Current-Generated Algebra and Mass Levels of the Hadrons

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The mass splittings of the $U(3)$ [or $SU(3)$] multiplets of the hadrons are investigated under the following assumptions: (i) The Hamiltonian decomposes into an invariant part plus an eighth component of an octet of $U(3)$, and the latter is a space integral of the scalar current transforming like $\frac{1}{2}(q^\dagger\beta\lambda_8 q)$, where q stands for the quark. (ii) An algebra of the positive parity operators generating a nonchiral $U(3)\times U(3)$ is a "good symmetry," where the generators consist of the scalar currents and the fourth components of the vector currents. It is shown that if a universal constant with the dimensions of a mass together with the average mass of the multiplet are given, then the splitting is calculated exactly for any multiplet under the above assumptions. A kind of competition of the kinematical group $U(3)$ and the dynamical group $U(3)\times U(3)$ can predict the average masses of $U(3)$ multiplets if they belong to the same multiplet of $U(3)\times U(3)$. The multiplets that concern us are the $\frac{1}{2}^+$ octet, $\frac{3}{2}^+$ decuplet, $\frac{3}{2}^-$ octet, 0^- nonet, 2^+ nonet, and 1^- nonet. The results show good agreement with experiment.

I. INTRODUCTION

IT has been emphasized that a set of equal-time commutation relations of various currents defines an algebraic (or a group) property that underlies the structure of the hadrons.[1] The currents consist of the space integrals of components of S, V, A, P, and T densities, which obey the same commutation rules as if the currents were equal to those of the quark Lagrangian model. It was further suggested that the algebra of the positive parity operators generating a nonchiral $U(6)\times U(6)$ should be a "good symmetry" of the hadrons at rest.[2] Many applications of the current algebra have been tried and many results were obtained in various fields.[2,3] One of the authors (K. K.) applied it to the mass-splitting problem of baryons and mesons in a previous paper,[4] hereafter referred to as (I), where it was shown that relations connecting the meson and the baryon mass splitting could be calculated although no symmetry higher than the approximate $U(3)$ was used.

It is the purpose of this paper to clarify the algebraic properties of the mass-splitting operator introduced in (I), and to extend the calculation method to obtain more complete mass formulas. The mass formulas will be improved in an exact form if a nonchiral $U(3)\times U(3)$ group, a subgroup of $U(6)\times U(6)$ introduced by some authors,[2,5] is a "good symmetry" of the hadrons at rest. We may lay emphasis, in the present work, on the analysis of the mass splittings of the $J^P=\frac{1}{2}^+$ baryons and the $J^P=0^-$ mesons, since some discrepancy with

the formula was found in (I). The predictions of the baryon mass splittings will be greatly improved. It will be pointed out that the splittings of the pseudoscalar mesons are difficult to treat satisfactorily.

In Sec. II the assumptions on which the general formalism is based will be presented. This formalism is used for obtaining the mass splittings and for seeing the connection between the splittings and the average masses of the $U(3)$ multiplets. In Sec. III the mass levels of the 1^- nonet and the 2^+ nonet will be discussed emphasizing the predictions for the $\varphi(1020)$ mass and $f'(1525)$ mass. In Secs. IV and V the $\frac{1}{2}^+$ octet baryons together with the $\frac{3}{2}^+$ decuplet and $\frac{3}{2}^-$ octet, and the 0^- nonet pseudoscalar (ps) mesons will be treated, respectively. The final section will be devoted to summarizing the results obtained. A few additional remarks will close this section.

II. GENERAL FORMALISM

The assumptions on which our work is based are that the Hamiltonian consists of an invariant part H_0 plus a term S_8 transforming like the eighth component of an octet of $U(3)$, and that the latter is the space integral of the scalar current transforming like $\frac{1}{2}(q^\dagger\beta\lambda_8 q)$:

$$H = H_0 - \delta m S_8, \qquad (2.1)$$

where δm is an unknown universal parameter. This assumption means that the $U(3)$ violation originates in the mass splitting of the quark. In connection with S_8, other scalar quantities are introduced, denoted by S_i, which are the space integrals of the scalar currents transforming like $\frac{1}{2}(q^\dagger\beta\lambda_i q)$ [$i=0,1,\cdots,7$]. If we can calculate all the matrix elements of S_8 between states with zero momentum, the mass splittings of given multiplets can be obtained by the diagonalization of the Hamiltonian (2.1).

To obtain the matrix elements we use the commutators of S_i and $V_{i,4}$,

$$[S_i, S_j] = i f_{ijk} V_{k,4}, \qquad (2.2)$$

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962) and Physics **1**, 63 (1964); R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters **13**, 678 (1964).

[2] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965); R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); **17**, 145 (1965).

[3] W. I. Weisberger, Phys. Rev. Letters **14**, 1047 (1965); S. L. Adler, *ibid.* **14**, 1051 (1965); H. J. Schnitzer, CERN Report (unpublished).

[4] K. Kikkawa, Progr. Theoret. Phys. (Kyoto) **35**, No. 2, 1966.

[5] K. Bardakci, J. M. Cornwall, P. G. O. Freund, and B. W. Lee, Phys. Rev. Letters **13**, 698 (1964); **14**, 48 (1965); **14**, 264 (1965).

$$[V_{i,4}, V_{j,4}] = if_{ijk}V_{k,4}, \qquad (2.3)$$

$$[V_{i,4}, S_j] = if_{ijk}S_k, \qquad (2.4)$$

where $V_{i,4}$ is the fourth component of vector current transforming like $\frac{1}{2}(q^\dagger \lambda_i q)$ and f_{ijk} is the structure constant of $U(3)$. These V's are the generators of the $U(3)$ group with which we are concerned, and are assumed to commute with the unperturbed part H_0 of the Hamiltonian. The relations (2.2), (2.3), and (2.4) define a group[6] $U(3) \times U(3)$ which is a subgroup of the nonchiral $U(6) \times U(6)$. Namely, defining

$$G_i^{(\pm)} = \frac{1}{2}(V_{i,4} \pm S_i), \qquad (2.5)$$

we obtain

$$[G_i^{(\pm)}, G_j^{(\pm)}] = if_{ijk}G_k^{(\pm)},$$
$$[G_i^{(\pm)}, G_j^{(\mp)}] = 0. \qquad (2.6)$$

The second assumption, which may, however, be removed in some part of our arguments, is that the group $U(3) \times U(3)$ is a "good symmetry" of the hadrons at rest, at least in classifying the particles. That is, when we apply S_i to a one-particle state with zero momentum belonging to an irreducible representation of $U(3) \times U(3)$, it leads mostly to a one-particle state belonging to the same irreducible representation. We refer the reader to other papers for a discussion of the validity of the assumption.[2]

Now, we are in a position to calculate the matrix elements of S_i. We take the matrix element of Eq. (2.2) between one-particle states of H_0, which have zero momentum. Since $V_{k,4}$ is a conserved quantity in the unperturbed system, the normalization of the right-hand side of Eq. (2.2) is readily determined. For example, $\langle \alpha | V_{8,4} | \alpha \rangle$ must be the hypercharge of $|\alpha\rangle$ times $\frac{1}{2}\sqrt{3}$. Inserting a complete set of states between the two operators on the left of Eq. (2.2), we get

$$\sum_\gamma \langle \alpha | S_i | \gamma \rangle \langle \gamma | S_j | \beta \rangle - \sum_\gamma (i \leftrightarrow j)$$
$$= if_{ijk}\langle \alpha | V_{k,4} | \beta \rangle. \qquad (2.7)$$

Comparing both sides we obtain sum rules of the scalar amplitudes. For example, let $|\alpha\rangle$ and $|\beta\rangle$ be particles of an octet. Then the intermediate state $|\gamma\rangle$ must belong to one of the representations $\mathbf{1}$, $\mathbf{8_1}$, $\mathbf{8_2}$, $\mathbf{10}$, $\mathbf{10^*}$, and $\mathbf{27}$, which in general may be a many-particle state. It

should be noted that all state vectors $| \;\rangle$ stand for the eigenstates of H_0, so that we can use the Wigner-Eckart theorem for[7] $\langle \alpha | S_i | \gamma \rangle$:

$$\langle 8_\alpha | S_i | 8_\gamma \rangle = F_{8_1}\begin{pmatrix} 8 & 8 & 8_1 \\ \gamma & i & \alpha \end{pmatrix} + F_{8_2}\begin{pmatrix} 8 & 8 & 8_2 \\ \gamma & i & \alpha \end{pmatrix},$$

$$\langle 10_\alpha | S_i | 8_\beta \rangle = F_{10}\begin{pmatrix} 8 & 8 & 10 \\ \beta & i & \alpha \end{pmatrix}, \quad \text{etc.} \qquad (2.8)$$

As will be shown in the Appendix, we have the following sum rules:

$$|F_{8_1}|^2 + |F_{8_2}|^2 + |F_1|^2/4 - 9|F_{27}|^2/4 = 3, \qquad (2.9)$$

$$-4|F_{8_1}|^2/5 + |F_1|^2/4 + (|F_{10}|^2 + |F_{10^*}|^2)/2$$
$$- 9|F_{27}|^2/20 = 0, \qquad (2.10)$$

$$2\,\mathrm{Re}(F_{8_1}F_{8_2}^*) + (\sqrt{5})(|F_{10^*}|^2 - |F_{10}|^2)/2 = 0. \qquad (2.11)$$

These sum rules are independent of the second assumption. But if the second assumption is taken into account, the intermediate states may be approximated by one-particle states at rest. Now suppose that $|\alpha\rangle$ and $|\beta\rangle$ belong to the octet representation of, say, $(6^*, 3^*) = 8 + 10^*$, where (m,n) stands for the representation of $U(3) \times U(3)$ and the single bold-faced number stands for that of $U(3)$. In this case there are 18 particles with the same spin parity but perhaps with different masses. The intermediate summation may be dominated by $\mathbf{8}$ and $\mathbf{10^*}$, and in that case the relations (2.9), (2.10), and (2.11) can be solved to get

$$F_{8_1} = -\sqrt{(5/3)}, \quad F_{8_2} = 2/\sqrt{3}, \quad \text{and} \quad |F_{10^*}|^2 = 8/3. \qquad (2.12)$$

Some manipulations (see Appendix) lead us to obtain another amplitude

$$G_{10^*} = -\sqrt{\tfrac{2}{3}}, \qquad (2.13)$$

where

$$\langle 10^*_\alpha | S_i | 10^*_\beta \rangle = G_{10^*}\begin{pmatrix} 10^* & 8 & 10^* \\ \beta & i & \alpha \end{pmatrix}. \qquad (2.14)$$

Quantities (2.12) and (2.13) are enough for us to get the eigenvalues of the total Hamiltonian. Namely,

$$\det|H-\lambda| = \begin{vmatrix} m_8 - \delta m\left\{F_{8_1}\begin{pmatrix} 8 & 8 & 8_1 \\ \alpha & 8 & \alpha \end{pmatrix} + F_{8_2}\begin{pmatrix} 8 & 8 & 8_2 \\ \alpha & 8 & \alpha \end{pmatrix}\right\} - \lambda, & -\delta m F_{10^*}\begin{pmatrix} 8 & 8 & 10^* \\ \alpha & 8 & \alpha \end{pmatrix} \\ -\delta m F_{10^*}\begin{pmatrix} 8 & 8 & 10^* \\ \alpha & 8 & \alpha \end{pmatrix}, & m_{10^*} - \delta m G_{10^*}\begin{pmatrix} 10^* & 8 & 10^* \\ \alpha & 8 & \alpha \end{pmatrix} - \lambda \end{vmatrix} = 0, \qquad (2.15)$$

where α stands for the isospin and the hypercharge of the particle concerned. Note that the unknown parameters contained in Eq. (2.15) are m_8, m_{10^*}, and δm. By

the average mass m_i we mean the expectation value of H_0 in the given state. Equation (2.15) gives all mass

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[6] The group $U(3)$, which commutes with H_0, should be distinguished from that when we refer to $U(3) \times U(3)$.

[7] Notations in the following papers are used throughout this one. J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963); Nuovo Cimento **31**, 420 (1964).

levels of 8 and 10* if we assume the three parameters are known. Conversely, if we give the three mass differences, say, of 8, the positions of the members of 10* are completely predicted. For other multiplets of $U(3) \times U(3)$ the levels can be obtained along the same lines.

The essential point of our model is that the quantity δm is common to all multiplets regardless of the spin and parity of the particles.[8] In the following sections we will show that any multiplet gives $\delta m = 140$ MeV ~ 165 MeV.

III. 1⁻ NONET AND 2⁺ NONET

Masses of the classical vector mesons (ρ, K^*, ω, and ϕ) and the recently discovered tensor mesons [$K^*(1430)$, $A_2(1310)$, $f(1250)$, and $f'(1525)$][9] are very good examples of our model. If we assume that both multiplets are represented by $(3,3^*)$ in $U(3) \times U(3)$, we get the following equations,

$$\begin{vmatrix} m_1 - \lambda, & (\sqrt{\tfrac{2}{3}})\delta m \\ (\sqrt{\tfrac{2}{3}})\delta m, & m_8 + \tfrac{1}{3}\sqrt{3}\delta m - \lambda \end{vmatrix} = 0 \quad (3.1)$$

for $I = 0$, $Y = 0$, and

$$\begin{aligned} \rho \quad \text{or} \quad A_2 &= m_8 - \tfrac{1}{3}\sqrt{3}\delta m, \\ K^* \quad \text{or} \quad K^*(1430) &= m_8 + \tfrac{1}{6}\sqrt{3}\delta m. \end{aligned} \quad (3.2)$$

Taking as inputs the masses of K^*, ρ, and ω for 1⁻, and of $K^*(1430)$, $A_2(1310)$, and $f(1250)$ for 2⁺, respectively, we obtained the ϕ mass $= 1031$ MeV and $\delta m = 148$ MeV for 1⁻, and the f' mass $= 1528$ MeV and $\delta m = 140$ MeV for 2⁺. These masses of ϕ and f' are in good agreement with the observed values. Of course m_8 takes different values for 1⁻ and 2⁺, respectively. We can also predict the ratios of ϕ-ω mixing and of f'-f mixing without knowing the physical masses of ϕ and f', since we already know the off-diagonal elements in our method. The results are

$$(\tan\theta)_{1^-} = 1.24, \quad (\tan\theta)_{2^+} = 1.95, \quad (3.3)$$

while the experimental values are

$$(\tan\theta)_{1^- \text{ exp.}} = 1.26 \quad \text{and} \quad (\tan\theta)_{2^+ \text{ exp.}} = 2.00.$$

The fact that nearly equal δm's are obtained for 1⁻ and 2⁺ encourages us to continue the analysis.

IV. ½⁺ BARYONS AND THEIR EXCITED STATES

In (I) the baryon octet was assigned to an (8,1) representation, and turned out to be a pure f-type

splitting, where the value of δm was also a little too large (~ 180 MeV). As another candidate we take $(15^*,3) = 8 + 10^* + 27$. All these must be ½⁺ baryons. Matrix elements of S_8 are obtained by a straightforward calculation, and the diagonalization of the total Hamiltonian gives the exact mass relations. Here, however, we calculate the mass levels in the perturbation theory assuming $\delta m/m_{10^*}$ and $\delta m/m_{27} \ll 1$. Second-order perturbation calculation gives

$$\begin{aligned} \lambda_\alpha(8) = {} & m_8 - \delta m \langle 8_\alpha | S_8 | 8_\alpha \rangle \\ & + [(\delta m)^2/(m_8 - m_{10^*})] |\langle 8_\alpha | S_8 | 10^* \rangle|^2 \\ & + [(\delta m)^2/(m_8 - m_{27})] |\langle 8_\alpha | S_8 | 27 \rangle|^2. \quad (4.1) \end{aligned}$$

The average masses m_{10^*} and m_{27} are supposedly very large, so we may neglect the second-order terms. Then the octet baryons are given by

$$\begin{aligned} N &= m_8 - (\sqrt{3}/30 + \tfrac{2}{3}\sqrt{3})\delta m, \\ \Lambda &= m_8 - (\sqrt{3}/15)\delta m, \\ \Sigma &= m_8 + (\sqrt{3}/15)\delta m, \\ \Xi &= m_8 - (\sqrt{3}/30 - \tfrac{2}{3}\sqrt{3})\delta m. \end{aligned} \quad (4.2)$$

Taking N and Ξ as inputs, we obtain $\delta m = 160$ MeV, $m_8 = 1137$ MeV, $\Lambda = 1118(1115)$ MeV and $\Sigma = 1155(1190)$ MeV. The figures in the parentheses are experimental values. They agree very well with the calculated values.

One might say about the above argument that the assumption of $U(3) \times U(3)$ as a good symmetry for the ½⁺ baryons would be violated, that the baryons should be assigned to (8,1) rather than to (15*,3) in the lowest order, and that the contributions from the low-mass continuum should be taken into account as the violation effect. This is one possibility. Here we rather assign the ½⁺ baryon to (15*,3). Using quarks (3,1) and antiquarks (1,3*) as a guide for constructing representations, it is easily seen that the state (15*,3) can be composed of five quarks and two antiquarks ($3^5 = 15^* + \cdots$, $3^{*2} = 3 + \cdots$). The F/D ratio of the mass splittings is given fairly well in this assignment.[10]

As other candidates we have examined $(6^*,3^*) = 8 + 10^*$ and $(8,10^*) = 8 + 10^* + 27 + 35^*$. The results of a calculation for the case of $(8,10^*)$ are as follows: If we use the masses of Ξ, Σ, and Λ as inputs, it gives $N = 900(940)$, $\Omega^+ = 1244$, $N^* = 1462$, $\bar{\Delta} = 1466$, and $Y_1 = 1536$. N^* may be identified with the observed $N^*(\tfrac{1}{2}^+)(1480)$, but the predicted Ω^+ is stable for the strong interactions. The situation is similar in the case of $(6^*,3^*)$. The constant δm is 165 MeV for $(8,10^*)$ or 230 MeV for $(6^*,3^*)$. The experimental absence of Ω^+ and the large calculated value of δm seem to rule out the possibility of these two representations.

[8] This was first suggested by S. Ishida from a different point of view. Soryusiron Kenkyu (in Japanese) 30, 372 (1964).

[9] S. L. Glashow and R. H. Socolow, Phys. Rev. Letters 15, 329 (1965); S. L. Glashow, Tokyo Summer Institute of Theoretical Physics, Oiso, Japan, 1965 (unpublished).

[10] Moreover if the new term S_0 in addition to S_8 is taken into account, the ratio of the average mass of the ½⁺ baryons to that of ³⁄₂⁺ baryons is reasonably interpreted under a certain assumption. [Details will be published Progr. Theoret. Phys. (Kyoto) by one of the present authors (Y. I.).]

We may remark that the $\frac{3}{2}^+$ decuplet and $\frac{3}{2}^-$ octet [$\Sigma^*(1810)$, $N^*(1512)$, $Y_1^*(1660)$, and $Y_0^*(1520)$] give $\delta m = 165$ MeV in both cases if we assign them to (10,1) and (8,1), respectively.[4] The octet must correspond to a pure f-type splitting in our model. The discrepancy of $Y_0^*(1520)$ with the prediction suggests that the $\frac{3}{2}^-$ baryons might be (8,1)+(1,1) and another Y_0^* might be found at 1660 MeV which is degenerate with Y_1^*. Finally we may remark that the three multiplets have given almost the same δm, though it is 10% larger than in the case of mesons.

V. 0^- NONET

Finally we go on to the case of 0^- nonet mesons [K, $\bar{K}$, π, η, $\chi(959)$], where the possible representations are $(3,3^*) = 1+8$, $(6,6^*) = 1+8+27$, etc. First we consider $(3,3^*)$. For K and π we can easily write down the following relations

$$K = m_8 + \tfrac{1}{6}\sqrt{3}\,\delta m, \quad \pi = m_8 - \tfrac{1}{3}\sqrt{3}\,\delta m. \quad (5.1)$$

There occurs mixing between η and χ and the eigenvalue equation for them is

$$\begin{vmatrix} m_1 - \lambda, & -(\sqrt{\tfrac{2}{3}})\delta m \\ -(\sqrt{\tfrac{2}{3}})\delta m, & m_8 + (\sqrt{\tfrac{1}{3}})\delta m - \lambda \end{vmatrix} = 0. \quad (5.2)$$

Taking $\delta m = 150$ MeV, which is consistent with the values for the vector mesons and baryons, and using the observed values of K and χ as inputs, the masses of η and π can be predicted. The results are

$$\eta = 506 \text{ MeV}, \quad \pi = 368 \text{ MeV}. \quad (5.3)$$

The η mass is predicted within an error of 8%, but we cannot be satisfied with the value of the π mass. If we use another value larger than 150 MeV for δm to increase the mass difference of K and π, then the η ends up with a smaller mass, and the situation cannot be improved.

In the case of $(6,6^*)$, using the same value for δm and taking the approximation of $m_{27} \rightarrow \infty$,[11] we can obtain a mass difference of K and π which is 7/5 as large as that for $(3,3^*)$. But if χ is fixed at 960 MeV, the predicted value of η is 465 MeV, which is too small, and the difficulty remains.

Another possible way to deal rather satisfactorily with this situation is to assign the 0^- mesons crudely to $(3,3^*)$ and take into account the effect of very massive scattering states by including the contribution from **27** of the unperturbed state.[12] In this approach the commutation relations (2.2)–(2.4) are surely satisfied, but this representation is not irreducible. However, if the 0^- mesons really belong to a higher dimensional representation than $(3,3^*)$ or $(6,6^*)$, then this approach may be meaningful as the approximation to it. Now using the same value for δm, we can consistently fit the masses of K, π, η to the observed ones, but for χ a too large value is obtained, i.e., $\chi = 2701$ MeV. Conversely, if we fit χ to the recently discovered resonance (1420 MeV),[13] we obtain the following result: $K = 498$ MeV, $\eta = 510$ MeV, and $\pi = 290$. So if we assign the $(\bar{K}K\pi)$ resonance (1420 MeV) rather than $\chi(960)$ to **1** of $(3,3^*)$, this result is satisfactory. Although the pion mass is larger than the observed one, it is smaller than that in the $(3,3^*)$ scheme (368 MeV).[14]

All the trouble comes from the fact that among the octet members the pion has a very small mass, and to our knowledge no one has succeeded in dealing satisfactorily with this difficulty by means of a linear mass relation.

VI. CONCLUSION

We have investigated the relations of the mass splittings for $\frac{1}{2}^+$, $\frac{3}{2}^+$, $\frac{3}{2}^-$, 0^-, 1^-, and 2^+ families, assuming that the $U(3)$-violating term in the Hamiltonian is the eighth component of the scalar current, and that the positive parity algebra generating $U(3) \times U(3)$ is a good symmetry of the hadrons at rest. The fact that most of the known multiplets give almost the same value of δm may be a strong support to our model. The predictions are shown in Table I. One of the interesting points is the interrelation between the meson and the baryon masses when the universal constant δm is given. Such a relation cannot be predicted by, say, the classical $SU(6)$. Miyazawa et al. and Cocho et al.[15] tried to unify the mesons and the baryons under a higher symmetry $SU(9)$, and obtained some mass formulas relating them. But a unification of that kind encounters a serious difficulty if a correct treatment of the statistics is attempted.

In Table II we show the masses of the particles of the various multiplets that are predicted, using the same value of δm ($\delta m = 160$ MeV).

We may remark that the quadratic mass should not be used in our formulas, since we have estimated the expectation values of the Hamiltonian between the states at rest. In the case of the mesons, the Gell-Mann–Okubo formula has been applied to the quadratic masses in conventional treatments. In our opinion it should be applied to the linear masses and should be calculated by adding the second-order correction. In

[11] Keeping m_{27} finite, we can obtain a set of solutions, $K = 517$, $\pi = 127$, $\eta = 530$, $\chi = 930$, but the masses of the members of **27** appear too small (the average mass $m_{27} \approx 300$ MeV) and it is very difficult to reconcile these results with the present experimental data.

[12] In this case m_{27} also becomes smaller than m_8, if we look for a solution with m_{27} finite, so that we take the very massive states ($m_{27} \rightarrow \infty$) as **27**.

[13] A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz, and M. Roos, Rev. Mod. Phys. **37**, 633 (1965).

[14] If we add the contributions from the **10** and **10*** states both masses of π and χ can be determined to fit in with the observation.

[15] H. Miyazawa and H. Sugawara, Progr. Theoret. Phys. (Kyoto) **33**, 771 (1965) and **34**, 263 (1965); G. Cocho and E. Chacon, Phys. Rev. Letters **14**, 521 (1965); Y. Iwasaki, S. Matsuda, H. Miyazawa, H. Okamura, and H. Sugawara (private communication); G. Cocho, Trieste (unpublished report).

TABLE I. The verification of the universality of δm and the predictions of masses. The numbers in parentheses stand for the observed values.

Input (MeV)		Output (MeV)	
1^-; $(3,3^*)=1+8$	$K^*=891$; $\rho=763$; $\omega=783$	$\phi=1031(1020)$;	$\delta m=148$
2^+; $(3,3^*)=1+8$	$K^*=1430$; $A_2=1310$; $f=1258$	$f'=1528(1525)$;	$\delta m=140$
$\frac{1}{2}^+$; $(15^*,3)=8+10^*+27$	$N=940$; $\Xi=1315$; $m_{10^*},m_{27}>2000$	$\Lambda=1118(1115)$; $\Sigma=1155(1190)$;	$\delta m=160$
$\frac{3}{2}^+$; $(10,1)=10$	$N^+=1238$; $Y_1^*=1385$	$\Omega^+-\Xi^*=\Xi^*-Y_1^*=145(145)$;	$\delta m=165$
$\frac{3}{2}^-$; $(8,1)=8$	$N^*=1512$; $\Xi^*=1810$	$Y_1^*=1661(1660)$; $Y_0^*=1661(?)$;	$\delta m=165$
0^-; $1+8+27$(scatt)	$\delta m=150$; $(K\bar{K}\pi)=1420$; $K=498$	$\eta=510(549)$; $\pi=290(140)$;	$\cdots$

TABLE II. The mass levels of the hadrons when $\delta m=160$ MeV. The numbers in parentheses stand for the observed values.

Input (MeV)			Output (MeV)
1^-; $(3,3^*)=1+8$		$K^*=891$; $\omega=783$	$\rho=756(763)$; $\phi=1040(1020)$
2^+; $(3,3^*)=1+8$		$K^*=1430$; $f=1250$	$f'=1551(1525)$; $A_2=1291(1310)$
$\frac{1}{2}^+$; $(15^*,3)=8+10^*+27$	$\delta m=160$	$N=940$; $m_{10^*},m_{27}>2000$	$\Lambda=1118(1115)$; $\Sigma=1155(1190)$; $\Xi=1315(1315)$
$\frac{3}{2}^+$; $(10,1)=10$			$\Omega^--\Xi^*=\Xi^*-Y_1^*=Y_1^*-N^*=140(145)$
$\frac{3}{2}^-$; $(8,1)=8$		$N^*=1518$	$Y_1^*=1660(1660)$; $Y_0^*=1660(?)$; $\Xi^*=1800(1810)$
0^-; $(6,6^*)=1+8+27$		$K=498$; $m_1,m_{27}>3000$	$\eta=562(549)$; $\pi=304(140)$

our model, however, a satisfactory prediction of the pion mass could not be so obtained. This may come from (i) the violation of the dynamical symmetry of $U(3)\times U(3)$ and/or (ii) the wrong choice of the group representation for the ps meson. It is our feeling that the light particles such as ps mesons would not be consistent with $U(3)\times U(3)$, since the internal kinetic energy part, which is not invariant under our group, would become comparable to the invariant part under the group.

Finally we would like to remark that in our approach we cannot go so far as to predict all the average masses of the various multiplets. However, if we are to be able to find a group or an algebra which may probably be noncompact that can predict all the average masses of the finite or infinite series of multiplets, then our method can predict the mass splittings of all the multiplets only if a single-value δm is given.

APPENDIX: THE MATRIX ELEMENTS OF S_i

In evaluating the matrix elements of S_i, we used two methods. In one, we used the sum rules obtained by employing crossing matrices. In the other, we calculated the matrix elements explicitly by tensor techniques.

1. Sum Rules

We obtain sum rules by rewriting the left-hand side of Eq. (2.7) by means of crossing matrices and comparing it with the right-hand side. For reference we write down here the sum rules in the case that the initial and final states belong to **8** [(A1)–(A3)], and **10** [(A4)].

$$|F_{8_1}|^2+|F_{8_2}|^2+|F_1|^2/4-9|F_{27}|^2/4=3, \quad \text{(A1)}$$

$$-4|F_{8_1}|^2/5+|F_1|^2/4+(|F_{10}|^2+|F_{10^*}|^2)/2$$
$$-9|F_{27}|^2/20=0, \quad \text{(A2)}$$

$$2\,\mathrm{Re}(F_{8_1}F_{8_2}{}^*)+\tfrac{1}{2}(\sqrt{5})(|F_{10^*}|^2-|F_{10}|^2)=0, \quad \text{(A3)}$$

$$16G_8^2+10G_{10}^2+9G_{27}^2-35G_{35}^2=60, \quad \text{(A4)}$$

where F's have been defined in the text, and

$$\langle C_\alpha|S_i|10^*{}_\beta\rangle=G_C\begin{pmatrix}10^* & 8 & C\\ \beta & i & \alpha\end{pmatrix}.$$

When a multiplet is represented by $(3,3^*)$, for example, this method is simple and powerful. In this case, only $|F_{8_1}|^2$ and $|F_1|^2$ are not zero. So using Eqs. (A1), (A2), and (A3), the matrix elements are determined uniquely except for trivial ambiguities.

2. Explicit Calculation

To obtain the matrix elements of S_i it is merely necessary to calculate the matrix representation of the generators of $U(3)\times U(3)$. For example we calculate the matrix element for the case of $(6,6^*)$.

Taking one of the states of $(6,6^*)$, say, $\pi^+(Y=0, I=1, I_3=1)$, we write it in terms of $U(3)\times U(3)$ as

$$\pi^+=(\sqrt{\tfrac{4}{5}})(T^{\{21\}}S_{\{11\}}+T^{\{22\}}S_{\{12\}}+T^{\{23\}}S_{\{13\}}), \quad \text{(A5)}$$

where T and S are the tensors transformed by G^+ and G^-, respectively. Let us calculate $\langle\pi^+|S_8|\pi^+\rangle$, the expectation value of

$$S_8=\tfrac{1}{2}\sqrt{3}(Y^+-Y^-). \quad \text{(A6)}$$

Here Y^+ and Y^- are hypercharge operators $[Y^\pm=\tfrac{2}{3}\sqrt{3}G_8^\pm]$.

On the other hand, from the transformation property of S_i, we can write

$$\langle \pi^+ | S_8 | \pi^+ \rangle = F_{8_1} \begin{pmatrix} 8 & 8 & 8_1 \\ \pi^+ & \eta & \pi^+ \end{pmatrix}. \qquad (A7)$$

Putting Eqs. (A5) and (A6) into the left-hand side of Eq. (A7), performing tensor calculations, and comparing the left-hand side with the right-hand side, we find $F_{8_1} = \frac{3}{5}\sqrt{3}$.

The other matrix elements can be obtained along the same lines.

Calculation of the Nucleon Magnetic Moments by Dispersion-Theory Methods

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The magnetic moments of the neutron and proton are calculated within the framework of the S-matrix perturbation theory recently developed by Dashen and Frautschi. In the present context, this method expresses the magnetic moments in terms of a dispersion integral involving photopion production. Evaluation of this integral in terms of contributions from appropriate low-mass intermediate states yields results for the individual magnetic moments which are larger than the experimental values by about a factor of two. The calculation does, however, give an approximately correct value for the ratio of the isovector moment to the isoscalar moment, and a value for the isoscalar moment that agrees with the experimental value to within about a factor of two.

I. INTRODUCTION

RECENTLY, Dashen and Frautschi[1,2] have suggested a method for finding the changes in the residues and positions of bound-state poles in the S matrix when the strong interactions are perturbed by the addition of another, weaker force. Dashen[3] applied this method to calculate the proton-neutron mass difference and obtained a result in good agreement with experiment. It is our purpose in this paper to apply these methods to calculate the magnetic moments of the nucleons.

To discuss the nucleon magnetic moments from this point of view, we need to study a scattering process in which the magnetic moments appear as a residue of a pole in the scattering amplitude. Photopion production in the $J=\frac{1}{2}^+$, $T=\frac{1}{2}$ channel has a nucleon pole whose residue, apart from kinematic factors, is proportional to the nucleon magnetic moments. Therefore, this is an appropriate process to study. Note, however, that it is the residue of the photoproduction amplitude, not a perturbation on this residue, which contains the quantity we want to calculate.

To understand in what sense this may be regarded as a perturbation calculation, one may consider a two-

channel S matrix in which channel 1 is the $J=\frac{1}{2}^+$, $T=\frac{1}{2}\pi N$ state, and channel 2 is the $J=\frac{1}{2}^+$, $T=\frac{1}{2}\gamma N$ state. When electromagnetism is turned off, channel 1 is coupled only to itself, through the strong interactions, and there is no scattering in the 22 amplitude. This defines the unperturbed problem. If the electric forces are turned on, and only terms of *first* order in e are kept, then the S matrix is *changed* only by the appearance of new nonzero matrix elements corresponding to photoproduction, which are of order e. It is in this sense that we speak of carrying out a perturbation calculation here.[4]

It may seem strange to think of the magnetic moments as in any way connected with an electromagnetic perturbation, because it is, of course, true that the magnetic moments are closely related to the nucleon form factors at $q^2=0$, which are determined by the strong interactions alone. It is, therefore, important to note that the basic formula to be used here for the magnetic moments can also be derived in a way which makes it clear that the magnetic moments are not of electromagnetic origin (see Sec. II).

Thus, we consider the $J=\frac{1}{2}^+$, $T=\frac{1}{2}$ partial-wave photoproduction amplitude. One would not normally expect to be able to determine the residue of this amplitude at the nucleon pole by purely S-matrix methods, simply because analyticity alone is compatible with any value whatever for this residue. However, if we *require* a suitably rapid convergence at high energies

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[1] R. F. Dashen and S. C. Frautschi, Phys. Rev. 135, B1190 (1964).

[2] R. F. Dashen and S. C. Frautschi, Phys. Rev. 137, B1318 (1965).

[3] R. F. Dashen, Phys. Rev. 135, B1196 (1964).

[4] This point is discussed in Ref. 2.