in case only the number but not the location of the sign changes of ρ is known.

As the last step one has to calculate the constants α_i in (B18) and (B19), which can be performed in the same manner.

We define

$$\phi_i^{(1)}(x)=(x-z_i)^{-1}, \qquad i=0,1,\cdots,n-1 \qquad \text{(B20)}$$

$$\phi_i^{(2)}(x)=(x-x_l)^{-i-1}, \quad i=0,1,\cdots,n-1. \qquad \text{(B21)}$$

By means of the Schmidt procedure one may then orthogonalize the linearly independent functions ϕ_i in the interval $x_1\leq x\leq x_2$.

Calling the new sets of functions $\psi_k^{(\sigma)}$, $\sigma=1, 2$, one thus has

$$\psi_k^{(\sigma)}=\sum_l U_{kl}^{(\sigma)}\phi_l^{(\sigma)}$$

$$(\psi_k^{(\sigma)},\psi_{k'}^{(\sigma)})=\int_{x_1}^{x_2}\psi_k^{(\sigma)}(x)\psi_{k'}^{(\sigma)}(x)dx=\delta_{kk'}. \qquad \text{(B22)}$$

The minimizing function $\tau^{(\sigma)}(x)$ is a linear combination of the $\psi_k^{(\sigma)}(x)$:

$$\tau^{(\sigma)}(x)=\sum_{k=0}^{n-1}\left(\int_{x_1}^{x_2}\lambda_p(x')\psi_k^{(\sigma)}(x')dx'\right)\psi_k^{(\sigma)}(x)$$
$$\times\sum_{i,k=0}^{n-1}U_{ik}^{(\sigma)}U_{il}^{(\sigma)}\left(\int_{x_1}^{x_2}\lambda_p(x')\phi_k^{(\sigma)}(x')dx'\right)$$
$$\times\phi_l^{(\sigma)}(x). \qquad \text{(B23)}$$

Substituting (B23) into (B17) one obtains

$$\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}-\sum_{l=0}^{n-1}\alpha_l^{(\sigma)}\phi_l^{(\sigma)}(x)\right)^2dx$$
$$\leq\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}-\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right)^2dx \qquad \text{(B24)}$$

with

$$\alpha_l^{(\sigma)}=\sum_{i,k=0}^{n-1}U_{ik}^{(\sigma)}U_{il}^{(\sigma)}\int_{x_1}^{x_2}\lambda_p(x)\phi_k^{(\sigma)}(x)dx; \qquad \text{(B25)}$$

Eqs. (3.7) and (3.8) are just (B24) and (B25) for $\sigma=1$.

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Unsubtracted Dispersion Relations and the Renormalization of the Weak Axial-Vector Coupling Constants*

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Assuming that the equal-time commutation rules for the vector and axial-vector-current octets proposed by Gell-Mann are valid and that the divergence of the $\Delta S=0$, $\Delta I=1$ axial current is a strongly convergent operator obeying unsubtracted dispersion relations and dominated by low-frequency contributions, we derive a sum rule for the renormalization of the neutron axial β-decay constant G_A, by the strong interactions. The result agrees with that previously obtained from the assumption that the axial-current divergence is proportional to the pion field. The results are generalized to the strangeness-changing leptonic decays in the context of Cabibbo theory and generalized Goldberger-Treiman relations, and are used to compute the d/f ratio for the weak baryon axial-current coupling and an independent value of G_A.

I. INTRODUCTION

RECENT calculations of the effects of the strong interactions in renormalizing the axial-vector coupling constant in β decay,[1,2] $g_A=G_A/G_V$, give good agreement with the experimental value. These results were derived from the following three assumptions.

(1) The equal-time commutators of the spatial integrals of the time components of the hadron currents measured to first order in the weak and electromagnetic interactions, the "charges" obey the algebra of $SU(3)\times SU(3)$ as postulated by Gell-Mann *et al.*[3]

(2) The effective Hamiltonian for leptonic decay of the hadrons is a current-current interaction which couples the appropriate members vector and axial-vector current octets of the strongly interacting particles to the usual $\gamma_\mu(1-\gamma_5)$ current of the leptons through the simple combination $V_\mu\pm A_\mu$.[4]

(3) Partially conserved axial current (PCAC) hy-

* Work supported by the U. S. Atomic Energy Commission.
[1] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965).
[2] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).
[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, (1964). R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).
[4] The relation of the algebraic relations to the specification of a universal weak coupling of leptons and hadrons has been discussed by M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N. Y.) **30**, 360 (1964).

pothesis. The divergence of the $\Delta S=0$ axial-vector current is proportional to the pion field.[5–8]

$$\partial^\alpha A_\alpha{}^i(x) = -(i\sqrt{2}\mu^2 M g_A/g_{\pi n})\varphi_\pi{}^i(x)\,,\quad i=1,2,3\,,\quad \text{(I.1)}$$

where $\varphi_\pi{}^i(x)$ is the renormalized Heisenberg field of the π mesons, μ is the pion mass, M is the nucleon mass, $g_{\pi n}$ is the rationalized renormalized π-nucleon coupling constant.

In this article, we derive the sum rule for g_A [Eq. (II.14)] from a more general form of PCAC analogous to that used by Bernstein *et al.*[9] to derive the Goldberger-Treiman relation. We assume that the divergence of the axial current is a highly convergent operator whose matrix elements satisfy unsubtracted dispersion relations in the four-momentum transfer squared q^2. For small q^2 and certain values of the other variables in the problem, these matrix elements may be dominated by nearby poles.

These notions will be made more precise in the theoretical development of Sec. II where we treat the problem of formulating an unambiguous definition and region of validity for pole dominance of matrix elements of axial current divergence when these matrix elements are functions of more than one invariant variable. In Sec. III the results are generalized to include the $\Delta S=1$ leptonic decays in the context of Cabibbo theory[10] and generalized Goldberger-Treiman[11] relations. The numerical evaluation of the sum rules is discussed in Sec. IV. The results give $|g_A|\simeq 1.2$, and a d/f ratio similar to other estimates. While there are considerable numerical uncertainties in the evaluation of the sum rule for $\Delta S=1$ decays, the general consistency with Cabibbo theory is good and is strong evidence against the explanation of the suppression of $\Delta S=1$ decays relative to $\Delta S=0$ decays as a strong-interaction renormalization effect.

II. THEORETICAL DEVELOPMENT FOR $\Delta S=0$ DECAYS

As a starting point we consider a matrix element of the time-ordered product of two components of the axial-vector current between one-proton states of equal momentum

$$R_{\alpha\beta} = \int d^4x\, e^{iq\cdot x}\langle P|T(A_\alpha{}^+(x)A_\beta{}^-(0))|P\rangle\,,\quad \text{(II.1)}$$

with $A_\alpha{}^\pm = A_\alpha{}^1 \pm iA_\alpha{}^2$.

A^i $(i=1,2,3)$ are the isovector members of the octet of axial-vector currents. The tensor $R_{\alpha\beta}$ is related to second-order forward scattering of a proton by an axial-vector field. From general invariance arguments, $R_{\alpha\beta}$ can be written as a sum of kinematic second-rank tensors formed from combinations of p, q, and γ matrices evaluated between Dirac spinors, each multiplied by appropriate normalization factors and a Lorentz-invariant scalar function. In the usual manner, the arguments of the scalar functions are chosen as the invariant variables in the problem, which in this case are

$$p^2 = M^2\,,$$
$$q^2\,,$$
$$p\cdot q = M\nu\,,$$

or some linear combinations of these three. ν can be considered as the energy of the particle incident on the proton in the rest system of the proton, the "laboratory system."

From Eq. (II.1) we obtain

$$q^\alpha R_{\alpha\beta}(q^2,\nu) = i\int d^4x\, e^{iq\cdot x}[\langle P|T(\partial^\alpha A_\alpha{}^+(x)A_\beta{}^-(0))|P\rangle + \delta(x_0)\langle P|[A_0{}^+(x),A_\beta{}^-(0)]|P\rangle]\quad \text{(II.2)}$$

and

$$q^\alpha q^\beta R_{\alpha\beta} = \int d^4x\, e^{-iq\cdot x}[\langle P|T(\partial^\alpha A_\alpha{}^+(0)\partial^\beta A_\beta{}^-(x))|P\rangle - \delta(x_0)\langle P|[\partial^\alpha A_\alpha{}^+(0),A_0{}^-(x)]|P\rangle + \delta(x_0)iq^\beta\langle P|[A_0{}^+(0),A_\beta{}^-(x)]|P\rangle]\,.\quad \text{(II.3)}$$

We have integrated by parts to cast Eq. (II.3) in the given form. Equation (II.3) is the basic equation for deriving our results. The sum rule is obtained from Eq. (II.3) as a low-energy theorem[12] in the limit $q^2\to 0$, $\nu\to 0$. We proceed to evaluate the terms in Eq. (II.3) up to first order in ν. For fixed space-like or light-like q^2, the invariant functions in the decomposition of $R_{\alpha\beta}$ can be shown from the axioms of local field theory[13] to satisfy dispersion relations in ν. For $\nu\simeq 0$ the only singular term as $q^2\to 0$ is the one-neutron pole at $q^2+2M\nu=0$. That is, the contribution to $R_{\alpha\beta}(\nu\simeq 0, q^2=0)$ from the cuts is finite in this limit. Therefore, if we consider $q^\alpha q^\beta R_{\alpha\beta}$ and take the $\lim q^\alpha\to 0$, the cut contributions are at least of second order, and the finite

[5] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960).
[6] J. Bernstein, M. Gell-Mann, and L. Michel, Nuovo Cimento **16**, 560 (1960).
[7] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960).
[8] S. L. Adler, Phys. Rev. **137**, B1022 (1965); **139**, B1638 (1965).
[9] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960). See also Ref. 3.
[10] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).
[11] M. Goldberger and S. Treiman, Phys. Rev. **110**, 1178, 1478 (1958).
[12] Similar methods have been used to deduce consequences of field theoretical versions of conserved and partially conserved axial currents. Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962); Y. Nambu and E. Shrauner, *ibid.* **128**, 862 (1962). See also Refs. 1 and 8.
Note added in proof. After completing this work the author learned that similar formalisms have been applied to current algebras independently by V. Alessandrini, M. A. B. Bég, and L. Brown, Phys. Rev. (to be published), and C. Bouchiat and Ph. Meyer (private communication).
[13] N. N. Bogoliubov and V. D. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959), Chap. IX.

and first-order terms on the left side of Eq. (II.3) come entirely from the one-neutron Born term.

This Born term will give a factor g_A^2. On the right side of Eq. (II.3), the term involving the time-ordered product of the axial-current divergences will be related to the forward π-p scattering amplitude on the mass shell via analyticity in q^2. The assumed equal-time commutation rules determine the last term on the right. The combination of these various factors leads finally from Eq. (II.3) to a sum rule for g_A^2, Eq. (II.14).

In deriving Eq. (II.3) we have integrated by parts with respect to space and time variables and discarded surface terms. The spatial surface terms give no contribution if we use wave packets. The temporal surface terms at $t=\pm\infty$ vanish in the same manner if all the intermediate states inserted in our expressions lead to oscillating time behavior, that is, if all intermediate states have different energy from the one-proton state.[14] For $q^2=0$, the only dangerous term comes from the one-neutron intermediate state; in our calculation we shall explicitly assume the neutron mass M_n to be different from the proton mass M_p. In final result we let $M_n=M_p$ and assume charge independence; the answer is insensitive to the order in which we let the various small quantities in the problem tend to zero. This procedure of keeping $M_n\neq M_p$ until the end of the calculation will have the additional advantage of allowing the derivation to be generalized immediately to renormalization of the strangeness changing decays (Sec. III), where the Born terms involve nucleon-hyperon transitions and the masses are manifestly unequal.

For reference we note that the matrix element of the axial-vector current between proton and neutron is given by

$$\begin{aligned}\langle P(p_1)|A_\alpha{}^+(x)|N(p_2)\rangle &= (2\pi)^{-3}[M_nM_p/(E_nE_p)]^{1/2}e^{iq\cdot x}g_A\\ &\quad\times\bar{u}_p(p_1)[\gamma_\alpha\gamma_5F_1(q^2)-q_\alpha\gamma_5F_2(q^2)]\tau^+u_n(p_2). \quad \text{(II.4)}\\ q^\alpha&=p_1{}^\alpha-p_2{}^\alpha;\quad F_1(0)=1.\end{aligned}$$

$\tau^+=\frac{1}{2}(\tau_1+i\tau_2)$ is a nucleon isotopic spin matrix.

If the effective Hamiltonian has $V-A$ coupling, then g_A equals G_A/G_V, the ratio of axial-vector to vector coupling constants measured in ordinary β decay. From Eq. (II.4),

$$\begin{aligned}\langle P|\partial^\alpha A_\alpha{}^+|N\rangle &= i(2\pi)^{-3}[M_nM_p/(E_nE_p)]^{1/2}\\ &\quad\times e^{iq\cdot x}g_AD(q^2)\bar{u}_p(p_1)\gamma_5\tau^+u_n(p_2). \quad \text{(II.5)}\\ D(q^2)&=(M_n+M_p)F_1(q^2)-q^2F_2(q^2).\end{aligned}$$

The assumption that $D(q^2)$ obeys an unsubtracted dispersion relation and that $D(0)$ is dominated by the one-pion pole at $q^2=\mu^2$ leads to a derivation of the Goldberger-Treiman (G-T) relation,

$$f_\pi=-\sqrt{2}g_AM/g_{\pi n}. \quad \text{(II.6)}$$

f_π is the decay constant of the charged pion defined by

$$\langle 0|\partial^\alpha A_\alpha{}^+(0)|\pi^-\rangle=i(2\pi)^{-3/2}(2E_\pi)^{-1/2}\mu^2f_\pi. \quad \text{(II.7)}$$

With these definitions the Born contribution to $R_{\alpha\beta}$ can be evaluated as

$$\begin{aligned}q^\alpha q^\beta R_{\alpha\beta}{}^{\text{Born}} &= N_pg_A{}^2[(M_n+M_p+\nu)F_1{}^2(q^2)-2F_1(q^2)D(q^2)\\ &\quad+D^2(q^2)(M_p-M_n+\nu)/(q^2+M_p{}^2-M_n{}^2+2M_p\nu)].\\ N_p&=(2\pi)^{-3}M_p/E_p. \quad \text{(II.8)}\end{aligned}$$

The last term on the right-hand side of Eq. (II.3) is determined from the assumed equal-time commutation rules:

$$\begin{aligned}\delta(x_0)[A_0{}^+(0),A_\beta{}^-(x)]&=2V_\beta{}^3(x)\delta^{(4)}(x)\\ &\quad+(\text{more singular terms}). \quad \text{(II.9)}\end{aligned}$$

$V_\beta{}^3$ is the third component of the total isotopic-spin current. We generalize the $SU(3)\times SU(3)$ algebra to include commutators of time components of currents with space components.

The more singular terms of the equal-time commutator involve derivatives of delta functions.[15] In the integral of Eq. (II.3a) these terms give polynomials in q. Since the results of interest will be obtained in the $\lim q^\alpha\to 0$, the derivatives of delta functions do not contribute in this calculation. From the delta-function term in Eq. (II.9) one has

$$\int d^4x\, e^{iq\cdot x}\delta(x_0)iq^\beta \times\langle P|[A_0{}^+(0),A_\beta{}^-(x)]|P\rangle=iN_p\nu. \quad \text{(II.10)}$$

Returning to Eq. (II.3) we have still to evaluate the first two terms on the right side. The equal-time commutator $[\partial^\alpha A_\alpha{}^+(0),A_0{}^-(\mathbf{x},0)]$ is presumably proportional to $\delta^{(3)}(\mathbf{x})$. This leads to a finite q-independent term in Eq. (III.3a),

$$C=\int d^4x\, e^{-iq\cdot x}\delta(x_0) \times\langle P|[\partial^\alpha A_\alpha{}^+(0),A_0{}^-(x)]|P\rangle. \quad \text{(II.11a)}$$

Let the first term on the left side of (II.3) be denoted by

$$R(q^2,\nu)=\int d^4x\, e^{-iq\cdot x} \times\langle P|T(\partial^\alpha A_\alpha{}^+(0),\partial^\beta A_\beta{}^-(x))|P\rangle. \quad \text{(II.11b)}$$

It is straightforward to show that $R(0,0)=C$. Thus, after evaluating $R(q,\nu)$, we need keep only terms pro-

[14] This is similar to the problem discussed by S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento (to be published), who derive dispersion sum rules analogous to those presented here by starting explicitly with the equal-time commutators for integrated charges.

[15] J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

proportional to ν.[15a] Since R involves matrix elements of the divergence of the axial current, we assume that for fixed ν, R satisfies an unsubtracted disperions relation in q^2. For $\nu \simeq 0$, $q^2=0$, we assume that R is dominated by nearby singularities. These are the one-neutron Born pole at $q^2+M_p^2-M_n^2+2M_p\nu=0$ and the one-pion poles at $q^2=\mu^2$.

There is, however, a possible ambiguity in defining the residues of the poles.[16] In this problem, the independent variables may be taken as q^2 and $\sigma=\nu+aq^2$, and we can disperse in q^2 with $\sigma=0$. As we vary the constant, a, different parts of the total dispersion relation for $R(0,0)$ are associated with the residues of the poles and the integral over the continuum. The problem is to choose a to give the best pole approximation, to put as much as possible of the contribution to $R(0,0)$ into the nucleon and pion poles and make the corrections due to the integral over the branch cut, which will be neglected, as small as possible.

In the context of dominance by nearby singularities there is a natural, if somewhat arbitrary, criterion for a best pole approximation, namely, choose a to keep the threshold of the cut as far from the poles as possible. The locations of the singularities in the Req^2-Reν plane which follow from perturbation theory are plotted in Fig. 1. For any fixed ν, R satisfies a dispersion relation in q^2. For q^2 fixed and not too time-like, R should obey a dispersion relation in ν with singularities on the Reν axis. The anomalous thresholds come from the reduced graph shown in Fig. 2. From Fig. 1, it is seen that the criterion given above leads to the value $a=0$, or $\sigma=\nu=0$, as the best choice of the fixed second variable for writing a pole-dominated dispersion relation for $R(0,0)$. For $\nu=0$, the cut has an anomalous threshold at $q^2 \simeq 8\mu^2$.

The choice of $\nu=0$, $(a=0)$, can be justified also by general symmetry arguments. The thresholds are determined by the masses of intermediate states in the s and u channels, where s, t, u are the usual Mandelstam variables. Here $t=0$, so s and u are related to q^2 and ν by $s=M^2+q^2+2M\nu$, $u=M^2+q^2-2M\nu$. For the purpose of specifying intermediate states in R, both s and u channels look like π-nucleon scattering and have the same intermediate states available. For a particular choice of a, denote the residue of the pion pole by $\bar{R}(\nu=-a\mu^2/M,\mu^2)$. It follows from the statements above that $\bar{R}$ is an even function of a. To retain the symmetry between the s and u channels one should disperse in q^2 with $a=0$.

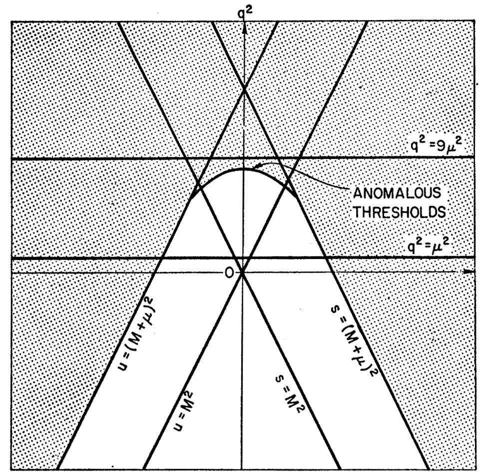

FIG. 1. Singularities of $R(q^2,\nu)$ in the Req^2–Reν plane which follow from perturbation theory.

For fixed $q^2 \simeq 0$, $R(q^2,\nu)$, which resembles a forward-scattering amplitude, should satisfy a dispersion relation in ν, and we can separate R into contributions from the Born and continuum terms of the ν dispersion relation. Thus, for small q^2, ν

$$R(q^2,\nu)=iN_p[g_A^2D^2(q^2)(M_p-M_n+\nu)/ (q^2+M_p^2-M_n^2+2M_p\nu)+\tilde{R}(q^2,\nu)]. \quad \text{(II.12)}$$

This Born term cancels the singular term of $q^\alpha q^\beta R_{\alpha\beta}{}^{\text{Born}}$, Eq. (II.8), and clearly satisfies an unsubtracted dispersion relation in q^2. Therefore, $\tilde{R}$ has no one-neutron pole and must itself obey an unsubtracted dispersion relation in q^2. $\tilde{R}$ has double and single one-pion poles at $q^2=\mu^2$ and a cut starting at $q^2 \simeq 8\mu^2$. In the spirit of our approach, the pole contributions dominate for $q^2=0$ and the integral over the branch cut is neglected. In the same manner it will be shown that the single-pole contributions are small. The result from keeping only the double-pion pole term is

$$\tilde{R}(0,\nu)=-f_\pi^2\tilde{T}_{\pi^-p}(\mu^2,\nu)\,, \quad \text{(II.13)}$$

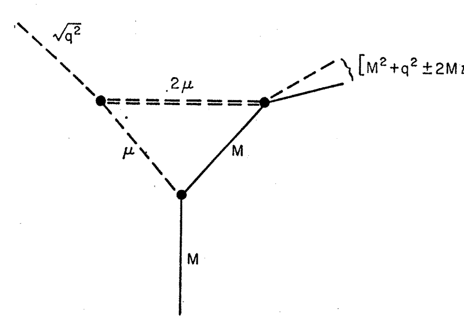

FIG. 2. Reduced graph producing the anomalous threshold shown in Fig. 1. Invariant masses of the external and internal lines are indicated for the s or u channel.

[15a] If $C=0$, the ν-independent terms yield Adler's consistency condition (see Ref. 8) on the even π-nucleon forward scattering amplitude at the crossing point. An analogous equal-time commutator appears in Adler's field theoretic derivation if the standard reduction techniques are used to remove the pions from the state vectors before continuing off the mass shell. The numerical success of the consistency condition can be taken as evidence that C must be small. The author acknowledges discussions with J. D. Bjorken on this subject.

[16] This point has been emphasized to the author in correspondence with S. L. Adler.

where $\tilde{T}_{\pi^- p}(\mu^2,\nu)$ is the invariant forward π^- proton scattering amplitude on the mass shell and with the Born terms subtracted. From the usual dispersion relations[17] for the forward π-nucleon scattering amplitude,

$$\tilde{R}(0,\nu)=-(f_\pi^2/\pi)\int_\mu^\infty d\nu'[A_{\pi^- p}(\mu^2,\nu')/(\nu'-\nu)+A_{\pi^- p}(\mu^2,-\nu')/(\nu'+\nu)]. \quad \text{(II.13a)}$$

From unitarity and crossing symmetry,

$$A_{\pi^- p}(\nu)=\operatorname{Im}T_{\pi^- p}(\nu)=k\sigma_{\pi^- p}(\nu),$$
$$A_{\pi^- p}(-\nu)=k\sigma_{\pi^+ p}(\nu),\quad \nu>\mu,$$

where σ's are total cross sections and k is the magnitude of the pion three-momentum in the laboratory system.

Recalling that we are interested in equating the terms $0(\nu)$ as $\nu\rightarrow 0$ in Eq. (II.3) we collect the results from Eqs. (II.8), (II.10), and (II.13a). Substituting we obtain the sum rule

$$g_A^2=1-(f_\pi^2/\pi)\int_\mu^\infty \frac{kd\nu}{\nu^2}[\sigma_{\pi^- p}(\nu)-\sigma_{\pi^+ p}(\nu)]. \quad \text{(II.14)}$$

If we eliminate f_π by the G-T relation, we can determine g_A from strong-interaction cross sections only.

$$\frac{1}{g_A^2}=1+\frac{2M_n^2}{\pi g_{\pi n}^2}\int_\mu^\infty \frac{kd\nu}{\nu^2}[\sigma_{\pi^- p}(\nu)-\sigma_{\pi^+ p}(\nu)]. \quad \text{(II.14a)}$$

We discuss the neglected single-pion-pole terms. These can be written as

$$\tilde{R}_{\text{s.p.p.}}(0,0)=f_\pi r(\mu^2,0)+\text{H.c.} \quad \text{(II.15)}$$

As an analytic function of q^2, r has no pion-pole

$$r(\mu^2,0)=\frac{1}{\pi}\int_{\sigma_0^2}^\infty \frac{\operatorname{Im}r(\sigma^2,0)}{\sigma^2-\mu^2}d\sigma^2. \quad \text{(II.16)}$$
$$\operatorname{Im}r\propto\sum_{m\neq\pi}\langle P\pi|j_P|m\rangle\langle m|\partial^\alpha A_\alpha|0\rangle.$$

If we considered the matrix element for forward creation of a pion from a nucleon by scattering of the axial-current divergence, pole dominance at $q^2=0$ (off-mass-shell pions) would imply

$$|f_\pi T_{\pi N}(\mu^2,0)|\gg|r(0,0)|.$$

Since $\sigma_0^2\sim 8\mu^2$, $r(\mu^2,0)\simeq r(0,0)$, and $r(\mu^2,0)$ is similarly unimportant compared to $f_\pi T_{\pi N}(\mu^2,0)$.

III. SUM RULES FOR $\Delta S=1$ DECAYS

The results of the preceding section can be applied to the $\Delta S=1$ decays in the context of the Cabibbo theory of weak interactions if one accepts the generalization of the G-T relation to K-meson pole dominance for the divergence of the strangeness-changing axial currents. We briefly review the Cabibbo theory of leptonic decays.

The $SU(3)\times SU(3)$ commutation rules fix the relative scale of the vector and axial-vector currents. The combinations,

$$Q_\pm{}^i=\int\tfrac{1}{2}(V_0{}^i(x)\pm A_0{}^i(x))d^3x,\quad i=1,\cdots,8,$$

form two mutually commuting octets of chiral charges. The hadron current which couples to the leptons and is measured in decay processes, is a component of one of these chiral octets

$$J_\mu{}^{\text{had}}=\cos\theta(V_\mu{}^{1+i2}-A_\mu{}^{1+i2})+\sin\theta(V_\mu{}^{4+i5}-A_\mu{}^{4+i5}). \quad \text{(III.1)}$$

The Cabibbo angle θ, which determines the suppression of the $\Delta S=1$ decays relative to the $\Delta S=0$ decays is an input parameter to the structure of the effective weak Hamiltonian. The problem of whether the right-handed or left-handed current appears is determined from experiment. In the limit of exact $SU(3)$ symmetry the vector currents are unrenormalized, and their matrix elements between one baryon states have only f-type coupling. For the corresponding matrix elements of the axial current, we have in the $SU(3)$ limit

$$\begin{aligned}\langle B^i(p)|A_\mu{}^k(0)|B^j(p)\rangle &=g_A{}^{B_iB_j}\bar{u}(p)\gamma_\mu\gamma_5 u(p)\\ &=g_A[(1-\alpha)f_{ijk}+\alpha d_{ijk}]\bar{u}(p)\gamma_\mu\gamma_5 u(p),\\ &(i,j,k=1,\cdots,8)\end{aligned} \quad \text{(III.2)}$$

where f and d are the usual Gell-Mann coupling coefficients, and we have neglected trivial kinematic factors.

Empirically[11,18] this description gives a satisfactory fit to the presently available data on leptonic decays even though $SU(3)$ is a badly broken symmetry. The origin of the small renormalization for the vector currents is suggested by the theorem of Ademollo and Gatto,[19] which shows that there is no renormalization of the vector currents to first order in the symmetry breaking because the space integrals of the time components of the vector currents are the generators of $SU(3)$. This theorem is not applicable to the axial-vector currents.

In order for (III.2) to be valid both the axial currents and the one-baryon states must transform as octets. If we believe that the commutation rules of

[17] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1337 (1957).

[18] N. Brene, B. Hellesen, and M. Roos, Phys. Letters **11**, 344 (1964); W. Willis *et al.*, Phys. Rev. Letters **13**, 291 (1964).

[19] M. Ademollo and R. Gatto, Phys. Rev. Letters **13**, 264 (1964); C. Bouchiat and Ph. Meyer, Nuovo Cimento **34**, 1122 (1964).

the vector and axial-vector currents are unchanged by the $SU(3)$-breaking interactions, then the axial currents transform exactly as an irreducible octet tensor even in the presence of symmetry breaking. The experimental success of Eq. (III.2) in describing the axial matrix elements suggests then that the one-particle states may be nearly pure octet despite the large mass splittings due to $SU(3)$ breaking.[20]

The divergence of the axial current, however, does not transform like a pure octet tensor if $SU(3)$ is broken. Indeed, the Cabibbo theory gives explicit $SU(3)$ violation of the matrix elements of $\partial^\alpha A_\alpha^i$ due to the mass splittings. The generalization of the G-T relations to the strangeness changing decays implies that the meson-baryon couplings have the same d/f ratio as the axial current–baryon vertex but the meson couplings show explicit dependence on the physical baryon masses. The results are the same as those of Freund and Nambu[21] who assumed that the currents are conserved but the states are not pure.

To check the consistency of this picture we obtain sum rules for the $\Delta S=1$ decays. The procedure is exactly the same as in Sec. II. Start with matrix elements of $T(A_\alpha^{4+i5}(x)A_\beta^{4-i5}(0))$. Consider matrix elements of this time-ordered product between both one-neutron states and one-proton states, respectively. In the first case the Born terms are due to a Σ^- pole. In the second both Σ^0 and Λ^0 contribute. These give the sum rules[22]

$$1=(g_A{}^{n\Sigma^-})^2+\frac{f_K{}^2}{\pi}\int_{\nu_0}^\infty\frac{d\nu}{\nu^2}[A_{K^-n}(\nu)-A_{K^+n}(\nu)],\qquad\text{(III.3a)}$$

$$2=(g_A{}^{p\Sigma^0})^2+(g_A{}^{p\Lambda^0})^2+\frac{f_K{}^2}{\pi}\int_{\nu_0}^\infty\frac{d\nu}{\nu^0}[A_{K^-p}(\nu)-A_{K^+p}(\nu)].\qquad\text{(III.3b)}$$

The A's are absorptive parts of forward scattering amplitudes. For $\nu>M_K$ they are proportional to total cross sections, but the K-nucleon dispersion relations have cuts in unphysical region owing to the hyperon-pion channels. f_K is a K-meson decay constant defined in analogy to Eq. (II.7). In Cabibbo theory, $f_K=f_\mu$, and the g_A's are given by Eq. (III.2). Making these substitutions in Eq. (III.3) and using the G-T relation, one obtains

$$\frac{1}{g_A{}^2}=(1-2\alpha)^2+\frac{2M_N{}^2}{\pi g_{\pi n}{}^2}\int_{\nu_0}^\infty\frac{d\nu}{\nu^2}[A_{K^-n}(\nu)-A_{K^+n}(\nu)],\qquad\text{(III.4)}$$

$$\frac{1}{g_A{}^2}=(1-2\alpha+\tfrac{4}{3}\alpha^2)+\frac{M_n{}^2}{\pi g_{\pi n}{}^2}\int_{\nu_0}^\infty\frac{d\nu}{\nu^2}[A_{K^-p}(\nu)-A_{K^+p}(\nu)].\qquad\text{(III.5)}$$

Evaluating the dispersion integrals in Eq. (III.4), one determines g_A, and α or the d/f ratio. By considering also the commutator of the $\Delta S=1$, $\Delta Q=0$ axial currents and taking all possible diagonal matrix elements of the three canonical commutators between baryon states, one can derive six more sum rules. They involve unmeasurable scattering processes, but in the limit of exact $SU(3)$, they can each be shown to be equivalent to Eqs. (II.14), (III.3a), or (III.3b).

IV. NUMERICAL RESULTS AND CONCLUSIONS

A. $\Delta S=0$ Axial Current

The sum rule for the $\Delta S=0$ axial current [Eq. (II.14a)] is evaluated[23] using tabulated values[24] of the experimental pion-nucleon cross sections to $k=5$ BeV/c. The very high-energy data is fitted with the exponential form[25]

$$\sigma_{\pi^+p}(\nu)-\sigma_{\pi^-p}(\nu)=b\nu^{-0.7}.\qquad\text{(IV.1)}$$

The convergence of the integral in Eq. (II.14a) depends on the validity of the Pomeranchuk theorem but the numerical result is insensitive to the details of the high-energy behavior. The result is

$$1-1/g_A{}^2=0.246\qquad\text{(IV.2a)}$$

or

$$|g_A|=|G_A/G_V|=1.15.\qquad\text{(IV.2b)}$$

The best value calculated from experimental β-decay measurements is[26]

$$(G_A/G_V)_{\text{exp}}=-1.18\pm0.025.\qquad\text{(IV.3)}$$

The theoretical uncertainties in Eq. (II.14) are due mainly to the continuum terms that have been discarded. This approximation is used for $\tilde{R}$, Eq. (II.13), and in deriving the Goldberger-Treiman relation. From the comparison of the G-T relation with experiments, the errors inherent in this type of approximation may be about 20% for the right-hand side of Eq. (III.2a)

[20] This implies that the off-diagonal matrix elements of the $SU(3)$-breaking interaction between initially nondegenerate states are small though the diagonal matrix elements may be large. In a naive potential picture this means that the second-order effects on the mass splittings will be very small.

[21] P. G. O. Freund and Y. Nambu, Phys. Rev. Letters **13**, 221 (1964).

[22] These sum rules have been discussed independently by D. Amati, C. Bouchiat, and J. Nuyts, Phys. Letters **19**, 59 (1965). Equation (III.3b) has also been evaluated by C. A. Levinson and I. J. Muzinich, Phys. Rev. Letters **15**, 715 (1965); L. K. Pandit and J. Schechter, Phys. Letters **19**, 56 (1965).

[23] We use the value $g_{\pi n}{}^2/4\pi=14.6\pm0.03$ given by J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).

[24] C. Hohlen, C. Ebel, and J. Giesecke, Z. Physik **180**, 430 (1964).

[25] G. von Dardel, D. Dekkers, R. Mermod, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters **8**, 173 (1962).

[26] C. P. Bhalla, Phys. Letters **19**, 691 (1966).

but only about 5% for g_A. For example, if we evaluate Eq. (II.14) using f_π as determined from experimental measurements of the lifetime of the charged pion,[27] the sum rule yields

$$|g_A|=1.21. \qquad \text{(IV.2b')}$$

Errors in the calculated value of g_A from uncertainties in the exact high-energy behavior of π-proton cross sections and the best experimental value of $g_{\pi N}$ are about 1%. It is the effect of the (3,3) resonance in Eq. (II.13) that makes $|g_A|>1$. In fact, the (3,3) resonance contribution alone gives $|G_A/G_V|\sim 1.35$, and the higher energy $I=\frac{1}{2}$ resonances reduce this value. Thus, the (3,3) resonance does not saturate the sum rule.

From Fig. 1 some general conclusions can be drawn about the expected domain of validity of pion-pole dominance in this problem. From the postulated equal-time commutation rules for the weak charges, one can obtain directly[1,14] a sum rule for g_A

$$g_A{}^2=1+\int_{\mu+\mu^2/2M}^{\infty}\frac{d\nu}{\nu^2}[D^+(0,\nu)-D^-(0,\nu)] \qquad \text{(IV.4)}$$

with

$$D^\pm(0,\nu)=\text{Im}R(0,\pm\nu)\,, \quad \nu>\mu+\mu^2/2M. \qquad \text{(IV.5)}$$

$D^\pm$ can be measured in high-energy neutrino reactions when the lepton is produced in the forward direction.[28]

This result is obtained by writing a dispersion relation in ν for $R(0,0)$ in Eq. (II.3). The result, Eq. (II.14), is obtained by first taking the pole approximation in q^2 for $R(0,0)$ and then dispersing the residue of the pion pole in ν. For the integrand in Eq. (IV.4), however, one cannot justify directly replacing the matrix element of $D^\pm$ by their pion-pole contributions.

This follows from Fig. 1 where it is seen that in the integration region for ν of Eqs. (II.14a) and (IV.4), the threshold of the cut in q^2 moves past the one-pion pole. That is, for physical ν there is no isolated pion pole, and multiparticle thresholds in q^2 are as close to $q^2=0$ as the one-pion state.[29]

Nevertheless, as $\nu\to\pm\infty$, a return to pion-pole dominance can be justified for $D^\pm(0,\nu)$. The position of the singularity from a state of invariant mass $M_j{}^2$ in the s or u channel is given by

$$q^2\pm 2M\nu=M_j{}^2-M^2.$$

Thus, as $|\nu|\to\infty$, the threshold for any state of finite mass moves off to $q^2=\pm\infty$. The only singularities remaining near $q^2=0$ are the one-pion pole and very heavy inelastic states. The normal threshold at $q^2=9\mu^2$ remains fixed. In the spirit of PCAC, with $\partial^\mu A_\mu$ assumed to be a highly convergent operator satisfying unsubtracted dispersion relations one expects that very heavy states are unimportant in the spectral functions for $\partial^\mu A_\mu$. Therefore, as $|\nu|\to\infty$, the only important contribution near $q^2=0$ comes from the pion pole at $q^2=\mu^2$. This leads to a derivation of Adler's proposed tests of PCAC in high-energy neutrino reactions.[28]. The preceding argument explicitly uses the PCAC hypothesis in dispersion theory as a physical assumption about small numerators as well as a geometrical statement about large denominators or far-away singularities. Faith in such arguments is needed also to justify K-meson pole dominance of the divergence of $\Delta S=1$ axial current.

B. Strangeness-Changing Currents

The numerical evaluation of the sum rules for the $\Delta S=1$ currents, Eq. (III.4), is complicated by the presence of an unphysical region in the $\bar{K}$-nucleon channel extending below the elastic threshold. Above the threshold the integral can be expressed in terms of total cross sections as in the π-nucleon case. The sum rules can be written as

$$1/g_A{}^2=(1-2\alpha)^2+2I(Kn)\,, \qquad \text{(IV.6a)}$$

$$1/g_A{}^2=(1-2\alpha+\tfrac{4}{3}\alpha^2)+I(Kp)\,. \qquad \text{(IV.6b)}$$

The contributions of the different energy regions to the dispersion integrals for $I(Kn)$ and $I(Kp)$ are summarized in Table I and discussed below.

(a) Unphysical region. $\nu<M_K$

We assume that the only important contributions come from the $I=1$ p-wave resonance $Y_1{}^*(1385)$, and the continuation of the $I=0, 1$ S waves below threshold. The $Y_1{}^*$ is a member of the decuplet of spin-$\frac{3}{2}$ resonances. To estimate the $Y_1{}^*$ contribution to the K-nucleon integrals we assume a phenomenological B^*BM (resonance-baryon-meson) coupling

$$\mathfrak{L}_{\text{eff}}(x)=-\lambda\bar{\psi}_\mu(x)\psi(x)\partial^\mu\varphi(x)\,, \qquad \text{(IV.7)}$$

where the $\psi_\mu(x)$, $\psi(x)$, $\varphi(x)$ are the field operators for the spin-$\frac{3}{2}$ resonance, spin-$\frac{1}{2}$ baryon, and pseudoscalar meson, respectively. The resonance is described by the Rarita-Schwinger formalism.[30]

The decay width for a resonance is related to the effective coupling constant λ by

$$\Gamma=(k^3\lambda^2/24\pi)[(M_{B^*}+M_B)^2-\mu_M{}^2]/M_{B^*}{}^2\,, \qquad \text{(IV.8)}$$

where k is the momentum of the baryon and meson in the center-of-mass system. M_{B^*}, M_B, and μ_M are the masses of the resonance baryon and meson, respectively.

TABLE I. Numerical contributions to the dispersion integrals in the sum rules for the $\Delta S=1$ axial current.

	Unphysical region		Physical region		
	$Y_1{}^*(1385)$	S waves	$0<P_k{}^{\text{lab}}$ <6 BeV/c.	$P_k{}^{\text{lab}}$ >6 BeV/c.	Total
$I(Kp)$	−0.010	0.106	0.205	0.048	0.349 (Kim)
	−0.010	0.126	0.198	0.048	0.362 (Sakitt)
$I(Kn)$	−0.021	0.055	0.134	0.027	0.195 (Kim)
	−0.021	0.055	0.107	0.027	0.168 (Sakitt)

[27] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. 36, 977 (1964).
[28] S. L. Adler, Phys. Rev. 135, B963 (1964).
[29] In a field-theoretic treatment these departures from simple pole dominance might be identified with the low-energy threshold corrections. See Ref. 1.
[30] W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

We assume that the coupling constants, λ, for all the B^*BM couplings are related by $SU(3)$. The $Y_1^*N\bar{K}$ coupling is computed from the observed width[27] for $\Delta(1235)\rightarrow N+\pi$. Then the Y_1^* is inserted as a pole in the $\bar{K}N$ scattering amplitudes. Various estimates[31] of the effects of $SU(3)$ breaking on the B^*BM couplings indicate that the Y_1^* contribution is uncertain within a factor of 2.

To evaluate the $I=0$ and $I=1$ the S-wave contribution below threshold we use the complex-scattering length, zero-effective range K-matrix formalism of Dalitz and Tuan.[32] Recent experiments on low-energy K^- proton scattering by Kim and Sakitt *et al.*[33] give very similar solutions for the two complex scattering lengths. Each of their solutions shows resonance behavior below threshold in the $I=0$ channel at the mass of the $Y_0^*(1405)$.

Both sets of scattering lengths were used to evaluate the S-wave contributions to the dispersion integrals below threshold. The integrals were truncated at the Σ-π threshold.

(*b*) *Physical region* $\nu>M_K$

In the physical region for KN and $\bar{K}N$ scattering, the integrands in the dispersion integrals can be expressed in terms of total cross sections. For low energies, $P_K{}^{\mathrm{lab}}<0.3$ BeV/c, we use the $\bar{K}N$ cross sections given by the Kim and Sakitt solutions. The KN cross sections at low energy have been measured; they are small and smoothly varying. We have collected the available experimental data[34] for the integrals up to $P_K{}^{\mathrm{lab}}=6$ BeV/c. For $P_K{}^{\mathrm{lab}}>6$ BeV/c, the experimental cross-section differences[35] occurring in Eq. (III.4) can be reasonably well-fitted with an exponential form as in Eq. (IV.1). For KN cross sections we use[36]

$$\begin{aligned}\sigma_{K^-n}-\sigma_{K^+n}&=b_n\nu^{-0.5},\\ \sigma_{K^-p}-\sigma_{K^+p}&=b_p\nu^{-0.5}.\end{aligned}\qquad\text{(IV.9)}$$

The contributions from the asymptotic region, $P_K{}^{\mathrm{lab}}>20$ BeV/c, are about 10% of the total integrals.

The possible numerical errors for these integrals are estimated to be $\sim$20%. This is in addition to errors introduced by the pole dominance approximation. Equations (IV.6a) and (IV.6b) can be solved simultaneously for α and g_A. There are two solutions to the resulting quadratic equation for α. One solution gives $\alpha\simeq0$ and $g_A\simeq0.85$ and is discarded. With the indicated errors the other solution is[37]

$$\begin{aligned}\alpha&=0.75\pm0.10,\\ |g_A|&=1.28\pm0.10.\end{aligned}\qquad\text{(IV.10)}$$

The solutions for the two sets of results in Table I are closer than the statistical errors indicated above. Correcting the I's for the errors in the G-T relation does not affect the solution[38] for α but gives

$$|g_A|=1.20\pm0.10.\qquad\text{(IV.10')}$$

The consistency with the value for g_A obtained from the $\Delta S=0$ sum rule is quite good. The solution for α agrees within error limits with the best fits of Cabibbo theory to experimental data on semileptonic decays.[18] These give

$$\begin{aligned}\alpha&=0.67\pm0.03\ \text{(Brene \textit{et al.})}\\ &=0.63\pm0.06\ \text{(Willis \textit{et al.})}.\end{aligned}\qquad\text{(IV.11)}$$

These results yield a consistent theoretical picture of low-energy semileptonic processes with only two imput parameters for the weak interactions, G_V and the Cabibbo angle θ. One should expect, however, that future precise measurements of semileptonic decays will show departures from complete $SU(3)$ symmetry of the matrix elements of the currents. The evidence is quite strong, however, that the suppression of the $\Delta S=1$ decays relative to $\Delta S=0$ decays by $\tan\theta$ lies in the structure of the weak interactions and cannot be explained as a strong-interaction renormalization effect.[39]

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[34] See Refs. 4 and 5 of R. Good and N. Xuong, Phys. Rev. Letters 14, 191 (1965); also, G. von Dardel, D. H. Frisch, R. Mermod, R. H. Milburn, P. A. Piroué, M. Vivargent, G. Weber, and K. Winter, Phys. Rev. Letters 5, 333 (1960); P. L. Bastien, J. P. Berge, O. I. Dahl, M. Ferro-Luzzi, J. Kirz, D. H. Millen, J. J. Murray, A. H. Rosenfeld, R. D. Tripp, and B. Watson, *Proceedings of the International Conference on High-Energy Physics, Geneva, 1962*, edited by J. Prenkti (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 373; W. F. Baker, R. L. Cool, E. W. Jenkins, T. F. Kycia, R. H. Phillips, A. L. Read, K. F. Riley, and H. Ruderman, *Proceedings of the Sienna International Conference on Elementary Particles* (Società Italiana di Fisica, Bologna, Italy, 1963), p. 634; V. J. Stenger, W. E. Slater, D. H. Stork, H. K. Ticho, G. Goldhaber, and S. Goldhaber, Phys. Rev. 134, B1111 (1964).

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[36] R. J. N. Phillips and W. Rarita, Phys. Rev. 139, B1336 (1965).

[37] For comparison we note the independent results, see Ref. 22.

$\alpha=0.73$ (D. Amati, C. Bouchiat and J. Nuyts);
$\alpha=0.63$ (C. A. Levinson and I. J. Muzinich).

[38] The value obtained for α is insensitive to these corrections because the contributions from the dispersion integrals nearly cancel when we take the difference between (IV.6a) and (IV.6b) to solve for α. Either of these equations can also be solved simultaneously with (II.14a) to obtain independent values for α. These results are generally consistent, but the precise answers are much more sensitive to the errors discussed above.

[39] R. Oehme, Ann. Phys. (N. Y.) 33, 108 (1965).