Algebra of Currents and Low-Energy $K\pi$ Interaction*

V. S. MATHUR† AND L. K. PANDIT‡
Department of Physics and Astronomy, University of Rochester, Rochester, New York
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Using the Fubini-Furlan-Adler-Weisberger technique, sum rules have been written down for strangeness-changing and strangeness-preserving axial-vector coupling constants. Coupled together, these sum rules provide information on the low-energy interactions of strongly interacting particles. In particular the $K\pi$ system is studied and a case made for the existence of an S-wave resonance near threshold in the $I=\frac{1}{2}$ channel. An evaluation of the $KN\Lambda$ and $KN\Sigma$ coupling constants required by the mutual consistency of the sum rules is also presented.

I. INTRODUCTION

A POWERFUL way of studying the algebra of currents has recently been developed by Fubini and Furlan.[1] Coupled with the hypothesis of the partially conserved axial-vector currents (PCAC),[2] this method has led to a remarkable evaluation of the renormalization of the β-decay axial-vector coupling constant by Adler[3] and Weisberger.[4] Several applications of this technique have been made by various authors.[5]

In the present paper we show how various sum rules obtained using the Fubini-Furlan-Adler-Weisberger technique may be coupled to obtain information on the low-energy interaction of strongly interacting particles. In particular we study here the K-π system. That the PCAC hypothesis allows weak interactions to impose consistency requirements on reaction amplitudes of strongly interacting particles has recently been emphasized by Adler.[6]

We start with the equal-time commutation relations suggested by the quark model for axial-vector charges of the strangeness-changing and strangeness-preserving currents. No use is made of $SU(3)$ or any higher symmetry. Various sum rules are obtained when matrix elements of these commutators are taken between one-particle baryon or meson states. The sum rules, which are related by the appearance in them of either the same coupling constants or the same cross sections, taken together lead to rather restrictive self-consistency requirements from which the unknown coupling constants or cross sections may be studied. This forms the basis of the calculations reported in this paper.

In Sec. II we state the sum rules and discuss our method in some detail. From the structure of the sum rules some general conclusions are drawn. In Sec. III we make a detailed study of the $K\pi$ system and present a case for a low-energy S-wave $K\pi$ resonance with isospin $\frac{1}{2}$. This prediction appears concomitantly with some restriction on the strong-interaction coupling constant $G_{KN\Lambda}$.

II. DISCUSSION

Denoting the octets of vector and pseudovector currents in a quark model by $(V_b{}^a)_\mu$ and $(P_b{}^a)_\mu$, we define the corresponding "charges" by

$$A_b{}^a(t)=i\int d^3x(V_b{}^a(\mathbf{x},t))_4\,, \tag{1a}$$

$$B_b{}^a(t)=i\int d^3x(P_b{}^a(\mathbf{x},t))_4\,. \tag{1b}$$

$A_b{}^a$, $B_b{}^a$ satisfy the equal-time commutation relations

$$[A_b{}^a,A_d{}^c]=\delta_d{}^aA_b{}^c-\delta_b{}^cA_d{}^a\,, \tag{2a}$$

$$[A_b{}^a,B_d{}^c]=\delta_d{}^aB_b{}^c-\delta_b{}^cB_d{}^a\,, \tag{2b}$$

$$[B_b{}^a,B_d{}^c]=\delta_d{}^aA_b{}^c-\delta_b{}^cA_d{}^a\,. \tag{2c}$$

Taking the matrix elements of suitable commutation relations of type (2) between single-particle states and following the procedure of Fubini-Furlan-Adler-Weisberger, one may easily write down a set of sum rules. The commutators we exploit in the present work are the following:

$$[B_2{}^1,B_1{}^2]=A_2{}^2-A_1{}^1=2I_3\,, \tag{3a}$$

$$[B_3{}^1,B_1{}^3]=A_3{}^3-A_1{}^1=Y+Q\,. \tag{3b}$$

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‡ On leave of absence from Tata Institute of Fundamental Research, Bombay, India.

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Taking the matrix element of (3a) between one-proton states, we obtain the well-known Adler- Weisberger[3,4] sum rule:

$$1-\frac{1}{(g_A{}^N)^2}=\frac{4M_N{}^2}{G_{NN\pi}{}^2(K_{NN\pi}(0))^2}\frac{1}{\pi}\times\int_{M_N+M_\pi}^{\infty}\frac{WdW}{W^2-M_N{}^2}[\sigma_p{}^{\pi^+}(W,0)-\sigma_p{}^{\pi^-}(W,0)], \quad (4a)$$

where $\sigma_p{}^{\pi^\pm}(W,0)$ denotes the $\pi^\pm$-p total cross section for a "zero-mass" pion at the center-of-mass energy W, and $g_A{}^N$ is the renormalized strangeness-preserving axial-vector coupling constant.

Again, taking the matrix elements of (3a) between one π^+ state and one K^+ state, we obtain, respectively,

$$\frac{2}{(g_A{}^N)^2}=\frac{4M_N{}^2}{G_{NN\pi}{}^2(K_{NN\pi}(0))^2}\frac{1}{\pi}\int_{2M_\pi}^{\infty}\frac{WdW}{W^2-M_\pi{}^2}[\sigma_{\pi^+}{}^{\pi^-}(W,0)-\sigma_{\pi^+}{}^{\pi^+}(W,0)]; \quad (4b)$$

$$\frac{1}{(g_A{}^N)^2}=\frac{4M_N{}^2}{G_{NN\pi}{}^2(K_{NN\pi}(0))^2}\frac{1}{\pi}\int_{M_\pi+M_K}^{\infty}\frac{WdW}{W^2-M_K{}^2}\times[\sigma_{K^+}{}^{\pi^-}(W,0)-\sigma_{K^+}{}^{\pi^+}(W,0)]. \quad (4c)$$

Similarly, on taking the matrix element of (3b) between one-proton states, the sum rule studied by Pandit and Schechter[5] is obtained, namely,

$$\frac{2}{(g_A{}^\Lambda)^2}-\left(\frac{g_A{}^{\Sigma^0}}{g_A{}^\Lambda}\right)^2=1+\frac{2(M_N+M_\Lambda)^2}{G_{KN\Lambda}{}^2(K_{KN\Lambda}(0))^2}\frac{1}{\pi}\times\int_{M_\pi+M_\Lambda}^{\infty}\frac{WdW}{W^2-M_N{}^2}\{\sigma_p{}^{K^-}(W,0)-\sigma_p{}^{K^+}(W,0)\}, \quad (5a)$$

where now $g_A{}^\Lambda$ and $g_A{}^{\Sigma^0}$ are the strangeness-changing axial-vector renormalized coupling constants for Λ and Σ^0 leptonic decays. In the same manner, using π^+ and K^+ states for taking the matrix elements, we obtain

$$\frac{1}{(g_A{}^\Lambda)^2}=\frac{2(M_N+M_\Lambda)^2}{G_{KN\Lambda}{}^2(K_{KN\Lambda}(0))^2}\frac{1}{\pi}\int_{M_\pi+M_K}^{\infty}\frac{WdW}{W^2-M_\pi{}^2}\times[\sigma_{\pi^+}{}^{K^-}(W,0)-\sigma_{\pi^+}{}^{K^+}(W,0)]; \quad (5b)$$

$$\frac{2}{(g_A{}^\Lambda)^2}=\frac{2(M_N+M_\Lambda)^2}{G_{KN\Lambda}{}^2(K_{KN\Lambda}(0))^2}\frac{1}{\pi}\int_{2M_\pi}\frac{WdW}{W^2-M_K{}^2}\times[\sigma_{K^+}{}^{K^-}(W,0)-\sigma_{K^+}{}^{K^+}(W,0]. \quad (5c)$$

Before we make any specific computation using these sum rules, we should like to make a few comments based on their general structure.

It is clear that these sum rules provide, in principle, a determination of the strong and weak coupling constants independently in terms of total cross sections of strongly interacting particles. The coupling among the strong-interaction attributes alone obtained from the behavior of weak current is, of course, due to the hypothesis of partially conserved axial-vector current (PCAC). This is completely in the spirit of the work of Adler.[6] As an example, consider the sum rules (4a) and (4b), from which we may eliminate the weak coupling constant $g_A{}^N$, obtaining

$$G_{NN\pi}{}^2=2M_N{}^2(I_\pi{}^\pi+2I_p{}^\pi), \quad (6)$$

where

$$I_p{}^\pi=\frac{1}{(K_{NN\pi}(0))^2}\frac{1}{\pi}\times\int_{M_N+M_\pi}^{\infty}\frac{WdW}{W^2-M_N{}^2}[\sigma_p{}^{\pi^+}(W,0)-\sigma_p{}^{\pi^-}(W,0)], \quad (7)$$

and $I_\pi{}^\pi$ is similarly defined. Since we do not have experimental information on $\pi\pi$ total cross sections entering Eq. (6) through $I_\pi{}^\pi$, we may now regard that equation as providing information on $\pi\pi$ interactions. The situation is particularly gratifying, since the other quantities ($G_{NN\pi}$ and $\sigma_p{}^\pi$) occurring in Eq. (6) are very well determined experimentally. Of course, a suitable correction has to be applied for the fact that the incident pion in $\sigma_p{}^\pi(W,0)$ and $\sigma_\pi{}^\pi(W,0)$ is off the mass shell. This point will be taken up in the application treated in the next section.

For an application of the kind outlined above, it is of particular importance to note that the integrals occurring in the sum rules are dominated by the low-energy region. This is because the difference of the cross sections contained in the integrand decreases with increasing energy as it approaches the zero Pomeranchuk value.[7] Thus a simple approximation scheme can be set up where the low-energy $\pi\pi$ amplitude may be represented in terms of resonances or effective-range parametrizations chosen to saturate the sum rule of type (6). Adler[8] has used a similar approach recently to suggest a large low-energy S-wave interaction in the $\pi\pi$ system. One can similarly investigate the sum rule (5c) to study the $\bar{K}K$ system. In the present paper we confine our attention only to the $K\pi$ interaction problem along the lines discussed here.

The $K\pi$ problem possesses a specially attractive feature. This arises because the sum rules (4c) and (5b) involve identical cross sections on account of the CPT theorem, ignoring off-mass-shell corrections for the moment. Thus, to obtain information on the low-energy $K\pi$ interaction, a self-consistency requirement can be set up by demanding that both (4c) and (5b) be satisfied.

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III. $K\pi$ SYSTEM

Here we study the low-energy $K\pi$ interaction along the lines indicated in the previous section making a simultaneous use of the sum rules (4c) and (5b).

As has already been mentioned, the sum rules are dominated by cross sections at low energies. We now make the simplifying assumption that these cross sections may be approximated by low-energy resonances in the $K\pi$ channel. In a zero-width resonance approximation[9] we may write the total $K\pi$ cross section for a particular isospin as

$$\sigma(W)=\sum_{l=0}^{\infty}\sigma_{l,I}(W), \tag{8}$$

$$\sigma_{I,l}(W)=4\pi(2l+1)\Gamma_{I,l}(\sqrt{s_{I,l}{}^{r}}/\nu_{I,l}{}^{r})\delta(s-s_{I,l})^{r}, \tag{9}$$

where we put

$$s=W^2,$$
$$\nu=(4s)^{-1}(M_K{}^2-M_\pi{}^2+s)^2-M_K{}^2; \tag{10}$$

$\Gamma_{I,l}$ and $s_{I,l}{}^r$ are the width and position of the resonance in the channel with isospin I and angular momentum l.

To make off-mass-shell corrections to the cross sections we employ the prescription used by Adler,[8] that is, we take

$$[\sigma_K{}^\pi(W,0)]_{I,l}=(K_{NN\pi}(0))^2(\nu_0/\nu)^l\sigma_{I,l}(W), \tag{11}$$

where ν_0 is the value of ν for a "zero-mass" pion. A similar correction is made for $\sigma_\pi{}^K(W,0)$. Equation (11) provides only a kinematical off-mass-shell correction due to the threshold behavior of the cross section. In the absence of a properly founded technique, this may be taken as a rough recipe.

Since no resonance states are known in the $I=\frac{3}{2}$ channel of the $K\pi$ system, we simply neglect the $I=\frac{3}{2}$ contribution. Then substituting (9) and (11) in the sum rules (4c) and (5b), we get

$$(g_A{}^N)^{-2}=(4M_N{}^2/G_{NN\pi}{}^2)I_K{}^\pi, \tag{12a}$$

$$(g_A{}^\Lambda)^{-2}=[2(M_N+M_\Lambda)^2/G_{KN\Lambda}{}^2]I_\pi{}^K, \tag{12b}$$

where, suppressing the isospin index ($I=\frac{1}{2}$), we have

$$I_K{}^\pi=\tfrac{4}{3}\sum_{l=0}^{\infty}(2l+1)(s_l{}^r-M_K{}^2)^{2l-1}\times[(M_K{}^2-M_\pi{}^2+s_l{}^r)^2-4s_l{}^rM_K{}^2]^{-l}\times\Gamma_l(\sqrt{s_l{}^r})/\nu_l{}^r, \tag{13a}$$

$$I_\pi{}^K=\tfrac{4}{3}\sum_{l=0}^{\infty}(2l+1)(s_l{}^r-M_\pi{}^2)^{2l-1}\times[(M_\pi{}^2-M_K{}^2+s_l{}^r)^2-4s_l{}^rM_\pi{}^2]^{-l}\times\Gamma_l(\sqrt{s_l{}^r})/\nu_l{}^r. \tag{13b}$$

TABLE I. S-wave resonance parameters.

$G_{KN\Lambda}{}^2/8\pi(g_A{}^\Lambda)^2$	$\sqrt{s_0{}^r}$ (MeV)	Γ_0 (MeV)
13	2.2×10^3	58.2×10^3
10	870	1120
8	690	115
7.5	667	55
7	644	≈0

In the $I=\frac{1}{2}$ channel we have the well-established p-wave resonance $K^*(888)$. We find that the contribution of this resonance alone to the right-hand side of (12a) is 0.085, whereas the left-hand side is known experimentally to be 0.725. It is thus clear that the K^* contribution is grossly inadequate to saturate the sum rule (12a). The essential reason for the smallness of the K^* contribution seems to be that it occurs much above the $K\pi$ threshold. Also, since low energies are emphasized in our problem, we do not expect higher partial waves to make any significant contribution. Further, any $I=\frac{3}{2}$ contribution, being in the wrong direction, cannot be invoked to save the situation. The sum rule can, therefore, be satisfied only if a predominant S-wave contribution exists in the $I=\frac{1}{2}$ channel. We investigate below two possibilities: (1) that there exists an S-wave $K\pi$ resonance state, (2) that there exists an S-wave $K\pi$ bound state.

Let us consider first the possibility of the S-wave resonance. In this case the contributions $I_K{}^\pi(l=0)$ and $I_\pi{}^K(l=0)$ are in the ratio

$$I_K{}^\pi(l=0)/I_\pi{}^K(l=0)=(s_0{}^r-M_\pi{}^2)/(s_0{}^r-M_K{}^2). \tag{14}$$

Since for the present case

$$s_0{}^r>(M_\pi+M_K)^2, \tag{15}$$

we obtain the inequality

$$I_K{}^\pi(l=0)>I_\pi{}^K(l=0)>I_K{}^\pi(l=0)(M_\pi{}^2+2M_KM_\pi)/(M_K{}^2+2M_\pi M_K). \tag{16}$$

Thus in order that both the sum rules (12a) and (12b) be simultaneously satisfied, we must have

$$6.8<G_{KN\Lambda}{}^2/8\pi(g_A{}^\Lambda)^2<13.5, \tag{17}$$

where the experimental values of $g_A{}^N=1.18$ and $(G_{NN\pi}{}^2/4\pi)=14.6$, have been used, and also the K^* contribution has been included. For each value of $G_{KN\Lambda}{}^2/8\pi(g_A{}^\Lambda)^2$ consistent with the limits (17) we may now obtain the required S-wave resonance parameters, $s_0{}^r$ and Γ_0. These are displayed in Table I. From this table we see that for the larger values of $G_{KN\Lambda}{}^2/8\pi(g_A{}^\Lambda)^2$ we get absurd values for Γ_0 and $s_0{}^r$. Clearly a resonance nearer threshold only can be reasonable. In view of the approximations involved in our model, the resonance parameters found can at best be considered as rough estimates. Experimentally there does seem to be some indication of an S-wave resonance at 725 MeV, but with a width much smaller than required by our calcu-

[9] Adler (Ref. 8) has shown that a zero-width approximation does not make an appreciable difference in the result.

lations. For $|g_A{}^\Lambda|\simeq 0.79$ one then expects $G_{KN\Lambda}{}^2/4\pi \approx 8$–$9$. This value for the $KN\Lambda$ coupling constant is much smaller than the $SU(3)$ symmetry result for the usually accepted F/D ratio. Our result may be taken as a possible measure of the extent to which $SU(3)$ is broken. However, this value is still in reasonable accord with the kaon photoproduction[10] data. The consistency of the PCAC hypothesis applied to strangeness-changing axial-vector currents as used by Pandit and Schechter,[5] implies a value for $G_{KN\Sigma}{}^2/4\pi \approx \mathbf{1}$, for a value of $g_A{}^{\Sigma^0} \approx 0.3$.

Next we consider the second possibility, that there exists an S-wave bound state rather than a resonance. In this case the sum rules (4c) and (5b) have to be modified to take account of the single-particle bound-state contribution. The sum rules then become

$$\frac{1}{(g_A{}^N)^2}=\frac{4M_N{}^2}{G_{NN\pi}{}^2}\left[\frac{1}{(K_{NN\pi}(0))^2}\frac{G_{\pi KK'}{}^2}{(M_{K'}{}^2-M_K{}^2)^2}+I_K{}^\pi\right], \tag{18a}$$

$$\frac{1}{(g_A{}^\Lambda)^2}=\frac{2(M_N+M_\Lambda)^2}{G_{KN\Lambda}{}^2}\times\left[\frac{1}{(K_{KN\Lambda}(0))^2}\frac{G_{\pi KK'}{}^2}{(M_{K'}{}^2-M_\pi{}^2)^2}+I_\pi{}^K\right], \tag{18b}$$

where K' denotes the bound state. As before, the requirement that

$$M_{K'}{}^2<(M_K+M_\pi)^2, \tag{19}$$

leads to the inequality

$$G_{KN\Lambda}{}^2/4\pi(g_A{}^\Lambda)^2<8.0. \tag{20}$$

The rather low values of $G_{KN\Lambda}{}^2/4\pi$ required by (20) will not satisfy the sum rule (5a). Thus, mutual consistency of the various sum rules seems to exclude this possibility. Also such a low value for $G_{KN\Lambda}{}^2/4\pi$ is hard to understand in a moderately broken $SU(3)$ symmetry scheme.

There is a third possibility we have not discussed here, namely, that there may be a large but non-resonant S-wave contribution at low energies to the $K\pi$ cross sections. Such a possibility will acquire special importance if a low energy resonance is definitely ruled out. In a general analysis involving all these possibilities, it will be expedient to make full use of the mutual compatibility of all the four sum rules (4a), (4c), (5a), and (5b), which are intimately related. Such an analysis will be reported elsewhere.

One last remark remains to be made. This concerns the suggestion made in the literature[11] that the Cabibbo angle may arise owing to $SU(3)$-breaking effects rather than as an intrinsic parameter of the theory. We would like to point out that our estimates of the integrals $I_K{}^\pi$ and $I_\pi{}^K$ occurring in Eqs. (12a) and (12b) rule out this possibility, in agreement with the conclusions of Pandit and Schechter[5] and of Sato and Sasaki.[5] From Goldberger-Treiman,[12] relations $(I_K{}^\pi)^{-1/2}$ and $(I_\pi{}^K)^{-1/2}$ are directly the decay constant of $\pi\to\mu\nu$ and $K\to\mu\nu$ decays, respectively. This leads to the small renormalization effect in the Cabibbo angle θ_A; so that now $\tan\theta_A$ turns out to about 0.21, in place of the unrenormalized value 0.27 obtained by Cabibbo[13] using $SU(3)$ symmetry.[14]

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