,1)+\left(\frac{2l-1}{2l}\right)\left(\frac{2l-2}{2l+2}\right)^{1/2}M_{l,l}{}^{l-1}(2,1)\Bigg].$$

$$J=2:\quad 4\pi a_2{}^r(2,1)=-\frac{1}{\sqrt{3}}\sum_l\Bigg\{f_{2;l,l}{}^r\Bigg[\frac{1}{(2l+2)}\left(\frac{2l+4}{2l}\right)^{1/2}M_{l,l}{}^{l+1}(2,1)$$
$$-2\sqrt{3}\left(\frac{2l+1}{(2l)(2l+2)}\right)\left(\frac{1}{(2l-1)(2l+3)}\right)^{1/2}M_{l,l}{}^{l}(2,1)-\frac{1}{(2l)}\left(\frac{2l-2}{2l+2}\right)^{1/2}M_{l,l}{}^{l-1}(2,1)\Bigg]$$
$$+\tfrac{1}{3}f_{2;l+2,l}{}^r\Bigg[\frac{1}{(2l+2)}\left(\frac{2l+6}{2l+4}\right)^{1/2}M_{l+2,l}{}^{l+1}(2,1)+\frac{\sqrt{2}}{(2l+2)}\left(\frac{(2l)(2l+1)}{(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l}(2,1)\Bigg]$$
$$-\tfrac{1}{3}f_{2;l-2,l}{}^r\Bigg[\frac{\sqrt{2}}{(2l)}\left(\frac{(2l+1)(2l+2)}{(2l-2)(2l-1)}\right)^{1/2}M_{l-2,l}{}^{l}(2,1)+\frac{1}{(2l)}\left(\frac{2l-4}{2l-2}\right)^{1/2}M_{l-2,l}{}^{l-1}(2,1)\Bigg]\Bigg\}.$$

$$J=3:\quad 4\pi a_3{}^r(2,1)=i\frac{\sqrt{2}}{\sqrt{3}}\sum_l\Bigg\{f_{3;l,l}{}^r\Bigg[\frac{1}{(2l+2)}\left(\frac{1}{(2l)(2l+4)}\right)^{1/2}M_{l,l}{}^{l+1}(2,1)$$
$$-\left(\frac{2l+1}{(2l)(2l+2)}\right)\left(\frac{3}{(2l-1)(2l+3)}\right)^{1/2}M_{l,l}{}^{l}(2,1)+\frac{1}{(2l)}\left(\frac{1}{(2l-2)(2l+4)}\right)^{1/2}M_{l,l}{}^{l-1}(2,1)\Bigg]$$
$$+f_{3;l+2,l}{}^r\Bigg[\frac{1}{(2l+2)}\left(\frac{1}{(2l+4)(2l+6)}\right)^{1/2}M_{l+2,l}{}^{l+1}(2,1)$$
$$-\frac{1}{(2l+2)}\left(\frac{2l+1}{2(2l)(2l+3)(2l+4)}\right)^{1/2}M_{l+2,l}{}^{l}(2,1)\Bigg]$$
$$+f_{3;l-2,l}{}^r\Bigg[\frac{1}{(2l)}\left(\frac{2l+1}{2(2l-2)(2l-1)(2l+2)}\right)^{1/2}M_{l-2,l}{}^{l}(2,1)$$
$$+\frac{1}{(2l)}\left(\frac{1}{(2l-4)(2l-2)}\right)^{1/2}M_{l-2,l}{}^{l-1}(2,1)\Bigg]\Bigg\}.$$

Next, we consider negative-intrinsic-parity amplitudes M^- where $l'+l$ is odd:

$$0+\tfrac{1}{2}\xrightarrow{-}0+\tfrac{1}{2}$$

$$J=1:\quad 4\pi a_1{}^r(\tfrac{1}{2},\tfrac{1}{2})=-i\sqrt{2}\sum_l\{f_{1,l+1,l}{}^rM_{l+1,l}{}^{l+\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})-f_{1,l-1,l}{}^rM_{l-1,l}{}^{l-\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2})\}.$$

$$0+\tfrac{1}{2}\xrightarrow{-}0+\tfrac{3}{2}$$

$$J=1:\quad 4\pi a_1{}^r(\tfrac{3}{2},\tfrac{1}{2})=-\frac{i}{4}\sum_l\Bigg\{f_{1;l+1,l}{}^r\Bigg[\left(\frac{2l+4}{2l+1}\right)M_{l+1,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})+(2l)\left(\frac{3}{(2l+1)(2l+2)}\right)^{1/2}M_{l+1,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\Bigg]$$
$$+f_{1,l-1,l}{}^r\Bigg[(2l+2)\left(\frac{3}{(2l)(2l+1)}\right)^{1/2}M_{l-1,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})+\left(\frac{2l-2}{2l+1}\right)^{1/2}M_{l-1,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\Bigg]\Bigg\}.$$

$$J=2:\quad 4\pi a_2{}^r(\tfrac{3}{2},\tfrac{1}{2})=-\sqrt{2}\sum_l\left\{f_{2;l+1,l}{}^r\left[\left(\frac{1}{(2l+1)(2l+4)}\right)^{1/2}M_{l+1,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})-\left(\frac{1}{3(2l+1)(2l+2)}\right)^{1/2}M_{l+1,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\right]\right.$$
$$\left.+f_{2;l-1,l}{}^r\left[\left(\frac{1}{3(2l)(2l+1)}\right)^{1/2}M_{l-1,l}{}^{l+\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})-\left(\frac{1}{(2l-2)(2l+1)}\right)^{1/2}M_{l-1,l}{}^{l-\frac{1}{2}}(\tfrac{3}{2},\tfrac{1}{2})\right]\right\}.$$

$0+1\rightarrow 0+0$

$$J=1:\quad 4\pi a_1{}^r(0,1)=-i\sum_l\left\{f_{1;l+1,l}{}^r\left(\frac{2l+3}{2l+2}\right)^{1/2}M_{l+1,l}{}^{l+1}(0,1)+f_{1;l-1,l}{}^r\left(\frac{2l-1}{2l}\right)^{1/2}M_{l-1,l}{}^{l-1}(0,1)\right\}.$$

$0+0\rightarrow 0+1$

$$J=1:\quad 4\pi a_1{}^r(1,0)=-i\sum_l\left\{f_{1;l+1,l}{}^r\left(\frac{2l+3}{2l+2}\right)^{1/2}M_{l+1,l}{}^{l}(1,0)+f_{1;l-1,l}{}^r\left(\frac{2l-1}{2l}\right)^{1/2}M_{l-1,l}{}^{l}(1,0)\right\}.$$

$0+2\rightarrow 0+0$

$$J=2:\quad 4\pi a_2{}^r(0,2)=\frac{\sqrt{2}}{\sqrt{3}}\sum_l\left\{f_{2;l+1,l}{}^r\left(\frac{2l+3}{(2l)(2l+2)(2l+4)}\right)^{1/2}M_{l+1,l}{}^{l+1}(0,2)\right.$$
$$\left.+f_{2;l-1,l}{}^r\left(\frac{2l-1}{(2l-2)(2l)(2l+2)}\right)^{1/2}M_{l-1}{}^{l-1}{}_{,l}(0,2)\right\}.$$

$0+0\rightarrow 0+2$

$$J=2:\quad 4\pi a_2{}^r(2,0)=-\frac{\sqrt{2}}{\sqrt{3}}\sum_l\left\{f_{2;l+1,l}{}^r\left(\frac{2l+3}{(2l)(2l+2)(2l+4)}\right)^{1/2}M_{l+1,l}{}^{l}(2,0)\right.$$
$$\left.+f_{2;l-1,l}{}^r\left(\frac{2l-1}{(2l-2)(2l)(2l+2)}\right)^{1/2}M_{l-1,l}{}^{l}(2,0)\right\}.$$

We note in passing that not all of the partial-wave amplitudes $M_{l',l}{}^j$ are independent for elastic scattering, because time reversal invariance implies the symmetry relation[11]

$$M_{l',l}{}^j(s',s)=M_{l,l'}{}^j(s,s').$$

APPENDIX IV: ANGULAR FUNCTIONS

Given the choice of the orthogonal tensor basis of Appendix I, we now extract from the compound spherical harmonic $Y_{J;l',l}{}^M(\hat{q}',\hat{q})$ the angular functions $f_{J;l',l}(\theta)$ for the even parity cases of $J=0$, 1, 2, 3 which are needed to complete the partial wave expansions of Appendix III. Using Eq. (5.11) we obtain $f_{J;l',l}$ by equating independently all values of M. For the choice $\hat{q}'=\hat{e}_3$ the basis vectors ($J=1$) are

$$T^{\pm1}(\hat{m})=\mp1/\sqrt{2}\,,\qquad T^0(\hat{m})=0\,,$$
$$T^{\pm1}(\hat{n})=-(i/\sqrt{2})\sin(\theta/2)\,,\qquad T^0(\hat{n})=\cos(\theta/2)\,,$$
$$T^{\pm1}(\hat{l})=(i/\sqrt{2})\cos(\theta/2)\,,\qquad T^0(\hat{l})=\sin(\theta/2)\,,$$

and using Eq. (5.12) we find for $J=2$

$$T_+{}^2=-\tfrac{1}{4}\,,\qquad T_-{}^2=\tfrac{1}{4}\cos\theta\,,\qquad T_0{}^2=\tfrac{1}{4}\sin\theta\,,$$
$$T_+{}^1=0\,,\qquad T_-{}^1=-(i/2)\sin\theta\,,\qquad T_0{}^1=(i/2)\cos\theta\,,$$
$$T_+{}^0=\tfrac{1}{4}\sqrt{2}/\sqrt{3}\,,\qquad T_-{}^0=\tfrac{3}{4}(\sqrt{2}/\sqrt{3})\cos\theta\,,\qquad T_0{}^0=\tfrac{3}{4}(\sqrt{2}/\sqrt{3})\sin\theta\,,$$

[11] See, for example, M. MacGregor, M. Moravcsik, and H. Stapp, Ann. Rev. Nucl. Sci. 10, 291 (1960).

with $T_{\pm}{}^M=\frac{1}{2}(T^M(\hat{n}\hat{n})\pm T^M(\hat{l}\hat{l}))$, $T_0{}^M=T^M(\hat{l}\hat{n})$ and for $J=3$

$$T_+{}^3=\tfrac{1}{8}\sqrt{2},\qquad T_-{}^3=-(\tfrac{1}{8}\sqrt{2})\cos\theta,\qquad T_0{}^3=-\tfrac{1}{8}\sqrt{2}\sin\theta,$$
$$T_+{}^2=0,\qquad T_-{}^2=(i/6)\sqrt{3}\sin\theta,\qquad T_0{}^2=-(i/6)\sqrt{3}\cos\theta,$$
$$T_+{}^1=-3/4\sqrt{30},\qquad T_-{}^1=-(5/4\sqrt{30})\cos\theta,\qquad T_0{}^1=-(5/4\sqrt{30})\sin\theta,$$

with $T_{\pm}{}^M=\frac{1}{2}(T^M(\hat{n}\hat{n}\hat{m})\pm T^M(\hat{l}\hat{l}\hat{m}))$, $T_0{}^M=T^M(\hat{l}\hat{n}\hat{m})$. Then we can write

$J=0,\ l'=l$: $\quad \sum_r f_{J;l',l}{}^r T_J(p^r)\to f_{l,l}.\quad (r\to 0)$

$J=1,\ l'=l$: $\quad \sum_r f_{J;l',l}{}^r T_J(p^r)\to f_{l,l}T(\hat{m}).\quad (r\to m)$

$J=2,\ l'=l,\ l\pm2$: $\quad \sum_r f_{J;l',l}{}^r T_J(p^r)\to f_{2;l',l}{}^{ll}T(\hat{l}\hat{l})+f_{2;l',l}{}^{nn}T(\hat{n}\hat{n})+f_{2;l',l}{}^{ln}T(\hat{l}\hat{n})\equiv f_{2;l',l}{}^+T_++f_{2;l',l}{}^-T_-+f_{2;l',l}{}^{ln}T_0,$

with

$$f_{2;l',l}{}^{ll}=\tfrac{1}{2}(f_{2;l',l}{}^+-f_{2;l',l}{}^-)\quad\text{and}\quad f_{2;l',l}{}^{nn}=\tfrac{1}{2}(f_{2;l',l}{}^++f_{2;l',l}{}^-)$$

$J=3,\ l'=l,\ l\pm2$: $\quad \sum_r f_{3;l',l}{}^r T_J(p^r)\to f_{3;l'l,}{}^{llm}T(\hat{l}\hat{l}\hat{m})+f_{3;l',l}{}^{nnm}T(\hat{n}\hat{n}\hat{m})+f_{3;l',l}{}^{lnm}T(\hat{l}\hat{n}\hat{m})$
$$\equiv f_{3;l',l}{}^+T_++f_{3;l',l}{}^-T_-+f_{3;l',l}{}^{lnm}T_0,$$

with

$$f_{3;l',l}{}^{llm}=\tfrac{1}{2}(f_{3;l',l}{}^+-f_{3;l',l}{}^-)\quad\text{and}\quad f_{3;l',l}{}^{nnm}=\tfrac{1}{2}(f_{3;l',l}{}^++f_{3;l',l}{}^+).$$

With a little algebra and the basic identity

$$P_{l+1}{}^{m+2}+(l-m)(l-m+1)P_{l+1}{}^m=P_{l-1}{}^{m+2}+(l+m)(l+m+1)P_{l-1}{}^m,$$

we obtain for the even parity f's (we omit the angular dependence of the $P_l{}^m$'s for clarity)

$J=0,\quad f_{0;l,l}=P_l{}^0,$

$J=1,\quad f_{1;l,l}=-\sqrt{2}P_l{}^1,$

$J=2,\quad f_{2;l+2,l}{}^+=-3[P_l{}^2+(l+1)(l+2)P_l{}^0],$
$f_{2;l,l}{}^+=-[3P_l{}^2-l(l+1)P_l{}^0],$
$f_{2;l-2,l}{}^+=-3[P_l{}^2+(l-1)lP_l{}^0],$
$f_{2;l+2,l}{}^-=P_{l+1}{}^2-3(l+1)(l+2)P_{l+1}{}^0,$
$f_{2;l,l}{}^-=P_{l-1}{}^2+l(l+1)P_{l-1}{}^0=P_{l+1}{}^2+l(l+1)P_{l+1}{}^0,$
$f_{2;l-2,l}{}^-=P_{l-1}{}^2-3(l-1)lP_{l-1}{}^0,$
$f_{2;l+2,l}{}^{ln}=-2(2l+3)P_{l+1}{}^1,$
$f_{2;l,l}{}^{ln}=0,$
$f_{2;l-2,l}{}^{ln}=2(2l-1)P_{l-1}{}^1,$

$J=3,\quad \sqrt{2}f_{3;l+2,l}{}^+=5[P_l{}^3+(l+2)(l+3)P_l{}^1],$
$\sqrt{2}f_{3;l,l}{}^+=5P_l{}^3-3(l-1)(l+2)P_l{}^1,$
$\sqrt{2}f_{3;l-2,l}{}^+=5[P_l{}^3+(l-2)(l-1)P_l{}^1],$
$\sqrt{2}f_{3;l+2,l}{}^-=-[3P_{l+1}{}^3-l(l+3)P_{l+1}{}^1],$
$\sqrt{2}f_{3l;,l}{}^-=-3[P_{l-1}{}^3+(l+1)(l+2)P_{l-1}{}^1],$
$\sqrt{2}f_{3;l-2,l}{}^-=-[3P_{l-1}{}^3-5(l-2)(l+1)P_{l-1}{}^1],$
$\sqrt{2}f_{3;l+2,l}{}^{lnm}=4(2l+3)P_{l+1}{}^2,$
$\sqrt{2}f_{3;l,l}{}^{lnm}=0,$
$\sqrt{2}f_{3;l-2,l}{}^{lnm}=-4(2l-1)P_{l-1}{}^2.$

For the odd-parity cases, the technique is similar and the results are for $J=1,\ 2$

$$J=1,\quad -i\sqrt{2}f_{1;l+1,l}{}^n=\frac{1}{\sin(\theta/2)}(P_l{}^1-P_{l+1}{}^1),$$
$$i\sqrt{2}f_{1;l+1,l}{}^l=\frac{1}{\cos(\theta/2)}(P_l{}^1+P_{l+1}{}^1),$$
$$-i\sqrt{2}f_{1;l-1,l}{}^n=\frac{1}{\sin(\theta/2)}(P_l{}^1-P_{l-1}{}^1),$$
$$i\sqrt{2}f_{1;l-1,l}=\frac{1}{\cos(\theta/2)}(P_l{}^1+P_{l-1}{}^1),$$

$$J=2,\quad if_{2;l+1,l}{}^n=\frac{1}{\sin(\theta/2)}(P_l{}^2-P_{l+1}{}^2),$$
$$-if_{2;l+1,l}{}^l=\frac{1}{\cos(\theta/2)}(P_l{}^2+P_{l+1}{}^2),$$
$$if_{2;l-1,l}{}^n=\frac{1}{\sin(\theta/2)}(P_l{}^2-P_{l-1}{}^2),$$
$$-if_{2;l-1,l}{}^l=\frac{1}{\cos(\theta/2)}(P_l{}^2+P_{l-1}{}^2).$$