Bound on the Coupling Constant from Unitarity and Analyticity*

H. D. Dahmen† and A. S. Reiner‡
Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 12 November 1965)

Unitarity of the S matrix is used to bound the sum of the pole term and the left-hand contribution appearing in a dispersion relation for a partial-wave scattering amplitude. An upper bound for the coupling constant is then constructed for the case of an arbitrary distribution of any finite number of sign changes in the left-hand cut discontinuity.

1. INTRODUCTION

In the following we address ourselves to the question whether unitarity and simple analytic structure of the S matrix are sufficient to restrict the coupling constant g.

The simplest example that comes to mind is, of course, the Lee model.[1] There a ghost state appears provided g^2 exceeds a critical value dependent on a cutoff function. Simultaneously one observes a violation of unitarity of the S matrix.

Unitarity as a minimal requirement to bound the coupling constant has further been used by Ruderman and Gasiorowicz[2] for both potential and static field theories. Cutoff functions have been introduced which, in contradistinction to the Lee model, are not necessary to insure convergence. Their existence is an assumption, strong enough to bound g.

Attempts to bound g for an unrestricted field theory were made by Geshkenbein and Joffe[3] and Meiman.[4] The function studied by those authors is not the scattering amplitude but the propagator D of a particle. Using complex analysis a bound for g has been given in terms of the masses of the particles at the simplest vertex. The necessary condition in their derivation is the absence of a zero in the propagator or, alternatively, of a pole in the vertex function Γ. Geshkenbein and Joffe assured this condition on the conjecture that the zero in D (or the pole in Γ) would make itself felt in the scattering amplitude. This conjecture, however, has been disproved for both nonrelativistic[5] and for relativistic theories.[6]

* This work has been supported by funds provided by the U. S. Atomic Energy Commission under contract AT(30-1)-2098.

† On leave from Institut für Theoretische Kernphysik, Technische Hochschule, Darmstadt, Germany.

‡ On leave from Weizmann Institute of Science, Rehovoth, Israel.

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954); G. Källén and W. Pauli, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, No. 7 (1955).

[2] M. A. Ruderman and S. Gasiorowicz, Nuovo Cimento **8**, 861 (1958).

[3] B. V. Geshkenbein and B. L. Joffe, Zh. Eksperim. i Teor. Fiz. **44**, 1211 (1963) [English transl.: Soviet Phys.—JETP **17**, 820 (1963)].

[4] N. N. Meiman, Zh. Eksperim. i Teor. Fiz. **44**, 1228 (1963) [English transl.: Soviet Phys.—JETP **17**, 830 (1963)].

[5] C. J. Goebel and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).

[6] S. D. Drell, A. C. Finn, and A. C. Hearn, Phys. Rev. **136**, B1439 (1964); Y. S. Jin and S. W. MacDowell, Phys. Rev. **137**, B688 (1965).

We further note attempts made by Barut and Sawada[7] to determine g^2 from maximal analyticity.

We discuss below the question of a bound on g using analytic properties of the scattering amplitude for a given partial wave, but only in the physical sheet. We consider the case that the amplitude has a pole corresponding to the bound state between the left- and right-hand branch points.

Both the real part of the scattering amplitude and the contribution from the physical cut can be bounded for a certain range of physical values of the energy. The resulting function, called the unitary bound in that region, constitutes an upper bound for the pole term and the left-hand contribution in the same energy interval.

We shall demonstrate that with square integrability of the left-hand discontinuity as sole condition, the bound for g^2 is infinite. In that case, the weight function of the left-hand cut necessarily has an infinite number of sign changes. A weight function having any distribution of a finite number of sign changes leads to a bound for g^2. (Rigorous information about the number of zeros of the left-hand discontinuity is scanty. For a lower bound see Ref. 8.)

Using unitarity we establish in Sec. 2 a bound for the real part of the scattering amplitude and the contribution from the physical cut. In Sec. 3 we discuss the competition between the left-hand contribution and the pole describing the bound state. We further present the actual construction of the bound on g^2 for the case of any distribution of a finite number of sign changes of the left-hand cut discontinuity.

2. UNITARY BOUND

We consider the simplest case of the scattering of two spinless particles with equal mass M, center-of-mass energy $s^{1/2}$, and momentum $q=\frac{1}{2}(s-4M^2)^{1/2}$.

It is assumed that for a given partial wave l, a bound state with strength g^2 is located at $s=m^2$ between the left- and right-hand branch points s_l, s_r.

Under general and weak conditions it has been

[7] See, for instance, A. O. Barut and T. Sawada, Phys. Letters **13**, 177 (1964); T. Sawada, Nuovo Cimento **38**, 1889 (1965).

[8] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

proved that if the scattering amplitude

$$F_l(s)=\frac{8\pi s^{1/2}}{q}e^{i\delta_l(s)}\sin\delta_l(s) \tag{2.1}$$

possesses the standard analytic structure, no subtraction is necessary in a dispersion relation for F_l.[9] However, in order to obtain a bound it will be necessary to apply a single subtraction at some point s_n. We thus start from

$$\frac{\mathrm{Re}F_l(s)-\mathrm{Re}F_l(s_n)}{s-s_n}=\frac{g^2}{(m^2-s)(m^2-s_n)}+\frac{P}{\pi}\int_{s_r}^{\infty}\frac{\mathrm{Im}F_l(s')}{(s'-s)(s'-s_n)}ds'+\frac{1}{\pi}\int_{-\infty}^{s_l}\frac{R_l(s')}{(s'-s)(s'-s_n)}ds', \tag{2.2}$$

where $R_l(s)$ denotes the discontinuity of $F_l(s)$ across the unphysical cut.

We first focus on the parts of (2.2) dependent on values of $F_l(s)$ for physical s. From

$$\mathrm{Im}F_l(s)=\frac{q}{8\pi s^{1/2}}\lambda_l(s)|F_l(s)|^2, \tag{2.3}$$

in terms of the inelasticity factor

$$\lambda_l(s)\equiv\sigma_{\mathrm{total}}^{(l)}(s)/\sigma_{\mathrm{el}}^{(l)}(s)\geq 1,$$

one readily derives

$$\left|\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}F_l(s)\right|\leq\frac{8\pi s^{1/2}}{q};\quad s>s_r. \tag{2.4}$$

Eq. (24) will now be used to bound the contribution to $F_l(s)$ coming from the physical cut. Normal threshold behavior implies for $s\gtrsim s_r$

$$\mathrm{Im}F_l(s)=\frac{8\pi s^{1/2}}{q}\sin^2\delta_l(s)\propto A_l(s-s_r)^l\frac{s^{1/2}}{q}. \tag{2.5}$$

We further assume $\mathrm{Im}F_l(s)$ to possess a maximum at some finite s_M and a minimum s_m, possibly at ∞.

One next chooses the subtraction point s_n and s, as well as finite intervals Δ_1 and Δ_2 around s_n and s, in an arbitrary way, subject only to (see Fig. 1)

$$\begin{aligned}&s_r<s_n-\Delta_1,\\&s_n+\Delta_1<s_M<s-\Delta_2,\\&s-\Delta_2<s_m.\end{aligned} \tag{2.6}$$

[9] R. Omnès, Phys. Rev. 133, B1543 (1964); A. Martin, Nuovo Cimento 38, 1326 (1965).

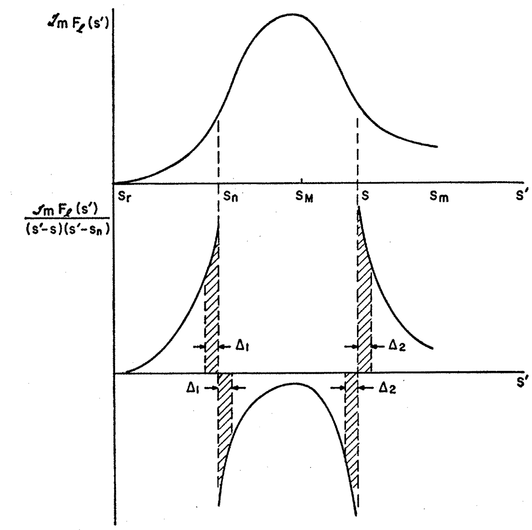

FIG. 1. $\mathrm{Im}F_l(s')$ and $[\mathrm{Im}F_l(s')]/(s'-s_n)(s'-s)$ as functions of s'. Shown is a choice of the subtraction point s_n and s with respect to s_M and s_m, the (not necessarily first) extrema of $\mathrm{Im}F_l$, satisfying the inequalities (2.6).

The contribution of the physical cut can then be split as follows:

$$\frac{P}{\pi}\int_{s_r}^{\infty}=\frac{1}{\pi}\int_{s_r}^{s_n-\Delta_1}+\frac{P}{\pi}\int_{s_n-\Delta_1}^{s_n+\Delta_1}+\frac{1}{\pi}\int_{s_n+\Delta_1}^{s-\Delta_2}+\frac{P}{\pi}\int_{s-\Delta_2}^{s+\Delta_2}+\frac{1}{\pi}\int_{s+\Delta_2}^{\infty}. \tag{2.7}$$

We first consider the first principal-value integral on the right-hand side of (2.7) by writing it as

$$\frac{P}{\pi}\int_{s_n-\Delta_1}^{s_n+\Delta_1}\frac{\mathrm{Im}F_l(s')}{(s'-s)(s'-s_n)}ds'=\frac{P}{\pi(s-s_n)}\int_{s_n-\Delta_1}^{s_n+\Delta_1}\left(\frac{1}{s'-s}-\frac{1}{s'-s_n}\right)\mathrm{Im}F_l(s')ds'. \tag{2.8}$$

Since $\mathrm{Im}F_l>0$, the first contribution will be negative. So will the second part, since $\mathrm{Im}F_l$ increases in the interval $(s_n-\Delta_1, s_n+\Delta_1)$ because of the choice of s_n, Eq. (2.6). The same reasoning holds for the fourth term in (2.7). Their contributions as well as that of the evidently negative third term can be discarded in establishing an upper bound for (2.7). By use of (2.4) one thus finds

$$\left|\frac{P}{\pi}\int_{s_r}^{\infty}\frac{\mathrm{Im}F_l(s')}{(s'-s)(s'-s_n)}ds'\right|\leq 16\left|\left(\int_{s_r}^{s_n-\Delta_1}+\int_{s_n+\Delta_2}^{\infty}\right)\times\frac{s'^{1/2}ds'}{(s'-4M^2)^{1/2}(s'-s)(s'-s_n)}\right|. \tag{2.9}$$

Defining

$$B(s,t)\equiv\{[s(s-4M^2)]^{1/2}+s-2M^2\}\times\left\{\frac{[s(s-4M^2)]^{1/2}+[t(t-4M^2)]^{1/2}}{s-t}+\frac{t^2-2M^2}{[t(t-4M^2)]^{1/2}}\right\}^{-1}\times\left(\frac{t}{t-4M^2}\right)^{-1/2}, \quad (2.10)$$

one establishes from (2.3), (2.5), (2.9), and (2.10), a unitarity bound:

$$\left|\frac{g^2}{(m^2-s)(m^2-s_n)}+\frac{1}{\pi}\int_{-\infty}^{s_l}\frac{R_l(s')}{(s'-s)(s'-s_n)}ds'\right| = \left|\frac{\mathrm{Re}F_l(s)-\mathrm{Re}F_l(s_n)}{s-s_n}-\frac{P}{\pi}\int_{s_r}^{\infty}\frac{\mathrm{Im}F_l(s')}{(s'-s)(s'-s_n)}ds'\right| \leq U(s,p_i,q_i) \quad (2.11)$$

with

$$U(s,p_i,q_i)\equiv U(s,M,s_r,s_n,\Delta_1,\Delta_2)=\frac{16}{s-s_n}\left[2\pi\left(\frac{s}{s-4M^2}\right)^{1/2}+\ln\frac{B(s_n-\Delta_1,\, s)B(s_r,s_n)B(s_n+\Delta_2,\, s_n)}{B(s_n-\Delta_1,\, s_n)B(s_r,s)B(s_n+\Delta_2,\, s)}\right]. \quad (2.12)$$

p_i is the set of parameters s_n, Δ_1, Δ_2, as yet free within the inequalities (2.6), while q_i denotes fixed parameters like M, s_r, s_M, s_m.

It is convenient to introduce the dimensionless variable $x=s/M^2$ and mass ratio $\mu=m/M$, and further two reduced functions (the index l will be suppressed from here on):

$$\rho(x)=\frac{M^2}{\pi g^2}\frac{\mu^2-x_n}{x-x_n}R_l(x), \qquad u(x,p_i,q_i)=(\mu^2-x_n)M^2U(s,p_i,q_i). \quad (2.13)$$

Equation (2.11) then becomes

$$\left(\frac{g}{M}\right)^2\left|\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right|\leq u(x,p_i,q_i) \quad (2.14)$$

with all parameters p_i and $x(s)$ satisfying the inequalities (2.6). Keeping for instance p_i fixed, Eq. (2.14) (now to be valid in an interval $x_1\leq x\leq x_2$) requires

$$\min_{\rho(x)}\max_{x_1\leq x\leq x_2}\left(\frac{g}{M}\right)^2\left|\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right| \leq \max_{x_1\leq x\leq x_2} u(x,p_i,q_i). \quad (2.15)$$

The minimum in (2.15) refers to that obtained for any as yet arbitrary function $\rho(x)$ for which the integral in (2.15) exists.

Instead of searching for the function $\bar{\rho}$, satisfying

$$\max_{x_1\leq x\leq x_2}\left|\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\bar{\rho}(x')dx'}{x'-x}\right| \leq \max_{x_1\leq x\leq x_2}\left|\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x')dx'}{x'-x}\right|, \quad (2.16)$$

one may alternatively try to determine $\tau(x)$, the best quadratic approximation of the pole term, satisfying

$$\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\tau(x')}{x'-x}dx'\right)^2dx \leq \int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right)^2dx. \quad (2.17)$$

Then,

$$\gamma\equiv\left[\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\tau(x')}{x'-x}dx'\right)^2dx\right]^{1/2} \quad (2.18)$$

certainly satisfies

$$\gamma\leq\min_{\rho}\max_{x_1\leq x\leq x_2}\left|\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x)}{x'-x}dx'\right|. \quad (2.19)$$

A bound for the coupling constant is then derived from (2.15), viz.

$$\left(\frac{g}{M}\right)^2\leq\min_{p_i}\max_{x_1\leq x\leq x_2}u(x,p_i,q_i)/\gamma, \quad (2.20)$$

where $\min_{p_i}$ refers to the sharpest bound obtainable for an optimal choice of the parameters p_i satisfying the inequalities (2.6).

3. OPTIMAL QUADRATIC APPROXIMATION OF THE POLE TERM BY THE LEFT-HAND CUT CONTRIBUTION

We now consider the best quadratic approximation to the pole term

$$\lambda_p(x)=1/(\mu^2-x) \quad (3.1)$$

by functions of the form

$$\lambda(x)=T\rho(x)\equiv\int_{-\infty}^{x_l}\frac{\rho(x')}{x-x'}dx'. \quad (3.2)$$

In other words, we look for the minimum value of the functional

$$A[\rho]=(\lambda_p-T_\rho)^2 \equiv\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}+\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right)^2dx. \quad (3.3)$$

With respect to the class of functions $\rho(x)$ we restrict ourselves in the following to $\rho\in\mathfrak{L}_2[-\infty, x_l]$, the Hilbert space of all functions square integrable in the interval $-\infty < x \leq x_l$. We further consider the linear transformation $\lambda = T\rho$, Eq. (3.2), the mapping of the Hilbert space $\mathfrak{L}_2[-\infty, x_l]$ into a set of functions $T(\mathfrak{L}_2[-\infty, x_l])$ defined on the interval $x_1 \leq x \leq x_2$.

Lemma:

(1) $T(\mathfrak{L}_2[-\infty, x_l]) \subset \mathfrak{L}_2[x_1, x_2]$, where $\mathfrak{L}_2[x_1,x_2]$ is the Hilbert space of functions $\lambda(x)$ square integrable in the interval $x_1 \leq x \leq x_2$.

(2) $T(\mathfrak{L}_2[-\infty, x_l])$ is dense in $\mathfrak{L}_2[x_1,x_2]$, i.e., for any $\epsilon > 0$ and any $\lambda \in \mathfrak{L}_2[x_1,x_2]$, there exists a $T\rho$, $\rho \in \mathfrak{L}_2[-\infty, x_l]$, with

$$(\lambda - T\rho)^2 < \epsilon.$$

(3) $\lambda_p - T\rho = 0$ has no solution.

(4) There is no $\rho \in \mathfrak{L}_2[-\infty, x_l]$ with

$$T\rho = 0.$$

(5) The inverse transformation T^{-1} exists, but is unbounded, i.e., there is no finite constant α, such that

$$\|T^{-1}\lambda\| \leq \alpha \|\lambda\|$$

for all $\lambda \in \mathfrak{L}_2[x_1,x_2]$.

The proof of this lemma is given in Appendix A. We use here the first and second statements of the lemma, which furnish the proof that in particular the pole term λ_p can be quadratically approximated with arbitrary accuracy. From (3.3), (2.18), and (2.20) we then infer that for a $\rho \in \mathfrak{L}_2[-\infty, x_l]$ the upper bound of g^2 is infinite.

The last two statements imply that for a solution of $\lambda_p - T\rho = 0$ one has to look for a ρ outside the Hilbert space. Indeed allowing distributions for ρ one finds,

$$\rho = \sum_{n=1}^{\infty} (-)^{n+1} \delta^{(n)}(x - x_l)(\mu^2 - x_l)^n / n! \qquad (3.4)$$

as solution of

$$\lambda_p(x) = \frac{1}{x - \mu^2} = \int_{-\infty}^{x_l} \frac{\rho(x')}{x' - x} dx'. \qquad (3.5)$$

The series (3.4) amounts to an expansion of $\lambda_p(x)$ in terms of poles of arbitrary order, all at the point $x = x_l$.

The series (3.4) is by no means unique. The choice

$$\rho(x) = \delta(x - x_l) - \sum_{n=1}^{\infty} \frac{(\mu^2 - x_l) \cdots (\mu^2 - x_l + n)}{n!} \times \sum_{k=0}^{n} \binom{n}{k} (-)^k \delta(x - x_l + n - k), \qquad (3.4a)$$

for instance, amounts to an expansion of λ_p in terms of a sum over a product of an increasing number of equidistant poles at $x_l, x_l - 1, \cdots$. (We are indebted to Nitzan-Nussinov for having brought to our attention the existence of the so-called Waring series.[10])

Equation (3.4) shows that an infinity of sign changes in ρ is required for a solution of Eq. (3.5). We now demonstrate that a ρ with any distribution of a *finite* number of sign changes leads to a finite bound on g^2.

Let first z_l, $i = 1, \cdots, n-1$ be the ordered set of zeros of $\rho(x)$

$$x_l \equiv z^0 > z_1, \cdots, > z_n \equiv -\infty. \qquad (3.6)$$

We then contend that

$$\int_{x_1}^{x_2} \left(\frac{1}{\mu^2 - x} - \sum_{l=0}^{n-1} \frac{\alpha_l^{(1)}}{x - z_l} \right)^2 dx \leq \int_{x_1}^{x_2} \left(\frac{1}{\mu^2 - x} - \int_{-\infty}^{x_l} \frac{\rho(x')}{x - x'} dx' \right)^2 dx \qquad (3.7)$$

with $\alpha_l^{(1)}$ given by

$$\alpha_l^{(1)} = \sum_{i,k=0}^{n-1} U_{ik}^{(1)} U_{il}^{(1)} \int_{x_1}^{x_2} \lambda_p(x)(x - z_k)^{-1} dx. \qquad (3.8)$$

The matrix elements $U_{ik}^{(1)}$ are determined by a Schmidt procedure such that the system

$$\psi_k^{(1)} = \sum_{l=0}^{n-1} U_{kl}^{(1)} \frac{1}{x - z_l}, \quad k = 1, \cdots, n-1 \qquad (3.9)$$

is an orthonormal one.

Comparison of (3.7) with (2.17) shows that

$$\tau^{(1)} = \sum_{l=0}^{n-1} \alpha_l^{(1)} / (x - z_l). \qquad (3.10)$$

One is then in a position to calculate γ and g^2, Eqs. (2.18) and (2.20), in a straightforward manner, leading to

$$\left(\frac{g^{(1)}}{M}\right)^2 \leq \min_{p_i} \max_{x_1 \leq x \leq x_2} u(x, p_i) \Big/ \left\{ [(x_1 - \mu^2)(x_2 - \mu^2)]^{-1} - (x_2 - x_1)^{-1} \left[\sum_{l=0}^{n-1} U_{kl}^{(1)} \left(\lambda_p, \frac{1}{(x - z_l)} \right) \right]^2 \right\}^{1/2}. \qquad (3.11)$$

If only the number of sign changes of $\rho(x)$ for $-\infty < x \leq x_l$ is known, but not their position, the best quadratic

[10] J. M. Milne-Thomson, *Calculus of Finite Differences* (The Macmillan Company, London, 1933), p. 293.

approximation is achieved by a linear combination of poles up to order $n-1$, all located at the left-hand branch point x_l ($\equiv z_0$).

The upper bound (3.11) becomes in this case

$$\left(\frac{g^{(2)}}{M}\right)^2 \leq \min_{p_i} \max_{x_1\leq x\leq x_2} u(x,p_i) \Big/ \left\{[(x_1-\mu^2)(x_2-\mu^2)]^{-1}-(x_2-x_1)^{-1}\left[\sum_{l=1}^{n-1} U_{kl}^{(2)}\left(\lambda_p, \frac{1}{(x-z_0)^l}\right)\right]^2\right\}^{1/2}, \quad (3.12)$$

where now $U_{kl}^{(2)}$ is determined by

$$\psi_k^{(2)}=\sum_{l=1}^{n-1} U_{kl}^{(2)}\frac{1}{(x-z_0)^l},$$
$$(\psi_k^{(2)},\psi_{k'}^{(2)})=\delta_{kk'}, \quad k=1,\cdots,n-1. \quad (3.13)$$

We shall prove the contention (3.7) here only for the case $n=1$, i.e., the case of a ρ with definite sign for $-\infty<x<x_l$. For the case $n>1$ see Appendix B.

By use of the mean value theorem one writes

$$\lambda(x)=\int_{-\infty}^{x_l}\frac{\rho(x')}{x-x'}dx'=\frac{\eta-\xi(x)}{x-\xi(x)}\lambda(\eta) \quad (3.14)$$

with $\xi(x)$ some point between $(-\infty, x_l)$ and η chosen to be larger than x_r. One sees that

$$\lambda(x)\leq\frac{\eta-x_l}{x-x_l}\lambda(\eta) \quad \text{for} \quad x<\eta$$
$$\geq\frac{\eta-x_l}{x-x_l}\lambda(\eta) \quad \text{for} \quad x>\eta. \quad (3.15)$$

Equation (3.15) establishes an inequality of the form (3.7) for $n=1$. One then determines α_0 by the choice of η, which minimizes

$$\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}-\frac{\alpha_0(\eta)}{x-x_l}\right)^2 dx.$$

Before concluding this section we wish to make the following remark. Since there exists a finite bound on g^2 for a finite number of sign changes of ρ, the function which approximates λ_p quadratically must necessarily possess an infinite number of sign changes. Of course, it is still possible that restrictions on the distribution of an infinite number of sign changes of ρ will lead to a finite bound for g^2.

4. CONCLUSION

Starting from a once subtracted partial-wave dispersion relation (2.2) we have calculated under certain conditions an upper bound for g^2, Eqs. (3.11), (3.12). The bound depends on details of the system and a limited number of parameters. The details of the system concern the position of a maximum in the imaginary part of the scattering amplitude and the number of sign changes (assumed to be finite) of the left-hand cut discontinuity. The parameters, on the other hand, are a subtraction point and arbitrary energy intervals, which only have to satisfy certain inequalities (2.6). These parameters can be adjusted in order to obtain the sharpest bound within the framework of the technique used.

The case of an infinite number of sign changes has been discussed, but no finite bound on g^2 could be established.

ACKNOWLEDGMENTS

The authors are deeply indebted to Professor F. Low for suggesting the problem treated above and for many helpful discussions. One of the authors (H.D.D.) wants to thank the Deutsche Forschungsgemeinschaft for a fellowship.

APPENDIX A

Proof of the lemma in Sec. 3: We prove the lemma point by point:

(1). By choosing correctly weighted Laguerre polynomials $r_\nu(x)=L_\nu(x_l-x)\sigma(x_l-x)^{1/2}$ as a basis of the Hilbert space $\mathcal{L}_2[-\infty, x_l]$ statement (1) is obvious, since

$$Tr_\nu\subset\mathcal{L}_2[x_1,x_2] \quad (A1)$$

for all ν.

(2). We have only to prove that there does not exist an element $\lambda\in\mathcal{L}_2[x_1,x_2]$, $\lambda\neq0$, which is orthogonal to $T(\mathcal{L}_2[-\infty, x_l])$. If for any $\rho(x)\in\mathcal{L}_2[-\infty, x_l]$,

$$0=\int_{x_1}^{x_2}\lambda(x)\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'dx$$
$$=\int_{-\infty}^{x_l}\rho(x')\int_{x_1}^{x_2}\frac{\lambda(x)}{x'-x}dxdx', \quad (A2)$$

then

$$\int_{x_1}^{x_2}\lambda(x)/(x'-x)dx=0 \quad (A3)$$

or

$$\lambda(x)\equiv0.$$

(3). Since the Hilbert integral equation

$$\int_{-\infty}^{+\infty}\frac{\phi(x')}{x'-x}dx'=\lambda_p(x) \quad (A4)$$

has a unique solution, which is $\phi(x)=\delta(\mu^2-x)$, there cannot be another solution, with $\phi(x)\equiv0$ for $x>x_l$, which would solve the equation $T\rho=\lambda_p$.

(4). This statement follows by an argument analogous to that for (3).

(5). T^{-1} exists, because of (4). The unboundedness of T^{-1} is equivalent to the following statement: There does not exist a $\delta>0$ with the property

$$\|T\rho\|\geq\delta\|\rho\| \quad \text{for all} \quad \rho\in\mathfrak{L}_2[-\infty,x_l].$$

Considering, e.g., the family of functions

$$\rho_n=\sin nx/x\,, \tag{A5}$$

then since

$$\int_{x_1}^{x_2}\left(\int_{-\infty}^{x_l}\frac{\sin nx'}{x'(x'-x)}dx'\right)^2dx\geq\delta_n\int_{-\infty}^{x_l}\frac{\sin^2 nx}{x^2}dx, \tag{A6}$$

we find

$$\lim_{n\to\infty}\delta_n=0, \text{ which proves (5).}$$

APPENDIX B

In order to prove the inequality (37) one represents $\lambda(x)$ as follows:

$$\lambda(x)=\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'=\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-x}dx' \tag{B1}$$

with

$$\rho(x)\neq 0 \quad \text{in} \quad z_\nu<x<z_{\nu-1}.$$

Using the mean-value theorem we get

$$\lambda(x)=\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}\frac{\eta_1-x'}{x-x'}dx'$$

$$=\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}dx'\frac{\eta_1-\xi_\nu(\eta_1,x)}{x-\xi_\nu(\eta_1,x)}, \tag{B2}$$

where $\eta_1>x_l$ is an arbitrary reference point outside the integration interval and $\xi_\nu(\eta_1,x)$ denotes the mean value,

$$z_\nu\leq\xi_\nu(\eta_1,x)\leq z_{\nu-1}. \tag{B3}$$

As a first step we prove for a given sequence of points $x_l<\eta_1<\eta_2<\cdots<\eta_n<\infty$ and corresponding values $\lambda(\eta_1)>\cdots>\lambda(\eta_n)$ the existence of a function $\bar{\lambda}(x)$ with the properties:

(1) $\bar{\lambda}(\eta_i)=\lambda(\eta_i)$.

(2) If $\Lambda=\{\tilde{\lambda}(x)\}$ is the set of all functions of the form

$$\tilde{\lambda}(x)=\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx' \text{ with } \rho(z)=0,\ \nu=1,\cdots,n-1 \tag{B4}$$

and

$$\tilde{\lambda}(\eta_i)=\lambda(\eta_i),$$

then the following statement holds for any $\tilde{\lambda}(x)\in\Lambda$:

$$\begin{aligned}\bar{\lambda}(x)&\geq\tilde{\lambda}(x) \quad \text{for} \quad x_l<x\leq\eta_1,\\ &\leq\tilde{\lambda}(x) \quad \text{for} \quad \eta_1<x<\eta_2,\\ &\geq\tilde{\lambda}(x) \quad \text{for} \quad \eta_2<x<\eta_3.\\ &\vdots\end{aligned} \tag{B5}$$

The n values $\lambda(\eta_\nu)$ are sufficient to eliminate the η unknown constants

$$\int_{z_\nu}^{z_{\nu-1}}\rho(x')/(x'-\eta_1)dx'$$

which appear in the expression (3.13) for $\lambda(x)$.

$$\begin{aligned}\lambda(x)-\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}dx'\frac{\eta_1-\xi_\nu(\eta_1,x)}{x-\xi_\nu(\eta_1,x)}&=0,\\ \lambda(\eta_1)-\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}dx'&=0,\\ \lambda(\eta_2)-\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}dx'\frac{\eta_1-\xi_\nu(\eta_1,\eta_2)}{\eta_2-\xi_\nu(\eta_1,\eta_2)}&=0,\\ &\vdots\\ \lambda(\eta_n)-\sum_{\nu=1}^{n}\int_{z_\nu}^{z_{\nu-1}}\frac{\rho(x')}{x'-\eta_1}dx'\frac{\eta_1-\xi_\nu(\eta_1,\eta_n)}{\eta_n-\xi_\nu(\eta_1,\eta_n)}&=0.\end{aligned} \tag{B6}$$

The solution $\lambda(x)$ of this system of $n+1$ linear equations is given by

$$D=\begin{vmatrix}\lambda(x) & \dfrac{\eta_1-\xi_1(\eta_1,x)}{x-\xi_1(\eta_1,x)} & \cdots & \dfrac{\eta_1-\xi_n(\eta_1,x)}{x-\xi_n(\eta_1,x)}\\ \lambda(\eta_1) & 1 & \cdots & 1\\ \lambda(\eta_2) & \dfrac{\eta_1-\xi_1(\eta_1,\eta_2)}{\eta_2-\xi_1(\eta_1,\eta_2)} & \cdots & \dfrac{\eta_1-\xi_n(\eta_1,\eta_2)}{\eta_2-\xi_n(\eta_1,\eta_2)}\\ \vdots & & & \\ \lambda(\eta_n) & \dfrac{\eta_1-\xi_1(\eta_1,\eta_n)}{\eta_n-\xi_1(\eta_1,\eta_n)} & \cdots & \dfrac{\eta_1-\xi_n(\eta_1,\eta_n)}{\eta_n-\xi_n(\eta_1,\eta_n)}\end{vmatrix}=0, \tag{B7}$$

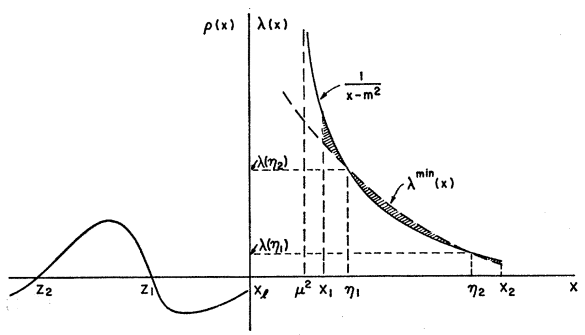

FIG. 2. Left-hand discontinuity $\rho(x)$ and left-hand contribution $\lambda(x)$ to scattering amplitude, illustrated, in the case where $\rho(x)$ has two zeros z_1 and z_2. Here $\lambda^{\min}(x)$ minimizes the area under $[\lambda(x)+1/(\mu^2-x)]^2$ between x_1 and x_2 under the condition that λ shall have the values $\lambda(\eta_1)$, $\lambda(\eta_2)$ at two points η_1, η_2.

which yields

$$\lambda(x)=\frac{1}{D_{00}}\sum_{i=1}^{n}(-1)^{i+1}D_{i0}\lambda(\eta_i). \quad \text{(B8)}$$

Here D_{ik} denotes the minor corresponding to the element ik ($i, k=0, 1, \cdots, n$) of the determinant D. In the construction of the abovementioned function $\bar{\lambda}(x)$ one exploits the freedom of the mean values $\xi_\nu(\eta_1,x)$ within the ranges $z_\nu<\xi(\eta_1,s)<z_{\nu-1}$. By regrouping the terms in this expression we find

$$\begin{aligned}\lambda(x)&=\sum_{i=1}^{n}(-1)^{i+1}\frac{D_{i0}}{D_{00}}\lambda(\eta_i)\\ &=\sum_{i=1}^{n/2}\frac{D_{2i-1,0}\lambda(\eta_{2i-1})-D_{2i,0}\lambda(\eta_{2i})}{D_{00}}, \quad n \text{ even}\\ &=\sum_{i=1}^{(n-1)/2}\frac{D_{2i-1,0}\lambda(\eta_{2i-1})-D_{2i,0}\lambda(\eta_{2i})}{D_{00}}+\frac{D_{n0}}{D_{00}}\lambda(\eta_n), \\ &\qquad n \text{ odd}. \quad \text{(B9)}\end{aligned}$$

We note that the elements in the determinants $D_{2i-1,0}$ and $D_{2i,0}$ are equal except for those in the $(2i-1)$th row, which leads to the identity

$$D_{2i-1,0}\lambda(\eta_{2i-1})-D_{2i,0}\lambda(\eta_{2i})$$

$$=\begin{vmatrix} \dfrac{\eta_1-\xi_1(\eta_1,x)}{x-\xi_1(\eta_1,x)} & \cdots & \dfrac{\eta_1-\xi_n(\eta_1,x)}{x-\xi_n(\eta_1,x)} \\ \vdots & & \\ \lambda(\eta_{2i-1})\dfrac{\eta_1-\xi_1(\eta_1,\eta_{2i-1})}{\eta_{2i-1}-\xi_1(\eta_1,\eta_{2i-1})}-\lambda(\eta_{2i})\dfrac{\eta_1-\xi_1(\eta_1,\eta_{2i})}{\eta_{2i}-\xi_1(\eta_1,\eta_{2i})} & \cdots & \lambda(\eta_{2i-1})\dfrac{\eta_1-\xi_n(\eta_1,\eta_{2i-1})}{\eta_{2i-1}-\xi_n(\eta_1,\eta_{2i-1})}-\lambda(\eta_{2i})\dfrac{\eta_1-\xi_n(\eta,\eta_{2i})}{\eta_{2i}-\xi_n(\eta,\eta_{2i})} \\ \vdots & & \\ \dfrac{\eta_1-\xi_1(\eta_1,\eta_n)}{\eta_n-\xi_1(\eta_1,\eta_n)} & \cdots & \dfrac{\eta_1-\xi_n(\eta_1,\eta_n)}{\eta_n-\xi_n(\eta_1,\eta_n)}. \end{vmatrix}. \quad \text{(B10)}$$

One may then convince oneself that (3.20) for $x_\eta<x<\eta$, consists of a sum of strictly positive terms which increase for increasing ξ_ν. In the case of odd n the unpaired term is positive and increasing; therefore we have proved the first inequality of (B5).

For the proof of the other inequalities we use the fact that from this first part if follows that

$$\partial\lambda(x)/\partial\xi_\nu(\eta,x)\geq 0, \quad x_l\leq x\leq\eta_1. \quad \text{(B11)}$$

By construction,

$$\partial\lambda(\eta_i)/\partial\xi_\nu=0, \quad i=1, \cdots, n \quad \text{(B12)}$$

so we conclude that $\partial\lambda/\partial\xi_\nu$ changes sign at $x=\eta_1$. Thus

$$\partial\lambda(x)/\partial\xi_\nu\leq 0, \quad \eta_1\leq x\leq\eta_2. \quad \text{(B13)}$$

Consequently, $\lambda(x)$ is smallest for $\xi_\nu(\eta_1,x)=z_{\nu-1}$. By repeating this argument one proves the proposition stated above.

To establish the connection with the original minimum problem we argue as follows:

Let η_i be the points of intersection of an arbitrary function $\lambda(x)\in\Lambda$ with the pole term $1/(x-\mu^2)$ and

$$\lambda(\eta_i)=\bar{\lambda}(\eta_i)=\frac{1}{\eta_i-\mu^2}, \quad i=1, 2, \cdots. \quad \text{(B14)}$$

Because of (B15), there cannot be more than n such points. So let us assume in the first place that there are just n intersections, ω of which may be located within x_1, x_2:

$$x_1\leq\eta_\tau\leq\eta_{\tau+1}\leq\cdots\leq\eta_{\tau+\omega}\leq x_2. \quad \text{(B15)}$$

With this adjustment of the points η_i we get

$$\left[\bar{\lambda}(x)+\frac{1}{\mu^2-x}\right]^2\leq\left[\bar{\lambda}(x)+\frac{1}{\mu^2-x}\right]^2; \quad x_1\leq x\leq x_2 \quad \text{(B16)}$$

and therefore

$$\int_{x_1}^{x_2}\left[\bar{\lambda}(x)+\frac{1}{\mu^2-x}\right]^2dx\leq\int_{x_1}^{x_2}\left[\bar{\lambda}(x)+\frac{1}{\mu^2-x}\right]^2dx. \quad \text{(B17)}$$

The minimizing function $\tau(x)$ is thus given by an appropriate superposition of n pole terms located at the positions

$$z_0, z_1, \cdots, z_{n-1};$$

$$\tau^{(1)}(x)=\sum_{i=0}^{n-1}\alpha_i^{(1)}/(x-z_i), \quad \text{(B18)}$$

where the α_i are to be determined by minimizing the integral A, Eq. (3.2). If the assumption of exactly n intersections is not fulfilled, the number of zeros can only be smaller than n. In this case $\tau^{(1)}$, Eq. (B18), gives certainly the smallest value of A, Eq. (3.3).

A trivial extension of this proof gives

$$\tau^{(2)}(x)=\sum_{i=1}^{n-1}\alpha_i^{(2)}/(x-x_l)^i \quad \text{(B19)}$$

in case only the number but not the location of the sign changes of ρ is known.

As the last step one has to calculate the constants α_i in (B18) and (B19), which can be performed in the same manner.

We define

$$\phi_i^{(1)}(x)=(x-z_i)^{-1}, \qquad i=0,1,\cdots,n-1 \qquad \text{(B20)}$$

$$\phi_i^{(2)}(x)=(x-x_l)^{-i-1}, \qquad i=0,1,\cdots,n-1. \qquad \text{(B21)}$$

By means of the Schmidt procedure one may then orthogonalize the linearly independent functions ϕ_i in the interval $x_1\leq x\leq x_2$.

Calling the new sets of functions $\psi_k^{(\sigma)}$, $\sigma=1, 2$, one thus has

$$\psi_k^{(\sigma)}=\sum_l U_{kl}^{(\sigma)}\phi_l^{(\sigma)}$$

$$(\psi_k^{(\sigma)},\psi_{k'}^{(\sigma)})=\int_{x_1}^{x_2}\psi_k^{(\sigma)}(x)\psi_{k'}^{(\sigma)}(x)dx=\delta_{kk'}. \qquad \text{(B22)}$$

The minimizing function $\tau^{(\sigma)}(x)$ is a linear combination of the $\psi_k^{(\sigma)}(x)$:

$$\tau^{(\sigma)}(x)=\sum_{k=0}^{n-1}\left(\int_{x_1}^{x_2}\lambda_p(x')\psi_k^{(\sigma)}(x')dx'\right)\psi_k^{(\sigma)}(x)$$
$$\times\sum_{i,k=0}^{n-1}U_{ik}^{(\sigma)}U_{il}^{(\sigma)}\left(\int_{x_1}^{x_2}\lambda_p(x')\phi_k^{(\sigma)}(x')dx'\right)$$
$$\times\phi_l^{(\sigma)}(x). \qquad \text{(B23)}$$

Substituting (B23) into (B17) one obtains

$$\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}-\sum_{l=0}^{n-1}\alpha_l^{(\sigma)}\phi_l^{(\sigma)}(x)\right)^2dx$$
$$\leq\int_{x_1}^{x_2}\left(\frac{1}{\mu^2-x}-\int_{-\infty}^{x_l}\frac{\rho(x')}{x'-x}dx'\right)^2dx \qquad \text{(B24)}$$

with

$$\alpha_l^{(\sigma)}=\sum_{i,k=0}^{n-1}U_{ik}^{(\sigma)}U_{il}^{(\sigma)}\int_{x_1}^{x_2}\lambda_p(x)\phi_k^{(\sigma)}(x)dx; \qquad \text{(B25)}$$

Eqs. (3.7) and (3.8) are just (B24) and (B25) for $\sigma=1$.

PHYSICAL REVIEW VOLUME 143, NUMBER 4 MARCH 1966

Unsubtracted Dispersion Relations and the Renormalization of the Weak Axial-Vector Coupling Constants*

WILLIAM I. WEISBERGER
Stanford Linear Accelerator Center, Stanford University, Stanford, California
(Received 20 October 1965)

Assuming that the equal-time commutation rules for the vector and axial-vector-current octets proposed by Gell-Mann are valid and that the divergence of the $\Delta S=0$, $\Delta I=1$ axial current is a strongly convergent operator obeying unsubtracted dispersion relations and dominated by low-frequency contributions, we derive a sum rule for the renormalization of the neutron axial β-decay constant G_A, by the strong interactions. The result agrees with that previously obtained from the assumption that the axial-current divergence is proportional to the pion field. The results are generalized to the strangeness-changing leptonic decays in the context of Cabibbo theory and generalized Goldberger-Treiman relations, and are used to compute the d/f ratio for the weak baryon axial-current coupling and an independent value of G_A.

I. INTRODUCTION

RECENT calculations of the effects of the strong interactions in renormalizing the axial-vector coupling constant in β decay,[1,2] $g_A=G_A/G_V$, give good agreement with the experimental value. These results were derived from the following three assumptions.

(1) The equal-time commutators of the spatial integrals of the time components of the hadron currents measured to first order in the weak and electromagnetic interactions, the "charges" obey the algebra of $SU(3)\times SU(3)$ as postulated by Gell-Mann *et al.*[3]

(2) The effective Hamiltonian for leptonic decay of the hadrons is a current-current interaction which couples the appropriate members vector and axial-vector current octets of the strongly interacting particles to the usual $\gamma_\mu(1-\gamma_5)$ current of the leptons through the simple combination $V_\mu\pm A_\mu$.[4]

(3) Partially conserved axial current (PCAC) hy-

* Work supported by the U. S. Atomic Energy Commission.

[1] S. L. Adler, Phys. Rev. Letters **14**, 1051 (1965); Phys. Rev. **140**, B736 (1965).

[2] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965).

[3] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962); Physics **1**, (1964). R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).

[4] The relation of the algebraic relations to the specification of a universal weak coupling of leptons and hadrons has been discussed by M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N. Y.) **30**, 360 (1964).