Echo Processes in a Plasma*

G. F. HERRMANN AND R. F. WHITMER
Lockheed Palo Alto Research Laboratory, Palo Alto, California
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The generation of echoes in classical systems is discussed with particular reference to cyclotron echoes in a plasma. A system of oscillators subjected to a sequence of pulses will produce an echo if the interaction contains appropriate nonlinear terms. A formal description of the echo mechanism is given for the case where nonlinear effects are confined to the times at which pulses are present, with emphasis on wave-propagation aspects. Diagrams are introduced for the classification of various nonlinear processes. The generation of echoes in a plasma interacting with microwave pulses propagating normal to an applied magnetic field is analyzed in a small-signal approximation. Echoes are found to arise from nonlinear terms associated with the convective motion of the electrons, and from the interaction with the magnetic component of the microwave pulses. Equations for the echo amplitude, in terms of the pulse widths and intensities, are given for transverse and longitudinal propagation.

I. INTRODUCTION

THE recent observation of an "echo" (an analog to "spin echo") at cyclotron resonance in a plasma[1] is interesting on several accounts. The magnitude of the echo indicates that the interaction must involve the electron orbit, rather than the spin, and therefore represents an entirely new effect. In addition the cyclotron echo provides a direct means of measuring and displaying relaxation processes such as momentum transfer, energy transfer, and diffusion. In itself it provides an example of a strictly classical system displaying a very pronounced manner a phenomenon normally associated with two-level quantum-mechanical systems.

The reader is assumed to have some familiarity with Hahn's original paper[2] on nuclear-spin echo. Briefly, a system of nuclear spins placed in a magnetic field is excited by a pulse at resonance and then left to precess. Because of slight inhomogeneities in the field, the phases of the precession of individual nuclei become rapidly incoherent and the macroscopic precession of the total magnetization disappears. At a time τ after the first pulse another pulse is applied, which causes a rearrangement of the phase of individual nuclear precessions, so as to cause the reappearance after the lapse of an additional interval τ, of a resultant component of the magnetization. This magnetization radiates a short electromagnetic pulse, referred to as an echo. Weaker echoes appear at other integral multiples of τ. In a similar manner, a sequence of three pulses gives rise to an echo whose separation from the third pulse equals the separation between the first two pulses. The very graphic description of the spin echo mechanism which Hahn presents is based essentially on the Bloch equation,[3] which treats an ensemble of spins as a classical system of precessing gyroscopes. Hahn's description can be easily extended to any ensemble of two-level quantum-mechanical systems. In fact, as shown by Feynman,[4] the evolution of any two-level system can be conveniently described in a fictitious three-dimensional space, where it obeys gyroscope equations completely analogous to those of a spin vector in real space. The various echo phenomena observed in the past, e.g., electron-spin echo,[5] echo from optical transitions,[6] and ferromagnetic echo,[7] all fall into this class, although the last of these involves a more nearly "classical" many-body problem.

A plasma, in contrast to the above, represents a truly classical system. At cyclotron resonance, its quantum mechanical description (in the nonrelativistic domain) is essentially analogous to that of an ensemble of harmonic oscillators with an infinite number of equally spaced energy levels, that is, systems as unlike as possible to two-level systems. Moreover, under normal conditions the average quantum number of the excitation is enormous. A classical treatment is therefore required.

One must first point out the obvious fact that in a truly linear system there is no echo. To clarify the use of the term linearity, let us consider the interaction as taking place in a "black box," with a series of pulses impinging from the outside. Each pulse by itself generates a response, e.g., in the form of a decay function. If it is now assumed that linear superposition holds, then a sequence of pulses will give rise to a response which is simply the sum of the individual responses and no more. An echo can appear only if the response to a given pulse depends on the "preparation" of the medium by previous pulses. This requires some nonlinearity. The deviation from linearity may take diverse forms, and the behavior of the echoes will vary accordingly. Experi-

* Supported by the Lockheed Independent Research Fund.
[1] R. M. Hill and D. E. Kaplan, Phys. Rev. Letters 14, 1062 (1965).
[2] E. L. Hahn, Phys. Rev. 80, 580 (1950).
[3] F. Bloch, Phys. Rev. 70, 460 (1946).
[4] R. P. Feynman, F. L. Vernon, Jr., and R. L. Hellworth, J. Appl. Phys. 28, 41 (1957). A step in the opposite direction is taken by Jaynes and Bloom in an elegant 2-dimensional spinor formalism for spin echo; see E. T. Jaynes, Phys. Rev. 98, 1099 (1955); A. L. Bloom, *ibid.* 98, 1105 (1955).
[5] J. P. Gordon and K. D. Bowers, Phys. Rev. Letters 1, 369 (1958).
[6] N. A. Kurnit, I. D. Abella, and S. R. Hartmann, Phys. Rev. Letters 13, 567 (1964).
[7] D. E. Kaplan, Phys. Rev. Letters 14, 254 (1965).

ments so far seem to indicate that in the formation of cyclotron echoes in a plasma more than one process is involved. The present discussion will confine itself to one particular class of nonlinearities, namely those which involve the direct interaction of the plasma with electromagnetic pulses, which relate to such other phenomena as harmonic generation and mixing.[8] This case presents the direct classical analog of the spin-echo process.

Since the nonlinear equations in question are not soluble in closed form, the treatment employs various approximations in order to simplify the mathematics and to expose the underlying physical principles. The presentation is thus confined to (a) a formal description of the echo mechanism in a system of oscillators whose interaction with an accelerating field deviates from strict linearity, and (b) an analysis of some nonlinear effects of this type in a plasma.

II. FORMAL DESCRIPTION OF ECHO PROCESS

In this section we deal in a formal way with some general aspects of echo generation in classical systems. The phenomena which we consider are generalizations of Hahn's spin echo,[2] with an unspecified oscillator in the role of the nuclear spin. To simplify the discussion we asume that the system is almost linear and regard the nonlinear effects as perturbations to be treated in the small-signal approximation.

The system under consideration is a very large ensemble of oscillators whose natural frequencies are spread in a narrow range $(\Delta\omega)_{\max}$ about a central frequency ω_0. [In the spin echo case, $(\Delta\omega)_{\max}$ would correspond roughly to Hahn's $1/T_2^*$.] The oscillators might consist of atomic entities, e.g., individual electrons, or of suitable normal modes in an interacting many-body configuration. Excitation is produced by a series of short pulses at ω_0, of duration t_w, such that $t_w(\Delta\omega)_{\max}\ll 1$. For a pulse of such short duration the excitation of each oscillator seems in effect to occur at resonance, since the Fourier spectrum of the pulse incorporates all the natural frequencies of the system. Between pulses, however, the phases of individual oscillators drift apart, resulting in the disappearance of coherent macroscopic moments after times large compared to $1/(\Delta\omega)_{\max}$. During these intervals the excitation of an oscillator can be described in terms of a complex amplitude A with a time dependence of the form $\exp[-i\omega_0 t-i\Delta\omega t]$, where $\Delta\omega$ is characteristic of the individual oscillator. (This behavior is of course modified by relaxation mechanisms of various kinds, e.g., collisions or diffusion.) Since $t_w\Delta\omega\ll 1$, $t\Delta\omega$ can be considered as constant for the duration of a pulse. We will refer to $\Delta\omega t$ as the relative phase angle, or when no confusion is possible, simply as the phase angle of the oscillator.

The macroscopic moment at any time is obtained by summing the amplitude A for all oscillators, or, in effect, integrating over the range of values assumed by $\Delta\omega$. For $(\Delta\omega)_{\max}t\gg 1$, such an integral is normally negligible since the oscillators can be expected to be evenly distributed among all phases. However, exceptions to this statement may occur at particular prescribed times. These arise in the following manner. Suppose we start with a pulse at $t=0$, and assume that it imparts equal amplitudes to all oscillators. Consider the ensemble at a time $t=\tau$, and in particular, that class of oscillators whose relative phases are within a narrow range about some angle θ. For $\tau(\Delta\omega)_{\max}\gg 1$ this class, henceforth referred to as a θ class, does not correspond to a single natural frequency. In fact, it includes a large selection of natural frequencies distributed over the entire available range. Hence, at times other than $t=\tau$ the phase angles in this class do not coincide and the total moment of the class vanishes. It will be also noted that at multiples of t, i.e., when $t=n\tau$, the phases in a given θ class will coincide again at a value $n\theta$. To a second pulse, introduced at $t=\tau$, all oscillators in a class will look the same and will therefore manifest an identical response. The total moment imparted to the θ class by the second pulse will rapidly disappear as oscillator phases separate to reappear only at integral multiples of τ. (This consideration must be somewhat modified if the natural frequencies are themselves affected by the pulses.) Thus, if a macroscopic moment (which is the sum of all θ-class moments) is to appear, it can do so only at $t=n\tau$, when each of the classes can contribute a finite moment. Similar considerations apply also to 3-pulse sequences. The existence of a finite macroscopic moment is of course not guaranteed and depends on the presence of a nonlinear mechanism which in some way will cause different θ classes to behave in a different manner.

In a first attempt to understand echo processes it is useful to assume that the nonlinearity is a small perturbation, so that at least over an interval of many cycles one can employ the linear concepts of natural frequency and phase angle. The following are examples of echo-producing nonlinearities:

1. A dependence of the natural frequency on amplitude. Since the amplitude after pulse 2 is a function of θ, the frequencies of various θ classes will have shifted by varying amounts. Each θ class will return to coincidence at $t=2\tau$, not at the angle 2θ but at a slightly shifted angle. This results in bunching of the phases and in the appearance of a resultant moment.

2. A dependence of dissipation on amplitude, e.g., through an amplitude dependent collision frequency. This process, suggested by Gordon,[9] involves the selective elimination, after pulse 2, of those θ classes which correspond to high collision frequencies.

3. Nonlinearity confined to the interaction of the oscillator with the pulsing field. This process differs from

[8] R. F. Whitmer and E. B. Barrett, Phys. Rev. 121, 661 (1961).

[9] J. P. Gordon (private communication).

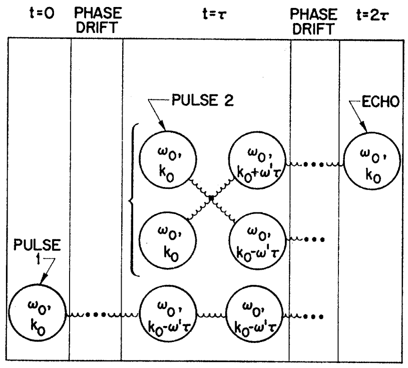

FIG. 1. Diagram for 2-pulse sequence.

the mechanisms of 1. and 2. in that the echo effect is instantaneous, i.e., it does not require a finite length of time τ in order to manifest itself. Of the various processes, this type is closest to representing the classical analog of spin echo.

In the present work we shall deal exclusively with processes belonging to the third type. Accordingly, we assume that the oscillators are strictly linear in the absence of a pulse. Moreover, we assume that the nonlinear interaction with the pulse is nondissipative, and can be treated in a small signal approximation.

The principal manifestation of this type of nonlinear interaction is the generation of "beat" signals whose phase angles are sums and differences of the phase angles of the various disturbances present in the medium. Such beats arise in the formalism whenever a product of signal amplitudes appears in the equations of motion. The relation of these beats to the echo problem is summed up in the following statement: whenever a particular beat-signal amplitude is independent of the value of $\Delta\omega$ characterizing the individual oscillators, an echo will appear. This is because integration over all $\Delta\omega$ of the ensemble will then amount to integration over a constant amplitude, and this must yield a finite moment. The identification of beat signals possessing this property therefore constitutes the first step in searching for echo conditions. The following example will clarify this procedure.

Consider the case of the so-called 3-pulse sequence. Pulse 1 at time $t=0$ is followed after an interval τ by pulse 2, to be succeeded by pulse 3 at $t=T$. Consider the phase angles associated with excitations of a particular oscillator at the time $t=T$ of pulse 3. The excitation produced by pulse 1 will have acquired the phase angle $\theta_1=\alpha+\omega t+\Delta\omega T$, that produced by pulse 2, $\theta_2=\beta+\omega t+\Delta\omega(T-\tau)$ and pulse 3 will have the phase $\theta_3=\gamma+\omega t$. (Here we have made use of the fact that, for the duration of the pulse, $\Delta\omega t$ can be regarded as constant. α, β, and γ are phase constants associated with the three pulses. Since they appear equally in the phases of all oscillators they could as well be set equal to zero.) Consider now a beat signal of phase θ_e, obtained by adding θ_2 to θ_3 and subtracting θ_1. We find

$$\theta_e=\theta_2+\theta_3-\theta_1=(\beta+\gamma-\alpha)+\omega t-\Delta\omega\tau$$

at the time of the third pulse. After the lapse of another interval τ, θ_e will have again increased by the drift angle $\Delta\omega\tau$, and will equal

$$\theta_e=(\beta+\gamma-\alpha)+\omega t.$$

Thus, at $t=T+\tau$, there appears a component whose phase is independent of the individual oscillator, and hence an echo.

In the mathematical formalism the beat signals arise from products of exponential terms, each multiplication producing a sum or difference of phase angles. The beat signal corresponding to the above echo results from the combination of three phase angles associated with each of the three pulses. It therefore corresponds to a third-order term in the expansion of the nonlinear interaction, and the echo is proportional to each of the three pulses.

A more elegant formal approach to the combination of phase angles can be obtained by considering the wave aspects of echo generation described in the next paragraphs.

Wave Picture of Echo; Diagrams

The generation of an echo in a plasma involves the interaction with a propagating electromagnetic wave, which establishes a phase pattern in the plasma. It is therefore useful to provide a visualization of the echo process in the context of wave propagation. At present we shall view the medium as consisting of independent, localized oscillators. A pulse traversing the medium leaves in its wake a disturbance characterized by the local resonance frequency ω_r, and the propagation constant k_0 of the pulse. We now assume that ω_r varies slightly within the medium, and for the sake of simplicity confine this variation to be along the direction of propagation (the generalization to arbitrary variation is easily made). At the end of the pulse each oscillator continues to oscillate at its own natural frequency, with a phase angle given by

$$\theta=k_0x-\omega_r(x)t, \tag{1}$$

where t is the time elapsed after the pulse.

Since ω_r is now itself a function of x, the apparent wave number (or wavelength) varies slowly with time. One can define a local, instantaneous wave number k, by putting

$$k=\partial\theta/\partial x=k_0-\omega_r'(x)t, \tag{2}$$

which, during short intervals, describes properly the phase relations in a given neighborhood of x, the dephasing of the medium is thus displayed as an adiabatic change of the wave number. Now in order for the medium to reradiate, a disturbance in the medium must itself be characterized by an identical wave number k_0; otherwise, the radiation is eliminated by destructive interference. Hence, an echo indicates the reappearance in the medium of a wave with $k=k_0$.

The nonlinearity which comes into play in the interaction of the pulses with the medium is displayed in the form of beat signals. We have assumed that the interaction is lossless. It is a well-known fact from the theory of parametric systems that in such interactions both frequency and wave number must be conserved.[10] In other words, if we characterize each wave by the pair (ω,k), then the outcome in mixing m such pairs (ω_1,k_1), (ω_2,k_2), $\cdots$, (ω_m,k_m) are waves of the form $(n_1\omega_1+n_2\omega_2+\cdots+n_m\omega_m,\ n_1k_1+n_2k_2+\cdots+n_mk_m)$, where the n_i are integers.

A graphic representation of the possible processes is given by the familiar diagrams representing each wave in terms of a "quantum" characterized by a given (ω,k) pair. At a vertex (i.e., at the time of a pulse) such "quanta" combine to give various mixing products, such that the sum of all ω's and of all k's is conserved. It should be noted that conservation applies only in the case of pulses which are short compared to the dephasing time. Between pulses, the value of k for each wave varies according to Eq. (2). An echo consists of a mixing product represented by the pair (ω_0,k_0) at the time of its appearance.

Examples of echo processes are described in Figs. 1 and 2. Figure 1 represents a diagram for the 2-pulse echo. Pulse 1 at $t=0$ produces a wave (ω_0,k_0). By the time $t=\tau$ the phase of this wave has drifted and it is now described by $(\omega_0,k_0-\omega'\tau)$, where $\omega'=\omega_r'(x)$, for short. The mixing process at $t=\tau$ is given by the equation

$$2(\omega_0,k_0)\rightarrow(\omega_0,k_0-\omega'\tau)+(\omega_0,k_0+\omega'\tau).$$

The quantum $(\omega_0,k_0+\omega'\tau)$ then changes into (ω_0,k_0) at $t=2\tau$ to give an echo. The diagram actually represents the stimulated emission of a quantum at $(\omega_0,k_0-\omega't)$ and an "idler" (or Raman) quantum at $(\omega_0,k_0+\omega't)$, and requires for its occurrence the presence of the stimulating quantum $(\omega_0,k_0-\omega't)$.[11]

Figure 2 represents two of the possible diagrams describing a 3-pulse sequence. The first, (a), is analogous to the diagram in Fig. 1. The interaction occurs entirely at the time $t=T$ of pulse 3, and the role of pump is played by the combination of a quantum $[\omega_0,k_0-\omega'(T-\tau)]$ arising from pulse 2 and a quantum (ω_0,k_0) from pulse 3. The second process, (b), involves an interaction at $t=\tau$ in which a pulse-2 quantum, (ω_0,k_0), splits into 2 quanta, $(\omega_0,k_0-\omega'\tau)$ and $(0,\omega'\tau)$. The latter pair presupposes the existence in the medium of zero-frequency (dc) resonances, and such resonances do in fact exist in a plasma. The wave number of this dc wave is unaffected by the phase drift associated with

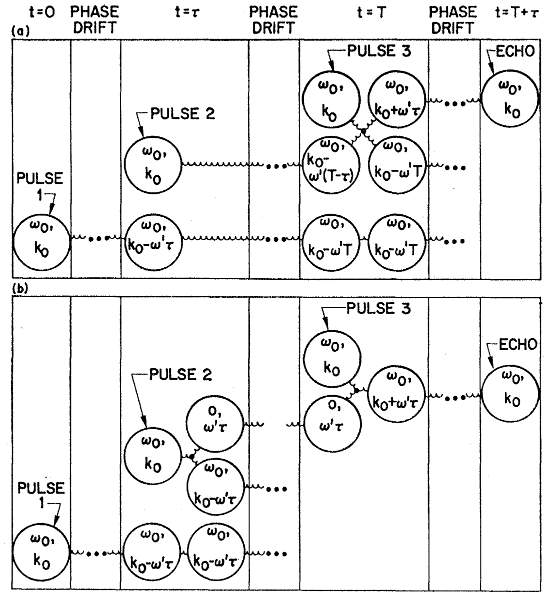

FIG. 2. Diagrams for 3-pulse sequence.

the spread of natural frequencies at ω_0, and therefore remains intact during the drift period. At the time $t=T$, it combines with a quantum (ω_0,k_0) of pulse 3 to give the echo. In either mechanism, the function of pulse 3 is to retrieve the information relative to the phase of individual oscillators at $t=\tau$, and restore it with opposite sign. Between pulse 2 and pulse 3 this information must be stored in some manner, and it is here that the chief distinction between the two processes is displayed. In the process of Fig. 2(a) the information is stored in terms of the amplitude of the oscillation, which depends on the difference between the phase of pulse 2 and the phase of the oscillation just prior to pulse 2. In the process of Fig. 2(b) the information is stored in the amplitude of the dc resonance.

In similar manner one can describe higher order echo processes. For example a second echo (at time $t=3\tau$) in a 2-pulse sequence can (if we ignore dc modes) be described by a fifth-order diagram. Three quanta (ω_0,k_0) combine with 2 quanta $(\omega_0,k_0-\omega t)$ at time $t=\tau$ to give, among others, a quantum $(\omega_0,k_0+2\omega'\tau)$ which, at $t=3\tau$, gives rise to an echo. The procedure can also be extended to signals at several frequencies. For example, a pulse at (ω_0,k_0) followed by a pulse at the second harmonic, $(2\omega_0,2k_0)$, will give rise to an echo at the fundamental.

The Echo Coefficient

The wave formalism developed above enables one to select in a given nonlinear expansion those terms which can be expected to give rise to an echo. The nonlinear interaction is, however, a property of the individual oscillator. For the type of echo processes discussed here, one can ignore the initial and final drift periods

[10] See, e.g., W. H. Louisell, *Coupled Mode and Parametric Electronics* (John Wiley & Sons, Inc., New York, 1960), Chap. 5.

[11] The analogy to parametric mixing (or stimulated Raman scattering) is obvious, with pulse 1, as modified by the drift period, in the role of signal, pulse 2 (or rather its second harmonic) in the role of pump, and the echo in the role of idler.

and describe the echo in terms of the response of any particular oscillator to the given sequence of pulses. This is done by introducing a so-called echo coefficient which relates the echo-generating component at the end of the last applied pulse to the amplitude of the original disturbance.

In the following we refer to the pulse (or pulses) whose echo is reproduced as the signal pulse and to the remaining pulses as pump pulses. We are concerned only with the latter, since the echo mechanism is implicit in the relation which these pulses establish between the initial and final excitation of an oscillator, a relation akin to a partial time reversal.

Let the initial excitation at the time of arrival of the first of the pump pulses be given by $A_i\exp(-i\omega_0 t)$ and at the end of the last pulse by $A_f\exp(-i\omega_0 t)$, where the phase-drift term $\exp(-i\Delta\omega t)$ is included in A_i and A_f. We shall define the echo coefficient by the expression

$$S=\sum A_f A_i/\sum A_i^* A_i, \qquad (3)$$

where A_i^* is the complex conjugate of A_i and where the summation is over all oscillators. That S indeed measures the effectiveness of the pulses in generating an echo can be seen as follows. Suppose that a short signal pulse arrives at $t=0$, to excite a uniform real amplitude A_0 which is the same for all oscillators. Let A_i represent the excitation of a particular oscillator at the time τ, i.e., just prior to the pump pulses. Then

$$A_i=A_0\exp(-i\Delta\omega\tau),$$

and according to (3)

$$S=(1/nA_0)A_f\sum\exp(-i\Delta\omega\tau),$$

where n is the number of oscillators. Now $A_f\exp(-i\Delta\omega\tau)$ represents the amplitude of the oscillator at a time τ following the last pulse. Hence $\sum A_f\exp(-i\Delta\omega\tau)$ is the echo moment at that time. On the other hand nA_0 is the initial moment. S therefore represents the ratio of the echo to the original signal. The calculation of an echo thus reduces to that of the echo coefficient S.

In general A_f is given as a function of initial conditions;

$$A_f=f(A_i,A_i^*), \qquad (4)$$

where the function f is determined by the nature of the pump pulses. (The dependence of A_f on A_i^*, as well as on A_i itself, arises in the complex formalism of oscillations whenever nonlinear interactions are included.) In the small signal approximation, Eq. (4) can be expanded in the form

$$A_f=A_i+Q+RA_i+SA_i^*, \qquad (5)$$

where Q, R, and S are functions of the pump pulses, and do not depend on the individual oscillator. The first two terms represent the linear superposition of the initial excitation and the excitation produced by the pulses. The remaining terms result from the nonlinear interaction. That S in Eq. (5) is indeed the same as in Eq. (3) can be seen by substituting the former into the latter equation and noting that $\sum A_i$ and $\sum A_i^2$ vanish for $\tau\Delta\omega\gg 1$. In this approximation the echo coefficient is thus defined as the coefficient of A_i^* in the expansion of A_f in terms of the initial excitation, and in calculating an echo we will simply seek to establish this coefficient. This approach no longer requires integration over the ensemble. The existence and magnitude of an echo can now be established from the study of the response of a single oscillator.

The echo coefficient has been defined and can be calculated without reference to the wave formalism previously developed. However, in the detailed mathematical manipulations, the diagrams are of considerable help in selecting, out of the multitude of terms in the nonlinear expansion, those which are associated with an echo.

Example

As an example we consider the case of a harmonic oscillator whose driving term depends on the absolute displacement from equilibrium. For very small amplitudes the equation can be put in the form

$$\ddot{x}+(\omega_0+\Delta\omega)^2x=E(1-\alpha x^2)[\exp(-i\omega_0 t)+\text{c.c.}], \qquad (6)$$

where α is a constant and $\alpha x^2\ll 1$. The term c.c. as used here, and subsequently, represents the complex conjugate of the preceding expression. E is in the form of a short rectangular pulse of duration t_w. For short t_w the deviation $\Delta\omega$ may be ignored in Eq. (6) for the duration of a pulse. It must however be included in describing the oscillation between pulses.

The solution during the presence of a pulse is obtained in the approximation of slowly varying coefficients. One puts

$$x=A(t)\exp(-i\omega_0 t)+\text{c.c.}+\text{other harmonics}. \qquad (7)$$

On substituting Eq. (7) into Eq. (6) one neglects time derivatives higher than first of the harmonic coefficients. By comparing coefficients in Eq. (6) one obtains a hierarchy of equations, of which we require only the first,

$$-2i\omega_0\dot{A}=(1-2\alpha AA^*)E, \qquad (8)$$

where only lowest order interaction terms are retained. Next, A is expanded as a power series in Et_w, which is substituted into Eq. (8). Comparison of the linear coefficients gives the first-order solution

$$A(t_0+t_w)=A(t_0)+(i/2\omega_0)[1-2\alpha A(t_0)A^*(t_0)]Et_w, \qquad (9)$$

where $t=t_0$ represents the onset of the pulse.

Equation (9) is sufficient for calculating the "three-pulse" echo effect. In such a sequence pulse 1 is regarded as a signal and pulse 2 and 3 as pump pulses. The initial excitation A_i as defined above thus represents A at a time just prior to pulse 2 and A_f represents A at the end of pulse 3. The only possible echo mechanism is the one described by the diagram in Fig. 2(a). The process is first order in each of the pulses and the nonlinear interaction (or mixing) takes place during pulse 3. Hence, just after pulse 2, using only the linear part of Eq. (9), we can put

$$A=A_i+(i/2\omega_0)E_2t_{w2},$$

where E_2 and t_{w2} characterize pulse 2. Just prior to pulse 3, on account of the phase drift associated with $\Delta\omega$,

$$A=[A_i+(i/2\omega_0)E_2t_{w2}]\exp[-i\Delta\omega(T-\tau)],$$

which substituted into Eq. (9) will yield A_f. Actually, to obtain the echo coefficient S, we require just the coefficient of A_i^* in the expression for A_f. This is given by

$$S=(\alpha/2\omega_0^2)(E^2t_{w2})(E_3t_{w3}), \tag{10}$$

which shows the echo as depending directly on the intensity and duration of each of the pump pulses and on the nonlinear coefficient α.

To obtain the echo coefficient for a 2-pulse sequence one must carry the calculation to the quadratic term in Et_w. One finds

$$S=(\alpha/4\omega_0^2)(E_2t_{w2})^2. \tag{11}$$

The 2-pulse echo can be obtained as a special case of the 3-pulse echo by allowing pulse 2 and 3 to coincide. A factor of 2 is introduced because of coherence considerations.

III. ECHOES IN A PLASMA

The interaction of an electromagnetic wave with a plasma, in the neighborhood of cyclotron resonance, is usually studied in one of two configurations. In the first the wave is propagated in a circularly polarized mode, in a direction parallel to the constant magnetic field $\mathbf{B}_0$. In the second, both the direction of propagation and of the electric vector are perpendicular to $\mathbf{B}_0$. Cyclotron echo has so far been observed primarily for transverse propagation.

In the following discussions a cold plasma is assumed; i.e., effects associated with the electron temperature are ignored.[12] This is justified by the fact that in the experiments the excitation energy produced by the pulses is large compared to thermal energy. In addition, the usual assumptions of small deviation from neutrality is made. Ion motion is neglected.

The equations describing this system are given by Maxwell's equations, coupled to the moments of the Boltzmann equation for an electron plasma. The latter, for a cold plasma, reduce to the Lorentz force equation as applied to the average electron velocities.[12] The exact solution, for a plane-wave pulse incident on the plasma at resonance, is exceedingly complicated. The difficulty is aggravated by the need for a nonlinear analysis and because of the presence of a nonuniform magnetic field. Whether such an exact solution, even if feasible, would increase our understanding of the physical processes is doubtful. We therefore make the simplifying assumption that the electromagnetic pulses can be described, at least locally, in terms of a wave with a given propagation constant k_0. Such an assumption is certainly valid for relatively low electron densities, but even at higher densities the qualitative conclusions of the analysis should be meaningful.

In an inhomogeneous field, the behavior of the disturbance during drift periods between pulses is roughly described by Eq. (2). It is seen that k rapidly becomes large, and the wave assumes the characteristics of an essentially longitudinal-type wave. Under these conditions different regions of the plasma are not strongly coupled, and in an inhomogeneous field will possess different local resonance frequencies. In this sense one can treat the plasma as a collection of independent oscillators, which can produce an echo, provided that there exists a proper nonlinear interaction with an electromagnetic pulse. Following the procedure indicated in the preceding section we shall calculate an echo coefficient from the response of a single "oscillator" to a sequence of pulses.

The Echo Coefficient in Transverse Propagation

For the extraordinary transversely propagating mode one can show from symmetry considerations that with B_0 along z, and propagation along x, the nonvanishing field variables are E_x, E_y, b_z, v_x, v_y, where E_x and E_y are electric field components, b_z the rf magnetic field, and v_x and v_y the components of the average electron velocity. The electromagnetic pulse is assumed to be given in the form $\exp[-i\omega(t-x/w)]$+c.c., and $b_z=(1/w)E_y$, where w is the velocity of the wave. E_y and b_z are considered as driving terms and v_x, v_y, and E_x (which results from space charge) as oscillator parameters.

Before writing the equations we introduce the following definitions (in the mks system):

$$\mathcal{E}_x=-(e/m)E_x,\quad \mathcal{E}_y=-(e/m)E_y,\quad \omega_c=-(e/m)B_0,$$
$$\Omega_z=-(e/m)b_z=(1/w)\mathcal{E}_y,\quad \omega_p^2=n_0e^2/m\epsilon_0,$$

where e and m are the electronic charge and mass, respectively, ϵ_0 is the permittivity of free space, and n_0 is the electron density at equilibrium.

The equations will be given in the "Lagrangian" form, where they follow the motion of a particular particle in the plasma. The particle in question is a fictitious "average electron" whose initial rest position is given by (x_0,y_0,z_0) and whose velocity is given by $(v_x,v_y,0)$. The equations are

$$\begin{aligned}\dot{v}_x&=-\omega_cv_y-\mathcal{E}_x-\Omega_zv_y,\\ \dot{v}_y&=\omega_cv_x-\mathcal{E}_y+\Omega_zv_x,\\ \dot{\mathcal{E}}_x&=\omega_p^2v_x,\end{aligned} \tag{12}$$

where the time derivative is with respect to the moving particle. The first two equations represent the Lorentz

[12] Formulation and detailed discussion of the assumptions can be found in Ref. 8.

force, and the third is equivalent to Poisson's equation. Before studying nonlinear aspects it is convenient to diagonalize the linear part of Eq. (11) by means of the substitution

$$v_x = -i\omega_0(a-a^*),$$
$$v_y = \omega_c(a+a^*)-b, \qquad (13)$$
$$\mathcal{E}_x = \omega_p^2(a+a^*)+\omega_c b,$$

where $\omega_0=(\omega_c^2+\omega_p^2)^{1/2}$ is the cyclotron resonance frequency as modified by space-charge effects (also known as upper hybrid frequency). The resulting equations are

$$(d/dt+i\omega_0)a = -\mathcal{E}_y\omega_c/2\omega_0^2 - i(\Omega_z/2\omega_0)(2\omega_c a - b), \qquad (14)$$

with its complex-conjugate equation, and

$$db/dt = \mathcal{E}_y\omega_p^2/\omega_0^2 + i(\omega_p^2\Omega_z/\omega_0)(a-a^*), \qquad (15)$$

where a represents the normal coordinate of a mode whose resonance is at ω_0, and b represents the coordinate of a dc mode. The resonance excitation of the latter mode consists of a standing wave in which the electron density is modulated longitudinally, along the x direction, whereas the electrons move transversely, along y, at such a velocity as to exactly balance the magnetic and electric forces. The two modes are coupled by the external electromagnetic field.

$\mathcal{E}_y$ and Ω_z are given explicitly as

$$\mathcal{E}_y = \mathcal{E}\exp[-i\omega_0(t-x/w)]+\text{c.c.},$$
$$\Omega_z = \Omega\exp[-i\omega_0(t-x/w)]+\text{c.c.} \qquad (16)$$

The dependence of $\mathcal{E}_y$ and Ω_z on the electron coordinate x introduces nonlinearities into Eqs. (14) and (15) which are associated with the convective motion of electrons along the direction of propagation. A second nonlinearity is associated with the products $\Omega_z a$ and $\Omega_z b$ and arises from the interaction with the rf magnetic field. Both give rise to an echo mechanism. For small orbits Eq. (16) can be expanded to second order in $(x-x_0)$,

$$\mathcal{E}_y = \mathcal{E}\exp[-i\omega_0(t-x_0/w)]$$
$$\times[1+i(\omega_0/w)(x-x_0)-(\omega_0/w)^2(x-x_0)^2]+\text{c.c.}, \qquad (17)$$

with a similar expansion for Ω_z.

We now apply the approximation of slowly varying coefficients and follow the procedure described in detail in connection with the example in Sec. II. We put

$$a = A\exp[-i\omega_0(t-x_0/w)]+\text{other harmonics},$$

and assume that A as well as b in Eq. (14) are slowly varying. In this approximation one can make the substitution $x-x_0 \simeq a+a^*$ in Eq. (17). A lowest order system of coupled equations is obtained from the coefficient of $\exp[-i\omega_0(t-x/w)]$ in Eq. (14) and the dc term in Eq. (15),

$$\dot{A} = -\omega_c\mathcal{E}/2\omega_0^2 + i\Omega b/2\omega_0 + (\omega_c/w^2)(\mathcal{E}+w\Omega)AA^*$$
$$+(\omega_c/2w^2)(\mathcal{E}-2w\Omega)AA, \qquad (18)$$

$$\dot{b} = i(\omega_p^2/\omega_0)\Omega(A-A^*). \qquad (19)$$

The third term in Eq. (18) is analogous to a similar term in Eq. (8), and represents an amplitude dependence of the excitation rate. It includes an echo process described by the diagram of Fig. 2(a). The second term of Eq. (18) together with Eq. (19) includes a process described by Fig. 2(b).

Continuing the procedure described in the example in Sec. II we find the echo coefficient for a 3-pulse sequence as given by

$$S = (\omega_p^2/2\omega_0^2)\Omega_2 t_{w2}\Omega_3 t_{w3} - (\omega_c^2/2\omega_0^2 w^2)\mathcal{E}_2 t_{w2}\mathcal{E}_3 t_{w3}$$
$$-(\omega_c^2/2\omega_0^2 w)\mathcal{E}_2 t_{w2}\Omega_3 t_{w3}, \qquad (20)$$

where the subscripts numbering Ω, $\mathcal{E}$, and t_w refer to the pulse in question.

So far we have purposely not made the substitution $w\Omega=\mathcal{E}$, in order not to disguise the physical origins of the various terms in Eq. (20). If we make the substitution we find that the two last terms are exactly equal and that S is given by

$$S = [(\omega_p^2-2\omega_c^2)/2\omega_0^2 w^2]\mathcal{E}_2 t_{w2}\mathcal{E}_3 t_{w3}. \qquad (21)$$

Now $\mathcal{E}t_w$ equals the total velocity imparted to an electron by a pulse. The echo coefficient is thus proportional to the square of the ratio of the velocity imparted by a pulse to the velocity of the wave (or to the ratio of cyclotron orbit to wavelength).

For very low electron densities $w=c$, the velocity of light, and the effects would be relativistic. However, at resonance the index of refraction for a plasma becomes infinite, and, except for extremely short pulses or very low density plasmas, the spectral distribution of a pulse near resonance must be associated with relatively large values of the refractive index. One might therefore expect that the disturbance left in the wake of a pulse is characterized by a wavelength considerably shorter than the free-space value.

Physical Interpretation

Since even the simplest of the echoes of the type discussed arise from third-order processes, it is not easy to provide a simple physical picture for the processes involved. There are three principal mechanisms which give rise to the echo coefficient in Eq. (20). These are presented respectively by terms proportional to (a) $(\mathcal{E}_2 t_{w2})(\mathcal{E}_3 t_{w3})$, (b) $(\mathcal{E}_2 t_{w2})(\Omega_3 t_{w3})$, and (c) $(\Omega_2 t_{w2})(\Omega_3 t_{w3})$. The processes (a) and (b) both arise from the term AA^* in Eq. (18), and result from the dependence of the interaction between the field and the electron on the absolute magnitude of the amplitude, or if one wishes, on the kinetic energy of the electron. The echo mechanism in such a situation operates as follows. At the arrival time of pulse 2, the excitation produced by pulse 1 has assumed various phases for different oscillators. The absolute value of the amplitude of a given oscillator, obtained by superimposing the excitations produced by pulses 1 and 2, depends on the relative phase of the two excitations. The information relative to the phase of the initial excitation is thus stored in terms of the absolute value of the amplitude (or energy) of the oscillator. Now the interaction with pulse 3 depends on this amplitude and thus on the initial phase angle θ. If we consider the class of oscillators which prior to pulse 2 were all at the phase θ, then the *average* displacement for this class after pulse 3 will be a function of θ. After the lapse of another interval τ, this average displacement will have drifted again by an angle θ, and there will be established at that time a definite correlation between phase and amplitude giving rise to a finite moment and to the appearance of an echo.

The nonlinear process in (a) which causes the amplitude dependence of the interaction arises in the following way. Because of forward and backward motion relative to the wave, the field which an electron sees is not strictly sinusoidal, but contains various harmonics. These harmonics appear at the expense of the fundamental and cause a reduction in the resonance interaction, proportional to the square of the amplitude.

The process (b) involves in part the interaction with the rf magnetic field. Since this field is parallel to B_0, it alternately adds to or subtracts from it. An electron is thus exposed during part of its cycle to a field smaller than B_0 and during the other part to a field in excess of B_0. For small orbits the two parts of the cycle are of equal duration. However, at large orbits, the two parts may differ in duration because of the motion along the direction of propagation. Depending on its phase, an electron will be subjected to an average field which may be smaller or larger than B_0, and its natural precession frequency will be accordingly modified. The class of electrons corresponding to a given absolute amplitude will therefore experience phase bunching giving rise to a resultant moment which again is proportional to the square of the amplitude.

The process (c) results from purely magnetic interactions and involves coupling to the dc mode. The initial phase is stored after pulse 2 in terms of the amplitude of the dc mode and then coupled back to the cyclotron-resonance mode. It is interesting to note that this process is proportional to ω_p^2, and hence a true plasma effect not exhibited by isolated electrons. In the experiment reported so far,[1] $\omega_p^2 \ll \omega_c^2$, and hence process (c) is not expected to contribute substantially to the observed echo.

There is an important difference between process (c) and processes (a) and (b) with regard to relaxation. In the latter processes the initial information is stored during the period between pulse 2 and pulse 3 in terms of the magnitude AA^*, which is independent of the phase of A. The echo therefore relaxes at a rate characteristic of energy relaxation, a process which is much slower than momentum relaxation. The energy is here analogous to the longitudinal magnetization in the spin-echo process, whereas the momentum would correspond to the transverse moment. In process (c), however, the information is stored throughout in terms of magnitudes which decay at the rapid rate of momentum relaxation.

Longitudinal Propagation

The formal analysis of the longitudinal right-circularly polarized mode is considerably easier than that of the transverse mode. Since the experimental observation of an echo in this configuration is still somewhat marginal, we shall dispense with a detailed analysis and merely describe the principal process. There are two natural resonances associated with this mode. The first consists of circular motion at ω_c in a plane perpendicular to B_0, and the second of a dc drift along the field lines. A possible echo mechanism is associated with the coupling of the two modes by means of the interaction with the rf magnetic field. This process in fact should give rise to an echo coefficient of magnitude

$$\tfrac{1}{2}\Omega_2 t_{w2}\Omega_3 t_{w3},$$

which is comparable to the coefficient for transverse propagation given by Eq. (21). The reason why the observed echo is so weak is not clear. The distinction between the two directions of propagation may be related to the role of electron diffusion, which takes place largely along magnetic field lines. If the wavelength is indeed very short, then, in the case of longitudinal propagation, diffusion can destroy the proper phase relation between the electron and the wave.

IV. CONCLUSION

We have described a certain class of echo-producing processes which are associated with nonlinear interactions between the electrons in the plasma and the electromagnetic pulses. This class is distinguished from other echo processes in that the effects take place completely during the presence of the pulses and do not require an additional finite waiting period τ in order for the echo to be generated.

Cyclotron echoes have now been observed under widely varied conditions, extending over many orders of magnitude in gas pressure and electron density. For the most part the behavior is so complex that one must conclude that diverse mechanisms are at play, with different ones predominating in various regimes of operation. The processes described in the present work are likely to play a role in some of these regimes. Further experimental work, and a more detailed knowledge of the plasma-wave interaction, is required for a better understanding of this interesting phenomenon.

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