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Field-Theoretic Formulation of the Optical Model at High Energies*

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Two approximation schemes are studied within the framework of the quantum electrodynamics of massive photons [Q.E.D.(W)], motivated by the semiclassical approximation in potential theory (i.e., the optical model) rather than Regge pole or cut expectations. In the first scheme we select a series of Feynman graphs (generalized ladder graphs) that form a natural extension of potential scattering at high energies s and finite momentum transfer t. (They moreover incorporate the effects of crossing symmetry in the s and u channels). In the second scheme we make a semiclassical approximation, without the use of conventional perturbation theory, involving classical currents in the T-product formula for the S matrix. After certain vertex-renormalization and self-energy effects are taken into account, in a general way, we find an S matrix that is identical with that of an optical model whose potential is a superposition of Yukawa potentials. Thus we find a sufficient (and perhaps necessary) condition for covariance of an optical model. Without taking any vertex and self-energy effects into account in either scheme, we have found agreement between the two schemes so far up to fourth order. To study the effect of spin in our models we introduce a spin-zero object "Π" and a spin-$\frac{1}{2}$ object "N." Because of the γ_5 invariance of the γ_μ vertex, the helicity-conserving amplitudes for the reactions $\Pi+N \to \Pi+N$ and $N+N \to N+N$ are the same as for $\Pi+\Pi \to \Pi+\Pi$, namely,

$$A(t)s+B(t)s\ln s.$$

We have verified that $B(t)\equiv 0$ up to sixth order. Taken by itself, this form violates unitarity unless $B(t)\equiv 0$ to all orders. We also study other ways to save unitarity in the whole of Q.E.D.(W). Such unitarity considerations are strongly indicative of the importance of an optical term $sf(t)$ with nonzero real and imaginary parts even when many other series of graphs are taken into account (although other contributions may compete with it near $t=0$). The optical contribution, taken alone, provides a natural qualitative explanation for the experimentally suggested constant behavior in the ratio of the real to imaginary part of the forward pp scattering amplitude above 8 BeV/c. Although a similar result is predicted for Kp scattering, it does not apply to πp.

I. INTRODUCTION

THE concept of a complex energy-dependent potential has been used recently[1,2] to describe the elastic diffraction scattering of elementary particles in the high-energy region. Despite the fact that such theories are derived usually in a very noncovariant way, such methods have met with some success in describing actual high-energy elastic[1] and inelastic[2] scattering data. Of special note is the fact that while the optical model leads naturally to a nonshrinking diffraction peak as observed in πp scattering,[3] such behavior apparently leads to complications in Regge theory.[4]

It seems important, then, to attempt to derive the optical model[5] in a more rigorous way. While formal studies indicate that scattering amplitudes arising from complex, energy-dependent potentials are consistent with the Mandelstam representation,[6,7] we have undertaken a more constructive approach. In this paper we seek to replace the optical-potential considerations with a particular field-theoretic model, incorporating, as much as possible, features that are expected of a correct theory: namely, covariance, retardation, recoil, the existence of crossing relations, etc.

We approximate the quantum electrodynamics of massive photons, Q.E.D.(W), in two ways. In the first approach, we select a series of planar and nonplanar ladder graphs that seem to be the natural generalization of potential scattering in the high-energy, fixed momentum-transfer limit (see Figs. 1 and 2). The second approach, which does not rely on perturbation theory, is a semiclassical approximation in the T-product formula for the S matrix.

The selection of graphs for the perturbation-theoretic model occurs in Sec. II after a brief outline of the optical-model formalism.

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[1] Y. Nambu and M. Sugawara, Phys. Rev. Letters **10**, 304 (1963); Phys. Rev. **132**, 2727 (1963); Robert Serber, Phys. Rev. Letters **10**, 357 (1963); Rev. Mod. Phys. **36**, 694 (1964); Phys. Rev. Letters **13**, 32 (1964).

[2] G. C. Summerfield, Nuovo Cimento **23**, 867 (1962); N. J. Sopkovitch, *ibid.* **26**, 186 (1962); K. Gottfried and J. D. Jackson, *ibid.* **34**, 735 (1965).

[3] K. J. Foley *et al.*, Phys. Rev. Letters **10**, 376, 543 (1963). See Ref. 65 for the most recent data.

[4] Y. Nambu and M. Sugawara, Ref. 1; P. G. O. Freund and R. Oehme, Phys. Rev. Letters **10**, 450 (1963); R. Oehme, *ibid.* **8**, 201 (1964); see also Sec. VII, part B of this article.

[5] The word "optical model" has been applied to a variety of related concepts. For the purposes of this paper, "optical model" will mean a certain semiclassical approximation in potential theory (see Sec. II) whether or not one assumes a complex, energy-dependent potential.

[6] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962); Y. Nambu and M. Sugawara (see Ref. 1); Roland Omnes, Phys. Rev. **137**, B653 (1965).

[7] J. N. Cornwall and M. A. Ruderman, Phys. Rev. **128**, 1474 (1965).

In Sec. III we study the reaction

$$\Pi+\Pi\rightarrow\Pi+\Pi,$$

where Π is a spin-zero particle in the perturbation model. We make extensive use of the methods and terminology of Tiktopoulos[8] in his two papers on the $\lambda\phi^3$ theory.[9] In contrast to his case, we have nontrivial numerators in our Feynman integrals. Naturally, the terms in the numerator which do not contain internal momenta (we call them trivial terms) may be easily studied by his methods. The nontrivial terms, as it turns out, are unimportant asymptotically. This is proved in Appendix B. Here we use the Mellin transformation method of Bjorken and Wu,[10] generalized in Appendix A, to incorporate the effects of internal momenta in the numerator. The over-all amplitude is of the form

$$A(t)s+B(t)s\ln s,$$

where $B(t)\equiv 0$ has been verified in our model up to and including sixth order. This amplitude, as it stands, violates unitarity unless $B(t)\equiv 0$. This might mean that $B(t)\equiv 0$ to all orders; however, we cannot exclude the possibility that unitarity may be satisfied in the whole Q.E.D.(W) in some other way. This other possibility, which is discussed in Sec. VII, also suggests the importance of an optical term $sf(t)$ even when many other important graphs are taken into account. Besides this it forms another "out" for our $s\ln s$ problem.

In Sec. IV we introduce an object N with spin $\frac{1}{2}$, and consider

$$\Pi+N\rightarrow\Pi+N,$$

and

$$N+N\rightarrow N+N.$$

The key to reducing the algebraic complexities to a manageable level in these reactions consists of rewriting the Feynman rules in such a way as to emphasize the helicity-conserving aspects. The result of Sec. III is reproduced in Sec. IV by the helicity-conserving amplitudes. The helicity flip amplitudes are negligible in this model. This is due to the γ_5 invariance of our vertices. If one introduces an anomalous magnetic moment term into our vertex one obtains important helicity flip amplitudes.[11]

In Sec. V, we derive the scattering amplitude of a specific optical model for purposes of comparison with Secs. III and IV as well as VI.

In Sec. VI we undertake the semiclassical approximation to the S matrix mentioned above. The basic approximation consists of regarding the source currents of the massive photons as classical currents. Assuming that the two particles move in undeflected straight-line paths, one finds an S matrix identical with the optical-model result of Sec. V. In this way we find a covariant optical model. Moreover these results agree with our perturbation-theoretic model up to and including fourth order.

The concluding Sec. VII, is divided into three parts. In part A we discuss the possible effects of other graphs in Q.E.D.(W) on our results. These considerations indicate that an optical term $sf(t)$, with both real and imaginary parts nonzero, is an important (if not dominant) term asymptotically. In part B we note that this term could explain the constant behavior of the real to imaginary part ratio of the forward pp scattering amplitude derived from small angle scattering data above 8 BeV/c and the assumption of a single spin-independent amplitude. In fact the negative sign of the observed constant ratio may be explained in the context of the covariant optical model. The same results should apply to Kp scattering but not πp. In part C we formulate the principles of *strong* and *weak asymptotic* γ_5 *invariance* (i.e., the asymptotic helicity conservation of our models). Unfortunately it is impossible to predict theoretically whether this symmetry will be obeyed in nature or not, although we actually assume strong γ_5 invariance in our discussion of part B. We discuss it, nevertheless, since: (1) strong γ_5 invariance gives some content to the often used *ad hoc* assumption that, even in the presence of spin, high-energy scattering may, in some limiting sense, be well represented by one complex amplitude; (2) the symmetry should be testable in the near future.

There is yet one more Appendix—Appendix C. Here we derive some results concerning the asymptotic limits of amplitudes satisfying dispersion relations that we have found useful throughout this work.

II. THE GENERALIZATION OF THE OPTICAL MODEL

The optical model proceeds from the assumption that high-energy scattering can be described by an effective complex energy-dependent potential.[12] Let us consider scattering in the center-of-mass system of two particles of mass m. One writes the Schrödinger equation

$$(\nabla^2+p^2-U)\psi=0. \qquad (1)$$

Assuming that U varies slowly (in a distance $1/p$) we

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[8] G. Tiktopoulos, Phys. Rev. **131**, 480 (1963); **131**, 2373 (1963). We shall call these Papers I and II, respectively. The earlier literature may be traced from these papers and that of Polkinghorne, Ref. 10. The basic technique for evaluating the asymptotic behavior of Feynman integrals is from J. C. Polkinghorne, J. Math. Phys. 4, 503 (1963).

[9] We have made one small change in terminology. Where Tiktopoulos uses either "essentially planar" or "crossed planar" we use the word "planar."

[10] J. D. Bjorken and T. T. Wu, Phys. Rev. **130**, 2566 (1963). See also T. L. Trueman and T. Yao, *ibid.* **132**, 2741 (1963); and J. C. Polkinghorne, J. Math. Phys. **5**, 431 (1964).

[11] The results of Secs. III and IV were reported at the December 1964 American Physical Society Meeting, except for the last two paragraphs of each section. See R. Torgerson, Bull. Am. Phys. Soc. **9**, 738 (1964). Note the misprint in the results in our abstract [cf. Eqs. (18) and (20)].

[12] For a detailed discussion of the optical model, see R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1958), Vol. I, p. 315. See this reference for the early original literature.

expect a wave function of the form $e^{ipz}\phi$, where ϕ is also a slowly varying function. One obtains

$$2ip\partial_z\phi+\nabla^2\phi=U\phi. \tag{2}$$

$\nabla^2\phi$ may be regarded as negligible in comparison to the first term. Equation (2), then, is easily integrated.

$$\phi=\exp\left[\frac{1}{2ip}\int_\infty^z U(\mathbf{b}+z\mathbf{k})dz\right]. \tag{3}$$

$b=|\mathbf{b}|$ is called the impact parameter since it is essentially the impact parameter of the classical two-body problem. Although ϕ is expected to be a good approximation only inside the range of U, this is all that is needed in the usual formula for the scattering amplitude. One indeed obtains

$$f=\frac{p}{2\pi i}\int d^2b\; e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{b}}(e^{i\chi(b)}-1), \tag{4}$$

where

$$\chi=-\frac{1}{2p}\int_{-\infty}^{\infty} U(\mathbf{b}+z\mathbf{k})dz. \tag{5}$$

In deriving (4) one needs to assume that $\mathbf{p}-\mathbf{p}'=\mathbf{\Delta}$, the momentum transfer, is small.[12,13] Indeed this approximation is already contained in (3) since we integrate over a straight-line path—not the "true" classical path. The straight-line approximation is valid only for large b and therefore for small momentum transfer. In studying the comparison with the field-theoretic case we may formally introduce into Eq. (4) the Lorentz-invariant amplitude $T=8\pi(s)^{1/2}f$, where $(s)^{1/2}$ is the total center-of-mass energy expressed in relativistic kinematics.

$$T(s,t)\sim\frac{2s}{i}\int d^2b\; e^{i\Delta\cdot\mathbf{b}}(e^{i\chi(b)}-1)$$

$$\sim 4\pi is\int_0^\infty b\; db\; J_0(\Delta b)(1-e^{i\chi(b)}). \tag{6}$$

In a typical optical-model study one could assume a potential whose asymptotic behavior in s is $\sqrt{s}$ times a function of the space coordinate.[1] This leads evidently to

$$T\sim sf(t). \tag{7}$$

This is the kind of optical-model behavior that we wish to duplicate since it leads to a constant total cross section and a nonshrinking diffraction peak. Usually one uses a purely imaginary potential which results in a purely imaginary scattering amplitude.[1] It is unclear if this is what one wants because of the evidence for the existence of a significant real part in high-energy elastic scattering. This point will be discussed more fully in Sec. VII.

One obvious way to approach Lorentz covariance is to replace Eq. (1) by the Klein-Gordon[7,14] equation. Then p^2 would stand for E^2-m^2, U would be interpreted as a Lorentz scalar, and the analysis would proceed as before. A better way would be to regard the interaction as due to a four-vector field emitted by the other particle (particle number 2). Let us go into the lab frame and regard particle 2 as fixed so that we do not have to worry about retardation. For large energies E we have

$$(\nabla^2+p^2-2EU')\psi\approx 0. \tag{8}$$

This leads to

$$\chi\sim-\int_{-\infty}^{\infty} U'(\mathbf{b}+z\mathbf{k})\,dz \quad\text{and}\quad T\sim sf(t),$$

although we assume no s dependence in U'. We shall call this optical model the K.G.-V. optical model.

Although we have the benefit of relativistic kinematics for particle number 1, this last version of the optical model is *apparently* also unacceptable, since there is no retardation or recoil. Thus one has no guarantee that the calculation is consistent with relativity. One should try to formulate the K.G.-V. optical model in the center-of-mass system but the effect of retardation which cannot be neglected makes things very difficult.

Cornwall and Ruderman[7] explain the energy-dependent potential in terms of the Fourier transform of the retarded interaction with respect to time. In fact, causality is satisfied if the energy-dependent potential satisfies a dispersion relation of the usual kind. However, the retarded interaction they discuss is the effect of the wave function at a prior time on the present wave function. They do not discuss the retarded effect of the wave function of another particle. The form of such effects is often sufficiently different so that their simple demonstration cannot be applied. This is especially in evidence in the K.G.-V. case. Since we are mostly interested in the two-body aspects of the problem handled as exactly as possible we are led to an entirely different approach.

Naively one would expect that it is much better to replace (8) or (1) by the Bethe-Salpeter equation, in the ladder approximation, since in it retardation and recoil between two particles, on an equal footing, are treated in a strictly covariant way. (See Fig. 1, line 1.) In order to obtain crossing symmetry, however, we should also

[13] One may define $\chi(b)$ for large s by $2\delta_l$, the l-wave phase shift, $(l+\frac{1}{2}=bp)$. However, here, one has to make a small-angle approximation. See, for example, P. T. Mathews and A. Salam, Nuovo Cimento **21**, 126 (1961). As they emphasize, their Eq. (2.12) makes no use of a potential. See also Ref. 12, pp. 349, 350. For an even more general definition, see W. N. Cottingham and Ronald F. Peierls, Phys. Rev. **137**, B147 (1965). They show that if one defines $h(b)$ as the Hankel transform of the scattering amplitude, then $(e^{2i\delta_l}-1)/2i\sim h(l+\frac{1}{2})/p$ for large s. This defines χ, in general, since their formula (2.2) is the same as our (6) with the identification: $h(b)=(e^{i\chi(b,s)}-1)/2i$.

[14] L. I. Schiff, Phys. Rev. **103**, 443 (1956). This work is more general than the usual optical model in that he finds an approximation suitable for large scattering angles as well. Note also that he studies a Dirac equation optical model.

introduce the crossed ladder graphs. (See Fig. 1, line 2.) We will find shortly that these series of graphs are not enough. We must also include nonplanar ladder-like graphs. (See Fig. 2.)

The idea is that the wavy lines in Figs. 1 and 2 replace the action of the potential U. Thus the model forms a simple generalization of potential scattering. This, in itself, suggests the fact that we should exchange vector particles since, in Eq. (8), the effective interaction was enhanced for large energies due to the transformation properties of a Lorentz four-vector, whereas in the scalar Klein-Gordon potential we had to put in the enhancement by hand.

We may argue also from the results of $\lambda\phi^3$ theory. According to Tiktopoulos' rules[8] these ladder graphs would give $(\ln s)/s^{n-1}$ for the n-photon exchange graph

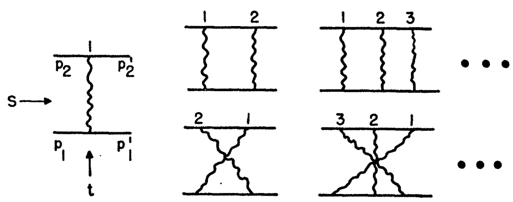

FIG. 1. In the first line we have the Bethe-Salpeter ladder series. In the second we have the associated crossed ladder graphs. The Mandelstam variables $s=-(p_1+p_2)^2$, $t=-(p_1-p_1')^2$, $u=-(p_1-p_2')^2$ are used. We study these graphs in the large-s, fixed-t limit.

interpreting all lines as scalars. Thus we would never get anything like Eq. (7) by exchanging scalar particles.

The simplest way to emit a vector field is from a conserved current. In that case, for example, the theory is renormalizable. We are led, therefore, to use the quantum electrodynamics of massive photons, which we shall call Q.E.D.(W). [W is the mass of the vector particles (W photons).]

III. THE SCATTERING OF TWO SPIN-ZERO PARTICLES

The simplest diagram is the pole term (see Fig. 1). One easily obtains

$$T=\frac{g^2(p_1+p_1')\cdot(p_2+p_2')}{W^2-t}\sim\frac{-2g^2s}{W^2-t}. \qquad (9)$$

Let us consider the $l+1$ photon ladder (l is the number of loops). The Feynman integral is of the form (see Fig. 3)

$$T=\frac{g^2(-ig^2)^l}{(2\pi)^{4l}}\times\int\frac{d^4q_1\cdots d^4q_l\mathfrak{N}}{\prod_{i=1}^{l+1}(\bar{q}_i^2+W^2)\prod_{j=1}^{l}(q_j^2-2p_1\cdot q_j)(q_j^2+2p_2\cdot q_j)}, \qquad (10)$$

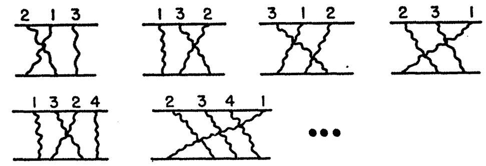

FIG. 2. Examples of the nonplanar ladder graphs in our model. The symbols (2,1,3), (1,3,2) etc., indicate a convenient way to label any graph in our theory. The graphs in Fig. 1 are also so labeled.

where

$$\bar{q}_1=q_1,\quad \bar{q}_2=q_2-q_1,\cdots,\quad \bar{q}_{l+1}=\Delta-q_l,$$

and

$$\mathfrak{N}=[(2p_2+q_1)\cdot(2p_1-q_1)]\cdots\times[(2p_2+q_l+\Delta)\cdot(2p_1-q_l-\Delta)]. \qquad (11)$$

When this product $\mathfrak{N}$ is multiplied out, we have one term that is composed entirely of products of the *external* momenta. Let us consider this term first. We shall call it the "trivial term" and the others "nontrivial."

$$\mathfrak{N}_{\text{tr}}=(4p_1\cdot p_2)^l[4p_1\cdot p_2+2\Delta\cdot(p_1-p_2)-\Delta^2]\sim(-2s)^{l+1}+O(s^l). \qquad (12)$$

The integral for this term is just the $\lambda\phi^3$ type Feynman integral. The Chisholm form[15] of this term is

$$T_{\text{tr}}\sim\frac{(-2g^2s)^{l+1}\Gamma(l+1)}{(4\pi)^{2l}}\int_0^1\frac{dx'\delta(1-\sum x')C^{l-1}}{D^{l+1}}. \qquad (13)$$

[The transition from the Feynman integral to the Chisholm form is actually derived in Appendix A.]

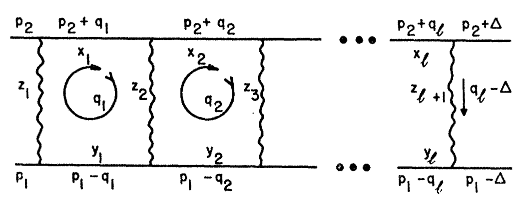

FIG. 3. The $l+1$ photon-exchange ladder graph with momenta labeled. The Feynman parameters $x_1\cdots x_l$, $y_1\cdots y_l$, $z_1\cdots z_{l+1}$ are also given.

In general, D may be written $fs+gt+h$, where f, g, and h are multinominals of the Feynman parameters which, incidentally, are homogeneous of degree $l+1$.[8,16] Using Tiktopoulos' method to get the asymptotic form, we make a "λ transformation" on the "t paths" $x_1\cdots x_l$ and $y_1\cdots y_l$.[8] That is: Let

$$x_i=\alpha\bar{x}_i,\quad y_i=\beta\bar{y}_i;\quad \sum\bar{x}_i=1,\quad \sum\bar{y}_i=1.$$

[15] R. Chisholm, Proc. Cambridge Phil. Soc. **48**, 300 (1952).
[16] R. J. Eden, Phys. Rev. **119**, 1763 (1960); see also Ref. 8, Paper II.

Then one obtains

$$T_l \sim 4\left(\frac{g}{4\pi}\right)^{l+1} \frac{\sigma^{l+1}\Gamma(l+1)}{(2\pi)^{l-1}} \int \frac{d\bar{x}\, d\bar{y}\, dz\, d\alpha\, d\beta(\alpha\beta)^{l-1}\delta(1-\sum \bar{x})\delta(1-\sum \bar{y})\delta(1-\sum z)}{C_0^{-l+1}[\alpha\beta\sigma\phi+g_0t+h_0]^{l+1}}, \tag{14}$$

where the zeros refer to the quantities in the "P-reduced graph." (See Ref. 8, Paper I.) More specifically,

$$C_0 = z_1 z_2 \cdots z_{l+1} \sum_{i=1}^{l+1} \frac{1}{z_i},$$

$$g_0 = -z_1 z_2 \cdots z_{l+1} = g,$$

$$h_0 = C_0 W^2 (z_1+z_2+\cdots z_{l+1}), \tag{15}$$

$\phi = -\tilde{f}$[$\tilde{f}$ is defined by Tiktopoulos (see Ref. 8, Paper I)],

$\sigma = -s$.

Now

$$\left(\frac{d}{d\sigma}\right)^{l-1} \frac{1}{[\alpha\beta\sigma+a]^2} = (-1)^{l+1} \frac{(\alpha\beta)^{l-1}\Gamma(l+1)}{[\alpha\beta\sigma+a]^{l+1}}, \tag{16}$$

where

$$a = (g_0 t + h_0)/\phi.$$

Since

$$\int_0^1 d\alpha \int_0^1 d\beta \frac{1}{[\alpha\beta\sigma+a]^2} = \frac{1}{\sigma a} \ln\left(\frac{\sigma+a}{a}\right),$$

and

$$\left(\frac{d}{d\sigma}\right)^{l-1} \frac{1}{\sigma} \ln\sigma \sim (-1)^{l-1}\Gamma(l) \frac{1}{\sigma^l} \ln\sigma + O(1/\sigma^l), \tag{17}$$

one obtains

$$T_l \sim (-s)\ln(-s) F_l(t) + G_l(t) s, \tag{18}$$

where we have inserted $s=-\sigma$ and used

$$F_l(t) = 4\left(\frac{g^2}{4\pi}\right)^{l+1} \frac{\Gamma(l)}{(2\pi)^{l-1}} \int d\bar{x} \int d\bar{y} \int dz \times \frac{\delta(1-\sum \bar{x})\delta(1-\sum \bar{y})\delta(1-\sum z)}{C_0^{-l+1}(g_0 t+h_0)\phi^l}. \tag{19}$$

In fact (18) is the correct asymptotic expression for the whole T_l since the nontrivial terms can be shown to give rise to negligible integrals in our model. This is proved in Appendix B.[17,18] Note that our ladder graphs are logarithmically divergent. The fact that the divergent terms are also negligible is discussed in Appendix B.

The analysis for the crossed-ladder graphs is quite similar. The result is

$$\bar{T}_l \sim [(-u)\ln(-u) F_l(t) + G_l(t) u](-1)^{l+1}, \tag{20}$$

where $u=-(p_1-p_2')^2$ is the third Mandelstam variable. The $(-1)^{l+1}$ factor essentially comes from the fact that in the crossed channel the "charge" g of one of the particles is opposite and it acts $(-1)^{l+1}$ times. Indeed $(-1)^{l+1}$ is our crossing matrix. It is, in fact, the charge conjugation number C in the t channel.

Since asymptotically $u\to s$ and $-s=e^{-i\pi}s$, the log term cancels in $T_l+\bar{T}_l$ for even numbers of exchanged photons. That is

$$T_l+\bar{T}_l \sim i\pi s F_l(t) \quad [\text{for even } l+1 \text{ only}]. \tag{21}$$

Here we see that we did not need to calculate the $G(t)s$ or $G(t)u$ term in Eqs. (18) and (20), since they cancel in the end.

For odd $l+1$ we retain $s \ln s$ terms. Thus we now consider nonplanar ladder graphs (see Fig. 2), in order to try to cancel these $s \ln s$ terms.

It is a general property that the f's of planar graphs have only one sign in the region of integration.[8] [This region is called the "cube."] However, the f's of nonplanar graphs will take on both signs in the cube. Therefore it is possible to have, besides the usual "t-path" contribution, a contribution to the asymptotic behavior from manifolds of $f=0$ which lie inside the cube. These have been studied rather systematically by Tiktopoulos in $\lambda\phi^3$ theory.[8] We cannot utilize his results directly, however, since for our graphs he gives only an upper bound Ks^{-1} *in* $\lambda\phi^3$ *theory*. Multiplying by our trivial numerator we get a bound Ks^{l+1} in our case.

There are two broad categories of manifolds for which $f=0$. A regular manifold is one for which $f=0$, but its gradient does not vanish. A singular manifold is one in which both f and its gradient vanish. Tiktopoulos has found the contribution due to a regular manifold (see Ref. 8, Paper II). In our case, the regular manifold produces at most constant terms even if we include nontrivial terms.

The singular manifolds are usually the most important ones. We have not been able to characterize our singular manifolds in a very complete way. We have found, however, that there is an infinite number of graphs that do not have singular manifolds inside the cube at all. To demonstrate this we make use of a theorem of Eden[16] for differentiation of the f's: *To obtain the derivative of* f *with respect to* x_i *one can remove the line in which* x_i *occurs in the graph. The "*f*" of the new graph is* $\partial_{x_i} f$.

[17] The properties of $F_l(t)$ that we have verified are as follows: (1) The integral [Eq. (19)] exists; (2) $F_l(t)>0$ for $t\leq 0$; (3) there is a branch point at $t=(l+1)^2W^2$. Explicitly $F_1(t)$ is

$$F_1(t) = \frac{8(g^2/4\pi)^2}{t(1-4W^2t^{-1})^{1/2}} \ln\frac{(1-4W^2t^{-1})^{1/2}-1}{(1-4W^2t^{-1})^{1/2}+1}.$$

[18] Notice that we do not include "seagull vertices"; that is the 4-lined vertex with two photons and two Π's. If one "shorts out" one or more Π lines in one of our ladders, thus introducing some seagulls, one can easily show that the resulting graph is unimportant asymptotically.

[Thus we shall call the new graph a *derivative graph.*] For example, in the case of the graph of Fig. 4 one easily obtains $\partial_{z_2}f=-x_2(y_1+y_2)$, which cannot vanish inside the cube. The topological reason for this is the fact that the derivative graph is planar. One can apply this method to many graphs. In particular, by choosing the appropriate photon line one can show that all the sixth-order graphs, in our model, have planar derivative graphs.

This can be done as well for almost all eight-order graphs. In the case of (2,1,4,3) [see Fig. 5] differentiation with respect to any photon line leads to a nonplanar graph. However, one finds a planar derivative graph by differentiation with respect to y_2. This is true also for (3,4,1,2). On the other hand (3,1,4,2) and (2,4,1,3) have no planar derivative graphs. Thus the method fails here. Let us call the graphs with at least one planar derivative *quasiplanar graphs.*

We have found that the derivatives of the f corresponding to (3,1,4,2) are linearly independent so that one might expect at most a discrete set[19] for the singular manifold. However, we can easily show that the Jacobian of the derivatives vanishes whenever all the derivatives do. Thus one cannot be absolutely sure. In higher orders it seems quite likely that all the derivatives of f will remain independent since there are in general many, many terms in each derivative with no simple relationship between them. Indeed, it would be strange if the non-quasiplanar graphs contributed more than $s\ln s$. For these reasons we expect the "t paths" to continue to dominate in the non-quasiplanar graphs as well. We shall assume this in the following.

Since the six order graphs have no singular manifolds we need to consider only the "t path" contributions. We may calculate the f's by use of Tiktopoulos' theorem (see Ref. 8, Paper I). Note first that the f's of (2,1,3) and (1,3,2) are the same. [Call it f_2.] f_1 will be the "f" of (1,2,3). We find

$$-f_1=x_1y_1(z_2+z_3+x_2+y_2)+x_2y_2(z_1+z_2+x_1+y_1)+x_1y_2z_2+x_2y_1z_2,$$

$$-f_2=x_2y_2(x_1+y_1+z_1+z_2)+z_1x_1y_2+z_2x_2y_1-z_3x_1y_1,$$

(see Figs. 3 and 4). The $\phi=-\tilde{f}$ are found in usual

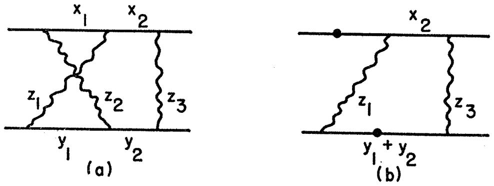

FIG. 4. The graph (2,1,3), (a), and (b), its "derivative graph" (with respect to z_2).

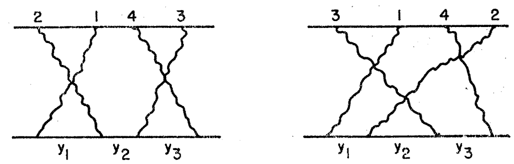

FIG. 5. Two eighth-order graphs discussed in Sec. III.

way.[8–I]

$$\begin{aligned}\phi_1&=xyz_1+(1-x)(1-y)z_3+z_2,\\ \phi_2&=yz_1+xz_2-(1-x)(1-y)z_3,\end{aligned}\tag{22}$$

where

$$x=x_2=1-x_1\quad\text{and}\quad y=y_2=1-y_1.$$

We may summarize the contribution from any of the six order graphs by the generality

$$T_i^{\mp}=-\frac{1}{\pi}\left(\frac{g^2}{4\pi}\right)^3 s^3\int F_i^{\mp}C_0\delta(1-z_1-z_2-z_3)d^3z,\tag{23}$$

where

$$F_i^{\mp}=-\frac{\partial}{\partial D_0}\int_0^1 dx\int_0^1 dy\int_0^1 d\alpha\int_0^1 d\beta\frac{\alpha\beta}{[\alpha\beta\phi_i\sigma+D_0]^2},\tag{24}$$

see Eq. (14). D_0 is g_0t+h_0. The $-$ in F^- and T^- refers to $\sigma=-s$ [for (1,2,3), (1,3,2), and (2,1,3)] while the $+$ refers to $\sigma=-u\sim s$ [for (3,2,1), (2,3,1), and (3,1,2)]. We have already taken the crossing factor $C=(-1)^{l+1}=-1$ into account in Eq. (23).

In general, ϕ is of the form A_iy+B. We integrate (24) first over y

$$F=\int dx\int d\alpha\int d\beta\frac{1}{\sigma A}\left[\frac{1}{[\alpha\beta B\sigma+D_0]^2}-\frac{1}{[\alpha\beta C\sigma+D_0]^2}\right]$$

$$=\frac{1}{D_0}\int dx\int d\alpha\frac{\alpha}{[\alpha B\sigma+D_0][\alpha C\sigma+D_0]},$$

where we use the notation $C=A+B$. We may now easily integrate over x since

$$A_1=x(z_1+z_3)-z_3,\quad B_1=(z_2+z_3)-xz_3,$$
$$A_2=(z_1+z_3)-xz_3,\quad B_2=x(z_2+z_3)-z_3.$$

Setting $\alpha\sigma=z$ in the result we obtain

$$\begin{aligned}D_0C_0\sigma^2F_1&=\int_0^\sigma dz\frac{C_0}{zC_0+(z_1+z_3)}\times\ln\frac{(z_2+z_3)z+D_0}{z_2z+D_0}\frac{(z_1+z_2)z+D_0}{z_2z+D_0},\\ D_0C_0\sigma^2F_2&=\int_0^\sigma dz\frac{C_0}{zC_0+z_3D_0}\times\ln\frac{z_2z+D_0}{-z_3z+D_0}\frac{z_1z+D_0}{(z_1+z_2)z+D_0}.\end{aligned}\tag{25}$$

[19] Using the constraint condition $\Sigma\, x+\Sigma\, y+\Sigma\, z=1$.

These logarithmic integrals may be integrated in terms of squared logarithms and Spence functions.[20] Fortunately, the Spence functions do not give rise to $s\ln s$ terms. Taking care to define the right branch by the prescription $s\to s+i\epsilon$ one eventually finds that

$$D_0C_0s^2F_1{}^{\pm}\sim\ln\frac{(z_2+z_3)(z_1+z_2)}{z_2{}^2}\ln s\,,$$

$$D_0C_0s^2F_2{}^{\pm}\sim\ln\left[\frac{z_1z_2}{(z_1+z_2)z_3}\right]\ln s\pm i\pi\ln s\,,$$

as $s\to\infty$. Consequently

$$D_0C_0s^2F^{\pm}\sim\ln\left[\frac{(z_2+z_3)z_1{}^2}{(z_1+z_2)z_3{}^2}\right]\ln s\pm 2i\pi\ln s\,,$$

where

$$F^{\pm}=F_1{}^{\pm}+2F_2{}^{\pm}\,.$$

In calculating $T^{\pm}$ the real part vanishes since the logarithm in the z's is antisymmetric with respect to z_1, and z_3 whereas the rest of the integrand is symmetric. Thus we are left with a pure imaginary term for the sum (1,2,3)+(2,1,3)+(1,3,2) which, as we see, cancels the imaginary term from (3,2,1)+(3,1,2)+(2,3,1).[21]

Thus at least to sixth order we have found the desired optical-model behavior. Because of the computational difficulties involved we have not been able to subject the higher order graphs to this kind of detailed scrutiny. The most reasonable outcome is that the $s\ln s$ terms will all cancel out. If they do not, our model will violate unitarity. To show this, we argue after the fashion of Freund and Oehme,[22] and Yamamoto.[23] In the asymptotic region one obtains the following inequality from unitarity:

$$T(s,0)\geq\frac{1}{16\pi s}\int_{-s}^{0}dt\,T^*(s,t)T(s,t) \tag{26}$$

by dropping the positive definite inelastic contribution. Limiting the lower limit of the integral to a fixed constant $-b$ only enhances the inequality. Inserting

$$T=Bs\ln s+As\,,$$

one finds

$$s(\ln s\,\mathrm{Im}B+\mathrm{Im}A)\geq\frac{s}{16\pi}\int_{-b}^{0}dt \times[|B|^2\ln^2 s+2\,\mathrm{Re}AB\ln s+|A|^2]. \tag{27}$$

Thus the only way to save unitarity is to take $B\equiv 0$. Consequently, if the $s\ln s$ terms cancel to all orders they will not do so by accident but in conformity to unitarity. Since Q.E.D.(W) is expected to obey unitarity, a lack of cancellation would mean that our set of graphs is incomplete.[24]

In Sec. V it will be found that the coefficient of the remaining term in sixth order, that goes like s asymptotically, will be of interest. To obtain this however much more elaboratore calculations are involved. In fact Eq. (14) involving the "P-reduced quantities" omits some terms that go like s.[25] We hope to be able to calculate this coefficient at a later time. Meanwhile we can at least derive the fact that the remaining s term will be real, using the restrictions implied by the dispersion relations as outlined in Appendix C. The sum of all six graphs is a $C=-1$ amplitude. If we assume that there is a term in the imaginary part of order s one finds $(-2s/\pi)\ln s$ in the real part. This contradicts the sixth-order cancellation that we have just found. A term of order s in the real part, on the other hand, gives no inconsistency.[26]

IV. THE COMPLICATIONS DUE TO SPIN

We now endow one or both of the scattered particles with spin $\frac{1}{2}$. Ordinarily this would entail a great amount of Dirac γ algebra. In desperation one might be tempted to neglect the mass of the spin-$\frac{1}{2}$ particles entirely since then one would have helicity conservation and the simpler σ algebra. This approximation might not be so bad since in comparison to the energy the mass is small indeed. In fact we will be able to show that this short cut is partially justified by rewriting the Feynman rules

[20] The Spence function is $L(z)=\int_0^z t^{-1}\ln(1-t)dt$. See K. Mitchell, Phil. Mag. **40**, 351 (1949); J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts), p. 462.

[21] At first the author, through an algebraic error, concluded that the $s\ln s$ cancellation did not occur. The author is indebted to Professor R. Oehme for showing him his own unpublished calculations showing cancellation. In this way, the author's error was found. This cancellation, apparently, was first discovered by A. Chester (Professor P. G. O. Freund, private communication; see also Ref. 66). These results disagree with the results of K. Ahmed, Phys. Rev. **138**, B1470 (1965). He calculates only the "pinch contribution" due to graphs (2,1,3) and (1,3,2) (that is, the imaginary terms of order $s\ln s$ in F_2^-) and, as a consequence, actually omits the real term of order $s\ln s$ in F_2^-. Naturally, without the real term in F_2^-, he cannot (and does not) find the cancellation of the real term in $F_1^-+2F_2^-$. Moreover, he makes a sign error in the imaginary part of the crossed graphs (3,1,2) and (2,3,1) and thus does not find the cancellation in the imaginary part. Lest the reader now worry that, perhaps, it was the present author who made the sign error, he may easily verify, using the $C=-1$ dispersion relations (see Appendix C) that an imaginary part of order $s\ln s$ leads to a real part of order $s\ln^2 s$. Because of its quasicontroversial nature we have derived this result in a somewhat more transparent, though lengthier, fashion than the usual technique which starts directly from (19). That is, we have taken $s\to\infty$ at the very last in order to satisfy doubts concerning the usual technique. The results are indeed the same for both methods.

Note added in proof. Chester's work has now been published: A. N. Chester, Phys. Rev. **140**, B85 (1965).

[22] P. G. O. Freund and R. Oehme, Phys. Rev. Letters **10**, 199, 315(E) (1963).

[23] K. Yamamoto, Phys. Letters **5**, 355 (1963).

[24] This means that these $s\ln s$ terms will have to combine with other terms in Q.E.D.(W) in such a way that the total asymptotic result for Q.E.D.(W) will not violate unitarity. We study a rather general example of this type of phenomena in Sec. VII part A.

[25] It is interesting to note that these terms of order s omitted by Eq. (14) actually depend on the masses of the scattered particles although the contributions from Eq. (14) do not.

[26] We should point out that all our results concerning cancellation would hold if we replaced our simple W-photon propagator by the more general Lehmann form, provided that the spectral density is sufficiently well behaved.

in such a way as to emphasize the helicity-conserving aspects. Let

$$U=\begin{bmatrix}\phi\\ \chi\end{bmatrix} \quad\text{and}\quad V=\begin{bmatrix}\bar\phi\\ \bar\chi\end{bmatrix}$$

be ordinary Dirac spinors normalized to $U^\dagger U=2E=V^\dagger V$. U represents a particle, V an antiparticle. We now display, for reference, a complete set of spinors $U_{\mathbf{p}}^{(h)}$; $\mathbf{p}$ is the 3-momentum and h the helicity.

$$\begin{aligned}U_{\mathbf{p}}^{(1)}&=(E+p)^{1/2}\begin{bmatrix}1\\ m/(E+p)\end{bmatrix}\alpha, & U_{\mathbf{p}}^{(-1)}&=(E+p)^{1/2}\begin{bmatrix}m/(E+p)\\ 1\end{bmatrix}\beta,\\ V_{\mathbf{p}}^{(1)}&=(E+p)^{1/2}\begin{bmatrix}-m/(E+p)\\ 1\end{bmatrix}\alpha, & V_{\mathbf{p}}^{(-1)}&=(E+p)^{1/2}\begin{bmatrix}1\\ -m/(E+p)\end{bmatrix}\beta,\end{aligned} \tag{28}$$

where

$$\alpha=\begin{bmatrix}1\\0\end{bmatrix} \quad\text{and}\quad \beta=\begin{bmatrix}0\\1\end{bmatrix}$$

are quantized in the direction of $\mathbf{p}$. Let

$$i\gamma_4\gamma_\mu=\Sigma=\begin{bmatrix}\sigma_\mu & 0\\ 0 & -\bar\sigma_\mu\end{bmatrix},$$

where $\sigma_\mu=(\boldsymbol{\sigma},i)$, $\bar\sigma_\mu=(\boldsymbol{\sigma},-i)$.[27,28] With these σ's one can form the products

$$A^-=\sigma_\mu A_\mu=\boldsymbol{\sigma}\cdot\mathbf{A}-A_0,$$
$$A^+=\bar\sigma_\mu A_\mu=\boldsymbol{\sigma}\cdot\mathbf{A}+A_0.$$

The Feynman rules are modified as follows: We write

$$-igU^\dagger(p_2)\Sigma_\mu U(p_1) \text{ instead of } -g\bar U(p_2)\gamma_\mu U(p_1) \quad\text{for the vertex} \tag{29}$$

and

$$\frac{-i[\gamma_4 m+\Sigma^\dagger\cdot p]}{p^2+m^2}=\frac{-i\begin{bmatrix}p^+ & m\\ m & -p^-\end{bmatrix}}{p^2+m^2} \text{ instead of } \frac{-i[-i\gamma\cdot p+m]}{p^2+m^2} \tag{30}$$

for the fermion propagator. These rules can be readily verified by considering an arbitrary fermion line in a Feynman graph expressed in terms of the usual rules. Upon transforming to the Σ's, the equivalence becomes obvious.

We now consider the pole term for scattering a spin-zero particle (Π) with a spin-$\frac{1}{2}$ particle (N). (We work explicitly in the center-of-mass system.)

$$\langle h_1'|T|h_1\rangle=[g^2/(W^2-t)](2p_2+\Delta)_\mu \times[(\phi_1')^\dagger\sigma_\mu\phi_1-(\chi')_1^\dagger\bar\sigma_\mu\chi_1], \tag{31}$$

where ϕ_1 means $\phi_{\mathbf{p}_1h_1}$ for short and $p_2'=p_2+\Delta$. Let $\mathbf{p}_1$ define the positive z axis and let the scattering occur in the xz plane. Consulting (28), we have

$$\phi_1=(E+p)^{1/2}\alpha,\quad \phi_1'=(E+p)^{1/2}e^{-i\sigma_y\theta/2}\alpha,\ \text{etc.},$$

where we have chosen $h_1=h_1'=1$. θ, of course, is the scattering angle. Note that

$$\begin{aligned}\alpha'=e^{-i\sigma_y\theta/2}\alpha&=\begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{bmatrix}\\ &=\frac{\alpha(-u)^{1/2}}{2p}+\frac{\beta(-t)^{1/2}}{2p}\sim\alpha+O(s^{-1/2}),\\ \beta'=e^{-i\sigma_y\theta/2}\beta&=\begin{bmatrix}-\sin(\theta/2)\\ \cos(\theta/2)\end{bmatrix}\\ &=\frac{-\alpha(-t)^{1/2}}{2p}+\frac{\beta(-u)^{1/2}}{2p}\sim\beta+O(s^{-1/2}),\end{aligned} \tag{32}$$

where we have used

$$t=-4p^2\sin^2(\theta/2),\quad u=-4p^2\cos^2(\theta/2).$$

Equation (31) becomes

$$\langle 1|T|1\rangle=\frac{g^2}{W^2-t}(E+p) \times\tilde\alpha'\left[2p_2^-+\Delta^- -\left(\frac{m}{E+p}\right)^2(2p_2^++\Delta^+)\right]\alpha.$$

[27] These covariant σ matrices are similar to the ones introduced by L. M. Brown, Phys. Rev. **111**, 957 (1958); *Lectures in Theoretical Physics*, (Interscience Publishers, Inc., New York, 1961), Vol. IV. See also M. Tonin, Nuovo Cimento **14**, 1108 (1959). The only difference is that we are using a Minkowski metric and Hermitian γ matrices whereas Brown and Tonin use Feynman's γ matrices and metric. As Brown points out, there are useful identities for the covariant σ algebra analogous to the ones for the γ matrices. Since the analogy is not perfect we list some of these identities in Ref. 28. Note that in the representation we use, $\gamma_4=\begin{bmatrix}0&1\\1&0\end{bmatrix}$.

[28] We provide here a handy list of identities satisfied by the covariant σ matrices:

1. $\sigma_\mu\bar\sigma_\nu+\sigma_\nu\bar\sigma_\mu=2\delta_{\mu\nu}$,
2. $\sigma_\mu\bar\sigma_\nu\sigma_\lambda=\delta_{\mu\nu}\sigma_\lambda-\delta_{\mu\lambda}\sigma_\nu+\delta_{\lambda\nu}\sigma_\mu+\epsilon_{\mu\nu\lambda\alpha}\sigma_\alpha$,
3. $\bar\sigma_\mu\sigma_\nu\bar\sigma_\lambda=\delta_{\mu\nu}\bar\sigma_\lambda-\delta_{\mu\lambda}\bar\sigma_\nu+\delta_{\lambda\nu}\bar\sigma_\mu-\epsilon_{\mu\nu\lambda\alpha}\bar\sigma_\alpha$,
4. $\sigma_\mu A^+\sigma_\mu=-2A^-$,
5. $\mathrm{Tr}A^+B^-=2A\cdot B$,
6. $\sigma_\mu\bar\sigma_\nu-\sigma_\nu\bar\sigma_\mu=-\epsilon_{\mu\nu\alpha\beta}\sigma_\alpha\bar\sigma_\beta$,
7. $\sigma_\mu A^+B^-\bar\sigma_\mu=4A\cdot B$,
8. $\sigma_\mu A^+B^-C^+\sigma_\mu=-2C^-B^+A^-$,
9. $\mathrm{Tr}A^+B^-C^+D^-=-2\epsilon_{\alpha\beta\gamma\delta}A_\alpha B_\beta C_\gamma D_\delta+2(A\cdot B)(C\cdot D)-2(A\cdot C)(B\cdot D)+2(A\cdot D)(B\cdot C)$,
10. $\sigma_\mu A^+B^-C^+D^-\bar\sigma_\mu=2\,\mathrm{Tr}A^+B^-C^+D^-$.

Now

$$p_2^- = \begin{bmatrix} -p-E & 0 \\ 0 & p-E \end{bmatrix} \quad p_2^+ = \begin{bmatrix} -p+E & 0 \\ 0 & p+E \end{bmatrix}, \quad (33)$$

$$\Delta^\pm = -2p\sin(\theta/2)e^{-i\sigma_y\theta/2}\sigma_x;$$

therefore

$$\langle 1|T|1\rangle \sim -2sg^2/(W^2-t),$$

the same as the result for the pole term in $\Pi\Pi\to\Pi\Pi$!

Trying $h_1'=-1$; one obtains

$$\langle -1|T|1\rangle = \frac{g^2}{W^2-t}2m\tilde{\beta}'(p_2^- - p_2^+)\alpha \sim \frac{2mg^2(-t)^{1/2}}{W^2-t}.$$

The helicity flip amplitude is negligible.

For $NN\to NN$ one gets

$$\langle 1,1|T|1,1\rangle \sim [g^2/(W^2-t)] \times s\tilde{\alpha}\sigma_\mu\alpha\tilde{\beta}\sigma_\mu\beta = -2g^2s/(W^2-t),$$

$$\langle -1,1|T|-1,1\rangle \sim [g^2/(W^2-t)] \times \tilde{\beta}(-\bar{\sigma}_\mu)\beta\tilde{\beta}\sigma_\mu\beta = -2g^2s/(W^2-t).$$

Similarly with a little patience one may verify that all helicity-conserving amplitudes give the same result, and that all helicity flip amplitudes are negligible.[29]

Encouraged by our success with the pole terms we study the general ladder graph in the case $\Pi+N\to\Pi+N$. T is the same as before [Eq. (10)] except that $\mathfrak{N}$ is different. Effectively (take $h_1=h_1'=1$)

$$\mathfrak{N} = (s)^{1/2}\tilde{\alpha}'(2p_2+q_l+\Delta)^-(p_1-q_l)^+ \times(2p_2+q_l+q_{l-1})^-\cdots(p_1-q_1)^+(2p_1+q_1)^-\alpha. \quad (34)$$

Note that we have truncated $\mathfrak{N}$ already. For a factor like $(p_1-q_1)^+$ we should really have $\gamma_4 m+\Sigma^\dagger(p_1-q_1)$; for $(2p_2+q_1)^-$ we should have $\Sigma(2p_2+q_1)$. We may omit all $\gamma_4 m$'s since they cannot compete with a p_1^+ which gives rise to a $\sqrt{s}$. Hence, each matrix in the $\mathfrak{N}$ is reduced to the direct sum of 2×2 matrices. The spinors we use have large upper components; thus the form (34) dominates. As in the $\Pi\Pi$ case we need to retain only the trivial part (see Appendix B).

$$\mathfrak{N}_{\rm tr} = (s)^{1/2}\tilde{\alpha}(2p_2+q)^-p_1^+(2p_2)^-\cdots p_1^+(2p_2)^-\alpha.$$

Using (33) and an analogous expression for p_1 one obtains $\mathfrak{N}_{\rm tr}\sim(-2s)^{l+1}$, the same as Eq. (12)! The helicity-conserving amplitude is thus given by Eq. (18). Similarly the crossed graph yields the result given in Eq. (20).

Furthermore, one can easily see that if we took a helicity flip amplitude instead of (34) we would be obliged to put in the complete 4 spinors and 4×4 matrices. Here, in any term, one is faced with the choice of having one or more $\gamma_4 m$ terms or to mix a large and small spinor. Either choice degrades the asymptotic behavior by $1/\sqrt{s}$ at least.[30]

One may do the same thing for NN scattering. Here the σ algebra is a bit more complicated[28] but the answer is the same as before. The helicity-conserving amplitudes, which again dominate, are given by (18) and (20).[31]

In the $\Pi\Pi$ case the interaction took place via the charge g alone. Since the answers are the same for the other two cases we see that effectively only the charge is involved in the asymptotic limit for the ΠN and NN cases as well.[32] More precisely, it is the Dirac type "charge" that is involved since the presence of the normal magnetic moment is absolutely necessary for this result. Thus, in particular, our conclusion of Sec. III that the resulting sixth-order term, that goes as s, will be real applies as well when spin is introduced. Moreover the simple crossing symmetry found in Sec. III applies here also.

It is well to consider briefly what would happen if we allowed vertex renormalization effects in our model. If we did, we would naturally find an anomalous magnetic moment. This anomalous term is capable of giving a significant helicity flip amplitude. For example, in ΠN scattering, one would obtain the following helicity flip amplitudes from the pole graph.

$$\langle\mp 1|T|\pm 1\rangle = \frac{\mp 2sg^2[(-t)^{1/2}/2m]K}{t-W^2}, \quad (35)$$

where K is the anomalous magnetic moment of particle 1.

In the NN case, double flip becomes possible as well. Obviously it would be extremely difficult to include vertex renormalization corrections in closed loop diagrams. Although we shall, in general, forget about the possibility of scattering via the anomalous magnetic moment, in our perturbation theory, we shall have more to say about it in Secs. VI and VII. We note, by the way, that the anomalous magnetic moment term does not contribute significantly to helicity-conserving amplitudes.

[29] In such calculations, one may cut down the labor by utilizing rotational invariance and time-reversal invariance. In general,

$$\langle h_2',h_1'|T|h_2,h_1\rangle = (-1)^{(h_1+h_1'+h_2+h_2')/2}\langle -h_2',-h_1'|T|-h_2,-h_1\rangle$$

from rotational invariance;

$$\langle h_2',h_1'|T|h_2,h_1\rangle_\theta = \langle h_2,h_1|T|h_2',h_1'\rangle_{-\theta},$$

from time-reversal invariance.

[30] In every special case that we have worked out, the "degradation" was $1/s$ at least.

[31] To minimize possible confusion, we point out that the asymptotic helicity conservation is valid to order s as well as $s\ln s$. This even applies to the terms omitted by Eq. (14) (see Ref. 25). We note, however, that it is possible for $G(t)$ to depend on the masses of the scattering particles.

[32] These results are independent of the statistics of the scattered particles, since the exchange graphs of the ones in our model may actually be neglected. For example, the exchange pole graph has a pole in u now—not t. Thus the pole amplitude $\to a$ constant, instead of s, as $s\to\infty$. Using the methods of Appendix B, it is easy to generalize this result to the exchange graphs corresponding to our ladders. The crucial point here is that under exchange both t paths (Ref. 8) increase by one unit so that the $s\ln s$ contribution from the "trivial term" becomes $\ln s$.

V. A CLOSER COMPARISON WITH THE OPTICAL MODEL

It is well known that the expression (5) may be inverted to give the potential from T.[33] However, since we cannot write down the complete $A(t)$, this fact is of little use to us. Instead, we compare our results with the Born series of the optical model. Let us study in particular the Klein-Gordon version of the optical model with a four-vector interaction [see Eq. (8)]. If one inserts the effective potential

$$U=2EU'=(2E)\frac{g^2}{4\pi}\frac{e^{-Wr}}{r}=\frac{(2E)g^2}{(2\pi)^3}\int d^3q\frac{e^{i\mathbf{q}\cdot\mathbf{r}}}{\mathbf{q}^2+W^2} \quad (36)$$

into Eq. (5), one obtains

$$\chi=\frac{-g^2}{(2\pi)^2\beta_L}\int d^2q\frac{e^{i\mathbf{q}\cdot\mathbf{b}}}{\mathbf{q}^2+W^2}$$
$$=\frac{-g^2}{2\pi}\frac{1}{\beta_L}\int_0^\infty\frac{qdqJ_0(bq)}{q^2+W^2}=\frac{-g^2}{2\pi\beta_L}K_0(bW), \quad (37)$$

where β_L is the laboratory velocity of the particle 1. (Particle 2 is at rest.) Expanding Eq. (6) to first order in χ, inserting (37), and using the Fourier integral theorem, one finds the usual Born term which is identical to our pole result [Eq. (9)].

Having found a potential that duplicates the pole term, we may test it again by finding the fourth-order term in (6). We find (using $\beta_L\to 1$ as $s\to\infty$)

$$T_1^{(0)}\sim 4is\left(\frac{g^2}{4\pi}\right)^2\int d^2q\frac{1}{(\mathbf{q}^2+W^2)}\frac{1}{(\mathbf{q}+\mathbf{\Delta})^2+W^2}.$$

Introducing Feynman parameters and integrating over d^2q, one obtains

$$T_1^{(0)}=4\pi is\left(\frac{g^2}{4\pi}\right)^2\int_0^1 dx\frac{1}{[W^2-tx(1-x)]}=i\pi F_1(t)s, \quad (38)$$

where $F_1(t)$ is defined in Eq. (19).[17] Comparison with (21) shows that this is precisely our fourth-order perturbation-theoretical result.

Similarly the $(l+1)$st term in the Born series can be written

$$T_l^{(0)}=8\pi is\left(\frac{-ig^2}{4\pi}\right)^{l+1}\frac{1}{\Gamma(l+1)}$$
$$\times\int\frac{dz\cdots dz_{l+1}(1-\sum z)}{(g_0t+h_0)}, \quad (39)$$

where g_0 and h_0 can be defined in terms of the "P-reduced graph" of the $l+1$ photon exchange ladder. [They are explicitly defined in Eq. (15).] The resemblance to Eq. (19) is quite striking.

[33] See p. 386 of Ref. 12; see also R. Omnes, Ref. 6.

It would be very exciting if our perturbation results would agree with (39) to all orders. This would mean that the effects of retardation, recoil, and crossing symmetry actually all cancel out in the end. However, this may be rather much to hope for in view of the fact that some terms of order s contain the masses of the scattered particles whereas (39) does not. (See Sec. III and Ref. 25.) This is why the sixth-order calculation of order s is of interest. If there is a disagreement in sixth order, one could easily modify the potential U given in (36) by adding a sixth-order contribution to it (in order to match T at least up to sixth-order). This will not lead to an unreasonable potential since the expression (39) has a threshold at $t=9W^2$ as do the graphs corresponding to that order. Thus whatever sixth-order potential is needed, it will be reasonable in that it will be of finite range. It is possible that this process may be carried out to higher orders but this is difficult to ascertain.

With the assumption that the higher order $s\ln s$ terms cancel, one can at least show that the alternation of reality properties as displayed in (39) will also be manifested in the perturbation-theory results. This follows, for the $C=-1$ graphs, from the same dispersion-theoretic argument given in Sec. III with reference to the sixth-order graphs. For the $C=+1$ terms, one may make a very similar argument by interchanging the roles of the real and imaginary parts. [See Eqs. (C-3) to (C-6).] This will show that only an imaginary term of order s is possible for $C=+1$. A real term of order s would lead to an imaginary term of order $s\ln s$.

VI. A SEMICLASSICAL DERIVATION OF THE OPTICAL MODEL FROM FIELD THEORY

At high momenta one expects particles to behave somewhat classically in wave mechanics since the de Broglie wavelength is small. In fact, the optical model (briefly reviewed in Sec. I) proceeds from this view and is, in fact, a semiclassical approximation. One wonders, indeed, if such a semiclassical view is ever appropriate in high-energy processes as described by a field theory. Obviously one would have to seek circumstances in which the many-body aspects of the process are ignorable. We may have such a case at hand. (For definiteness, let us consider NN scattering in the center-of-mass system.)

Before the two particles come into the mutual range of interaction $\sim 1/W$, a classical, single-particle description of each of the two N's individually should be legitimate, provided that one uses their renormalized masses. When they are in the interaction region, they exchange any number of massive photons. Since we work at finite momentum transfer, the average or effective momentum transfer involved with the emission (absorption) of any single "photon" will be rather small in comparison with the average energy of the N

emitting (absorbing) it. This circumstance indicates that the emission (absorption) of a single virtual massive photon does not, on the average, push the N far off the mass shell, at least relative to the magnitude of its energy. That is to say, we might imagine it to be quasifree. Thus we will treat the two N's as classical sources of the W-photon field. That means that we start from the assumption that $[j_\mu(x),j_\nu(y)]=0$, where j_μ is the total source current $j^{(1)}+j^{(2)}$ for our massive photons.[34] In the beginning we shall entirely neglect off-the-mass-shell effects and the momentum dependence of the vertex, and work only with the renormalized mass and charge g. We have actually done the same thing in our perturbation-theoretic discussions of Secs. II–IV. Later on in this section we shall try to compensate for these approximations.

The S matrix

$$S=T\exp\left(i\int j_\mu(x)A_\mu(x)d^4x\right), \tag{40}$$

may be simplified when j is a classical current,[35] i.e.,

$$S=\exp\left(-\frac{1}{2}\int j_\mu(x_1)G(x_1-x_2)j_\mu(x_2)d^4x_1d^4x_2\right)\times\mathbf{N}\exp\left(i\int d^4z\; j_\mu A_\mu\right), \tag{41}$$

where $G(x-y)$ is the "contraction" or propagator for the photon field and $\mathbf{N}$ stands for the normal product. This result is useful in the theory of the infrared divergence problem.[35] Since we have no photons in the initial and final states, the normal product exponential becomes 1. Thus

$$S=\exp\left(-\frac{1}{2}\int j_\mu(x)G(x-y)j_\mu(y)d^4xd^4y\right). \tag{42}$$

G is given by

$$G(x_1-x_2)=\frac{-i}{(2\pi)^4}\int d^4p\frac{e^{i(x_1-x_2)\cdot p}}{p^2+W^2-i\epsilon}=\frac{1}{(2\pi)^3}\int d^3p\frac{e^{i(\mathbf{p}\cdot(\mathbf{x}_1-\mathbf{x}_2)-|t_1-t_2|E)}}{2E}, \tag{43}$$

where $E=(p^2+W^2-i\epsilon)^{1/2}$. $j_\mu(x)$ is the sum of two contributions, one from each particle; $j=j^{(1)}+j^{(2)}$, where

$$j_\mu^{(i)}(x)=g\int u_\mu^{(i)}\delta^{(4)}[x_i(\tau)-x]d\tau. \tag{44}$$

$x_i(\tau)$ is the path of the ith particle; τ is its proper time, and $u_\mu^{(i)}$ is its four-vector velocity. For simplicity, we shall take the following straight-line undeflected paths in the center-of-mass system:

$$\begin{aligned}x_1(\tau)&=(0,\tfrac{1}{2}b,\beta t_1,it_1),\\x_2(\tau)&=(0,-\tfrac{1}{2}b,-\beta t_2,it_2),\end{aligned} \tag{45}$$

with the $u_\mu^{(i)}$'s defined accordingly. We expect these paths to be sufficiently good for large impact parameters.

When one inserts j into (42), one gets $j^{(1)}j^{(1)}$ and $j^{(2)}j^{(2)}$ self-interaction terms which are divergent purely imaginary constants independent of the impact parameter b. This is a typical self-energy divergence due to the point-like character of the charge.[36] These divergent integrals have a $(1-\beta^2)^{1/2}$ factor multiplying them. Thus, if one cuts them off in any way whatsoever, they will vanish as $\beta\to 1$. The physical S matrix, then, is given by the $j^{(1)}j^{(2)}$ cross terms, viz.:

$$S=\exp\left(-\int j_\mu^{(1)}(x)G(x-y)j_\mu^{(2)}(y)d^4x\,d^4y\right). \tag{46}$$

This defines the S matrix even if we do not use such a point-like current as in Eq. (44).

Using (44) and (45) and the 4 velocities corresponding to (45), one obtains

$$S=\exp\frac{g^2(1+\beta^2)}{(2\pi)^3}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\times d^3p\frac{\exp i[\mathbf{p}_\perp\cdot\mathbf{b}+p_z\beta(t_2+t_1)-E|t_1-t_2|]}{2E}$$

$[\mathbf{p}_\perp$ lies in the xy plane$]$. Using $t_1-t_2=t$ and $t_1+t_2=T$,

$$S=\exp\frac{g^2(1+\beta^2)}{(2\pi)^3}\int_{-\infty}^{\infty}dT\int_0^{\infty}dt\times\int d^3p\frac{\exp i(\mathbf{p}_\perp\cdot\mathbf{b}+p_z\beta T-Et)}{2E}. \tag{47}$$

Since E has a negative imaginary part, the integral with

[34] $[j_\mu^{(1)}(x),\, j_\nu^{(2)}(y)]=0$ is quite natural, even if N_1 and N_2 are identical, because during most of the scattering act the interval between N_1 and N_2 is space-like [see Eq. (45)]. This is not true about $[j_\mu^{(1)}(x),j^{(1)}(y)]\equiv 0$ however, since the interval is mostly time-like. This seems to be the weakest point of these considerations. In the present situation, it seems that this is more a self-interaction effect than a many-body effect, since time-like noncommutation happens also for a free N. We shall assume that we may compensate for this approximation by using the renormalized mass.

[35] Johan G. Belinfante, Am. J. Phys. **32**, 655 (1964) has perhaps the simplest derivation of this fact [see his Eq. (6)]. See also R. J. Glauber, Phys. Rev. **84**, 394 (1951); J. M. Jauch and F. Rohrlich, Ref. 20, Chap. 16; N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, London, 1959), Chap. VII.

[36] In field-theoretic language, this divergence is due to "vacuum fluctuations" since the nonquantized currents are part of the "vacuum."

respect to t converges. Replacing $z=\beta T$,

$$S=\exp\left[-i\frac{g^2(1+\beta^2)}{2(2\pi)^3\beta}\int_{-\infty}^{\infty}dz\int d^3p\frac{\exp i(\mathbf{p}_\perp\cdot\mathbf{b}+p_z z)}{\mathbf{p}^2+W^2}\right]$$

$$=\exp\left[-i\frac{g^2}{(2\pi)^2}\frac{1+\beta^2}{2\beta}\int d^2p\frac{\exp(i\mathbf{p}_\perp\cdot\mathbf{b})}{\mathbf{p}_\perp{}^2+W^2}\right]. \tag{48}$$

It is well known in the context of the optical model that as $s\rightarrow\infty$

$$S(b)=e^{i\chi(b)}. \tag{49}$$

In fact, this relationship may be defined in a manner independent of the concept of a potential.[13] Consequently, we find

$$\chi=\frac{-g^2}{(2\pi)^2}\left(\frac{1+\beta^2}{2\beta}\right)\int d^2p\frac{e^{i\mathbf{p}\cdot\mathbf{b}}}{\mathbf{p}^2+W^2}$$

$$=-\frac{g^2}{2\pi}\left(\frac{1+\beta^2}{2\beta}\right)K_0(bW). \tag{50}$$

This is identical to (37) since $\beta_L=2\beta/(1+\beta^2)$. Thus, although one might have actually violated Lorentz covariance in the K.G.-V. optical model due to the neglect of recoil and retardation, we find this not to be the case since we obtain precisely the same result in our covariant formalism. This present formalism serves to replace the K.G.-V. optical model when one does not work in the lab system. This is fortunate since the K.G.-V. optical model presents formidable difficulties in the center-of-mass system. In fact, we have a simple means to improve the calculation; namely, by using a better approximation to the true classical path. We hope to study this problem in a future publication. In this connection, the reader may have already noticed that this formulation bears a strong resemblance to Feynman's functional integral formulation of quantum mechanics and quantum electrodynamics.[37] Indeed, we have estimated the S matrix $S(b)$ by considering only one path rather than an average over all possible paths. Note however when we insert $S(b)$ into Eq. (6) we are effectively integrating over a continuous set of paths, although each one is a straight line.

The ideas involved in this semiclassical approach are very much the same as our perturbation-theoretic model. In fact, the perturbation theory served to motivate the classical current approximation. In both cases closed fermion loops are forbidden. The open fermion lines were retained, however, in the perturbation theory and apparently they are somehow effectively retained in the semiclassical approach. The fact that statistics is not relevant is found in both models.[32,34] In this model we neglected the internal structure including the spin of the scattered particles, treating them as point-like charges. This corresponds to the fact that in our perturbation-theory model we were able to show that we were scattering by means of the charge alone. This indicates that the two models might be different versions of the same thing; so far, as we see from the last section, this has been tested up to fourth order in the coupling constant.

It is easy to generalize these results. We may replace $G(x)$ by

$$G'(x)=\int G(x,W^2)\rho(W^2)dW^2. \tag{51a}$$

In this way we will be able to represent covariantly a superposition of Yukawa potentials in the ordinary optical model. The form (51a) looks very much like the Lehmann spectral form for the exact propagator. However, $\rho(W^2)$ may account for certain vertex renormalization effects as well as photon self-energy effects, which have been neglected entirely up to now. As we mentioned in Sec. V, we may need to modify the potential taken in the optical model to establish a connection between it and our perturbation-theory model. The form (51a) may also incorporate such effects.

Generalizing further, it is interesting to speculate upon the possibility of allowing $G'(x)$ to depend on s. In potential scattering one understands the s dependence in the potential in terms of a retarded interaction of a particle on itself. (See Sec. II.) In the present formulation this may not seem necessary since we have already included retardation in Eq. (51a). However, the retardation, so accounted for, is essentially the retarded interaction of one particle with another. In the above discussion we assumed that the retarded self-interaction could be accounted for by simply using the renormalized mass. Thus it would seem that an s dependence in G' should allow, in some degree, for off-the-mass-shell self-energy effects. Another way to view this is that $N_1(N_2)$ travels through an inhomogeneous W-photon field produced by $N_2(N_1)$; thus its propagation characteristic (mass) is subject to dispersion. Clearly, the natural place to introduce the s dependence is in the distribution $\rho(W^2)$; to wit

$$G'(x,s)=\int_{W_0{}^2}^{\infty}G(x,W^2)\rho(W^2,s)d(W^2). \tag{51b}$$

This automatically retains manifest covariance, something that would become dubious if we insisted on an s dependence in $G(x,W^2)$ itself.

Such a form (51) accommodates only the vertex corrections corresponding to the Dirac form factor. (It is also suitable for $\pi\pi$ scattering.) In the presence of spin,

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[37] R. P. Feynman, Phys. Rev. **80**, 440 (1950); see also Bogoliubov and Shirkov, Ref. 35, Chap. VII. The author is deeply indebted to Professor Y. Nambu who first noticed that the results of Secs. II–IV should be related to Feynman's path integral formulation. This suggestion, in fact, led the author to find the simple relationship discussed in this section.

vertex renormalization should lead as well to an anomalous magnetic moment. It is possible to study the effect of the anomalous magnetic moment and therefore helicity flip in the context of the formalism of this section, although to attempt this in perturbation theory would be quite complicated (see Sec. IV). This may be done by simply adding a magnetization term involving the σ matrices to the classical current j_μ. We will discuss this point at another time. For our present purposes it is sufficient merely to note that it is possible to write down such a formalism. The result should qualitatively be the same as the result in the nonexplicitly covariant optical model.[38]

VII. DISCUSSION OF THE RESULTS

A. The Optical Term and Trajectories in Q.E.D.(*W*)

The present study may be viewed as a theoretical "experiment" in the context of Q.E.D.(*W*) as advocated by Gell-Mann *et al.*[39,40] Usually one tests Regge-type ideas whereas, here, we have tested an optical-model idea. For example the N in Q.E.D.(*W*) was found to Reggeize[39–43] while the Π (regarded as a spinless nucleon) does not.[43] Furthermore the vector meson does not Reggeize. Indicative evidence for this is given by the cancellation in $s \ln s$ of the 6 sixth-order graphs (see Sec. IV).[21] A more general argument given by Mandelstam apparently demonstrates this more completely.[42]

As of yet there is little information concerning the supposed Pomeranchuk trajectory which might come about from graphs such as (1,2), (2,1) and those planar and nonplanar ladders made by inserting internal Fermion loops (see Fig. 6). Such graphs are important even from our point of view since they can give an indication of the corrections to our optical term $sf(t)$.

Such graphs are expected to go as $s \ln^n s$ where n may be quite large for long ladders. One can perform this estimate by methods similar to the discussion in Appendix B, or else by considering the $\lambda\phi^3$ result for the planar ladder[8] and multiplying by our usual trivial numerator. There may be a Γ-pole enhancement (see Appendix B) or the possibility of cancellations. Thus one cannot tell what n is exactly without explicit calculations. We should point out that the same $s \ln^n s$ behavior will apply to many other graphs (see, e.g., Fig. 7).

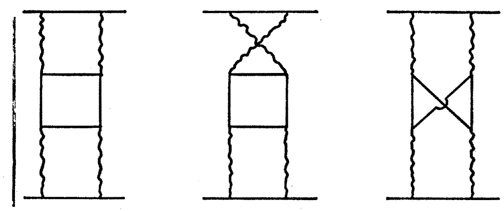

FIG. 6. Eighth-order graphs involved in the Pomeranchuk problem.

There may be some concern, here, lest the Froissart bound $s \ln^2 s$ be exceeded. However one expects that various groups of diagrams will somehow be summed together to form contributions of the form $g(t)s^{\alpha(t)} \times (\ln s)^{\beta(t)}$ as happened in $\lambda\phi^3$ theory [for $\beta(t)=0$].[8,10] (We shall call such contributions "trajectories.") Something of the sort must happen to the $s \ln^n s$ terms ($n \geqslant 1$) in order that unitarity be satisfied. Thus, for example, if the $s \ln s$ terms in our perturbation-theoretic model of Secs. II–V do not cancel, apparently, they will have to be absorbed into some such sum. We could call this summation process the process of "absorption into trajectories" even if the results are not exactly of the form $g(t)s^{\alpha(t)}(\ln s)^{\beta(t)}$ and call the results of such summation trajectories. These considerations, however, lead one to worry also about the prospect that s terms as well may be "absorbed into trajectories."[44] We note immediately, however, that due to the fact that the vector meson does not Reggeize, not all real $C=-1$ s terms may be absorbed. The question really is, what about the imaginary $C=+1$ terms? Let us assume for simplicity that one "trajectory" is more important than all the others at $t \approx 0$. Then we may write

$$T \sim sf(t) + g(t)s^{\alpha(t)}(\ln s)^{\beta(t)}. \qquad (52)$$

If all the imaginary s terms are absorbed, $\operatorname{Im} f(t) \equiv 0$. This case has been investigated by Freund and Oehme,[22] and Yamamoto[23] using inequalities based on unitarity. Yamamoto's inequality is more powerful and yields the restriction that $\alpha(0)=1$ and $\beta(0)=1$, implying a logarithmically increasing total cross section. One may avoid this unphysical possibility if one takes $\operatorname{Im} f(0) \neq 0$. In this case Yamamoto's condition states

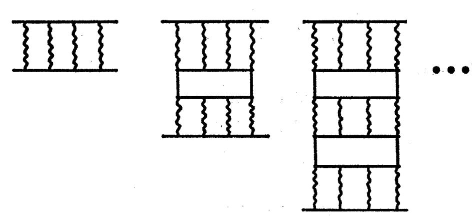

FIG. 7. Another series of important graphs, possibly also involved in the Pomeranchuk problem.

[38] Ref. 12, pp. 360, 361. See also G. Alexander, A. Dar, and U. Karshon, Phys. Rev. Letters **14**, 918 (1965) for a recent use of the black-sphere model incorporating the effects of spin.
[39] M. Gell-Mann and M. L. Goldberger, Phys. Rev. Letters **9**, 275 (1962); **10**, 39(*E*) (1963).
[40] M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).
[41] J. C. Polkinghorne, J. Math. Phys. **5**, 1491 (1964).
[42] S. Mandelstam, Phys. Rev. **137**, B949 (1964).
[43] M. Gell-Mann, M. L. Goldberger, F. E. Low, and F. Zachariasen, Phys. Letters **4**, 264 (1963); M. Gell-Mann, M. L. Goldberger, F. E. Low, V. Singh, and F. Zachariasen, Phys. Rev. **133**, B161 (1964).

[44] One may also worry about cancellations of the graphs in our model with others. We have studied all second- and fourth-order graphs and found no cancellations. In fact as far as skeleton graphs are concerned, the graphs in our model dominate up to fourth order.

that $\max[s^{-1},s^{\alpha(0)-2}(\ln s)^{\beta(0)-1}]$ is not to be dominated by $\max[s^{-1},s^{2\alpha(0)-3}(\ln s)^{2\beta(0)-2}]$. Thus under these circumstances one may only conclude that: I) either $\alpha(0)<1$ and $\beta(0)$ may be anything at all, or II) $\alpha(0)=1$ and $\beta(0)\leq 1$. In case II, naturally, one may wish to require $\beta(0)\leq 0$ so that the total cross section will not grow as $s\to\infty$.[45]

Another possible objection is that an imaginary term of order s violates the well-known theorem of Oehme that poles in the complex angular momentum plane should not be constant functions of t.[46] The observation that such behavior is not rigorously excluded has been made more than once.[47–50] It is interesting to note that our expression (52) when projected into the complex angular momentum plane is a special case of the counter example of Gribov[48] provided that we choose $\alpha(t)$ and $\beta(t)$ appropriately. In any event the analyticity properties in the complex angular momentum plane must be investigated rather thoroughly [in Q.E.D.(W)] before one can establish a contradiction. In fact we could even use our results as an indication that somehow this theorem does not apply [in Q.E.D.(W)] to our optical term.[49]

We conclude, then, that the existence of an optical term $sf(t)$ in the asymptotic scattering amplitude is not at all in contradiction to field theory. In fact after finding such terms in perturbation theory we show, with some reasonable assumptions[45] that they should not disappear by "absorption into trajectories" but will be a "persistent" effect. Probably the most important of these assumptions is the strong but physical assumption that $\sigma_{\text{total}}\sim$const in Q.E.D.($W$). Our arguments however give little or no indication as to the importance of the leading trajectory, i.e., the actual values of $\alpha(0)$ and $\beta(0)$.

B. Actual High-Energy Elastic Scattering and Our Models

A characteristic aspect of the optical term $sf(t)$ that we have found in either in perturbation theory or in the semiclassical approximation of Sec. VI is the presence of a real part (with $C=-1$). This is interesting because of the recent evidence for the existence of significant real terms for $t\approx 0$ and $p_{\text{lab}}>8$ BeV/c in pp and πp elastic scattering.[51] In particular, if one assumes that only one amplitude describes pp elastic scattering one finds from the data[51] that α_+ [which is $\text{Re}T(pp)/\text{Im}T(pp)$ for $t=0$] looks very much like a negative constant over the range from 8 to 24 BeV/c. This is, in fact, the behavior that we predict asymptotically, that is, inserting (50) into (6) one obtains

$$\alpha_+=-\int_0^\infty x\,dx\,\sin(g^2/2\pi)K_0(x)\Bigg/\int_0^\infty xdx[1-\cos(g^2/2\pi)K_0(x)]. \quad (53)$$

While it is obvious that the denominator is positive it takes some work to show that the numerator is negative. We have verified numerically that the numerator is negative for $0<g^2/4\pi<100$. [It probably holds for even larger couplings.] Naturally, we predict that α_- (for $\bar{p}p$)$=-\alpha_+$. We have the same conclusions for Kp scattering.

The case of πp scattering, however, is a different story. In this case the evidence for a real part rests on much firmer ground. The data indicates that $\alpha_+\to 0$, indicating that our real term of order s is absent.[51] More convincing evidence for the absence of this term comes from the study of $\pi^-+p\to\pi^0+n$. Since by isospin invariance

$$\sqrt{2}T(\pi^-p\to\pi^0n)=T(\pi^+p)-T(\pi^-p)\,, \quad (54)$$

our real $C=-1$ term contributes whereas the $C=+1$ imaginary term cancels out. However, the data from 6 to 20 BeV/c may be represented by a Regge-like amplitude of the form $F(t)s^{\gamma(t)}$ where γ is about 0.5 or 0.6 at $t=0$ and probably less for $t<0$.[52–55] This excludes our $C=-1$ s term to a relatively high degree of accuracy.[56] This result excluding a $C=-1$ term of order s is appar-

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[45] This argument can be generalized to include a more complicated leading trajectory

$$g(t)s^{\alpha(t)}(\ln s)^{\beta(t)}(\ln\ln s)^{\gamma_1(t)}\times(\ln\ln\ln s)^{\gamma_2(t)}\cdots$$

without changing the conclusions. We wish to mention that we are assuming $\alpha'(0)\neq 0$ in (52).

[46] R. Oehme, Phys. Rev. Letters **9**, 358 (1962). See also P. G. O. Freund and R. Oehme, *ibid.* **10**, 450 (1963).

[47] Masao Sugawara and Yoichiro Nambu, Phys. Rev. **131**, 2335 (1963).

[48] V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 [English transl: Soviet Phys.—JETP **14**, 1395 (1961)].

[49] Y. S. Jin, Institute for Advanced Study report (unpublished).

[50] T. Kinoshita, private communication and *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1965), Vol. VII B, pp. 144–171.

[51] For the latest small-angle πp results, see K. J. Foley *et al.*, Phys. Rev. Letters **14**, 862 (1965). For pp see Bellettini *et al.*, Phys. Letters **14**, 164 (1965). See also S. J. Lindenbaum, Ref. 65.

[52] I. Mannelli *et al.*, Phys. Rev. Letters **14**, 408 (1965).

[53] A. V. Stirling *et al.*, Phys. Rev. Letters **14**, 763 (1965).

[54] Robert K. Logan, Phys. Rev. Letters **14**, 414 (1965) has fitted the data of Manelli *et al.* (Ref. 52) with a ρ Regge pole. The very similar data (with better statistics) of Stirling *et al.*, (Ref. 53) may be similarly fitted. However, Stirling *et al.* (Ref. 53) warn that the shrinkage is *within* their present experimental errors. On the other hand, the "distorted Born approximation," although very successful in a variety of inelastic reactions, notably fails in this reaction, H. Högaasen and J. Högaasen, CERN report, 1965 (unpublished). A more detailed experiment study of this reaction is clearly indicated, not to mention theoretical effort.

[55] W. Galbraith *et al.*, Phys. Rev. **138**, B913 (1965).

[56] Note also that one may argue against the existence of a $C=+1$ real amplitude of order s as well. If present, it leads via the dispersion relations (see Appendix C) to an $s\ln s$ term in the imaginary part, i.e., a logarithmically increasing cross section. See also Ref. 57.

ently unobtainable on general theoretical grounds.[57] One must either learn it from the data or, hopefully, derive it from some specific model.

This gross discrepency with our results may be understood once we realize that of all the known vector mesons only the ρ_0 may couple to the $\bar{\pi}\pi$ vertex because of G parity. Thus one badly violates isospin independence by not including ρ_+ and ρ_-. That is, for the πp or $\pi\pi$ scattering problem we really should use the Yang-Mills theory[58] which is much more complicated than Q.E.D.(W). In such a theory we can no longer guarantee the $s \ln s$ cancellation in the low-order graphs. Thus this theory may explain the Regge form $F(t)s^{\gamma(t)}$ for the charge exchange amplitude as well as an imaginary term of order s with $C=-1$ (at least at $t=0$).

Since the ϕ and ω couple to the $\bar{p}p$ and $\bar{K}K$ vertices, our Q.E.D.(W) results should apply to pp and Kp elastic scattering even though the effect of the ρ meson is still present. Presuming that the Yang-Mills part of the theory supplies a Reggeized ρ and perhaps a $C=+1$ term of order s, the dominant asymptotic features may qualitatively be given by the Q.E.D.(W) theory alone. The $C=+1$ term would be only changed numerically and the ρ pole would be dominated by the optical term. Certainly the study of the Yang-Mills theory would be very interesting if one would find out that it predicts a Reggeized ρ. However, the theory is marred by the unpleasant circumstance that it is unrenormalizable owing to the presence of the "bad" term in the propagator for the vector particle, i.e., $-iq_\mu q_\nu/[W^2(q^2+W^2)]$ which may be avoided in Q.E.D.(W).

One may wonder if such a view is consistent with $SU(3)$ [specifically the "eightfold way"] since we are suggesting that one part of the vector-meson octet Reggeizes (the ρ) while another part (mostly the ϕ)[59] does not. To satisfy $SU(3)$ one should replace the isospin symmetry of the Yang-Mills theory with unitary symmetry.[58] However it still would be consistent to treat the unitary singlet (mostly the ϕ) as we did in Q.E.D.(W) and have a theory in which the singlet vector meson does not Reggeize whereas the octet of vector mesons does. One should study both generalizations of Q.E.D.(W) in order to see if one (or both) of these two models may be ruled out on experimental grounds.

We may call the above considerations a *literal* interpretation of our theory. We may also interpret it *figuratively*. That is, instead of associating the massive photons in our theory with real vector mesons, one could regard our massive photons as merely a convenient device for developing a covariant phenomenological optical model. In this second view we explain rather little. We are only able to conclude that covariance is connected with a superposition of Yukawa potentials in the usual optical model. One merely inserts various trial $\rho(W^2,s)$ in Eq. (51b) to fit the data. This second *figurative* interpretation would become relevant if it is found that neither of the Yang-Mills+Q.E.D.(W) studies suggested above agrees with the data. Since this proposal may take some time to carry out and besides, is made somewhat ambiguous through nonrenormalizability, it may be well to proceed in the meantime by making various ansätze about $\rho(W^2,s)$ [see Eq. (51b)]. This second approach isn't necessarily inconsistent with the first. However there is obviously considerably more freedom in the figurative approach. For example one need not associate the $G'(x,s)$ or covariant generalized potential with single-particle exchange. The recent work of Arnold[60] may actually be viewed this way. Although he defines χ in the optical model as the Fourier-Bessel transform of a sum of Regge poles, these Regge poles are regarded by him as an approximation to the Chew-Frautschi potential which is indeed more complicated than single-particle exchange.

Such a theory may in principle be capable of comprehending the quasi-asymptotic elastic scattering data that we are now confronted with. In this connection we should mention an older idea that may be the natural extension of the optical model, that is, the statistical approach.[61–63] We are impressed by the fact that statistical models seem to give rise to a variety of phenomena for example: expanding, shrinking, or stationary diffraction peaks.[61,62,64]

It is clear moreover, that some such extension is necessary. A pure optical term $sf(t)$ is obviously inadequate. It is also clear that a pure Regge-pole theory is incomplete since it cannot fit the present data. Even if it would be augmented by the addition of an optical term with the usual Regge poles it would not be able to explain the expansion of the $\bar{p}p$ diffraction peak,[65] al-

[57] N. N. Khuri and T. Kinoshita, Phys. Rev. **137**, B 720 (1965) have given an argument from analyticity properties and physical assumptions that an elementary vector meson is impossible. This does not follow from their work, however, since a vector meson has $C=-1$; the theory that they develop is only valid for $C=+1$ amplitudes. Such a behavior, $T\sim s$, could come, for example, from the subtraction term in a $C=-1$ dispersion relation with two subtractions. See Appendix C. Similarly the $C=-1$ term of order s does not contribute to the $C=+1$ sum rule of the above authors, Phys. Rev. Letters **14**, 84 (1965). See also A. Martin, CERN report, 1964 (unpublished).

[58] C. N. Yang and R. L. Mills, Phys. Rev. **96**, 191 (1954). See S. L. Glashow and M. Gell-Mann, Ann. Phys. (N. Y.) **15**, 437 (1961) for the generalization of the Yang-Mills theory; see also V. I. Ogievetskij and I. V. Polubarinov, *ibid.* **25**, 538 (1963).

[59] J. Sakurai, Phys. Rev. Letters **9**, 472 (1962); Phys. Rev. **132**, 434 (1963); S. L. Glashow, Phys. Rev. Letters **11**, 48 (1963).

[60] Richard C. Arnold, private communication and Phys. Rev. **140**, B1022 (1965).

[61] A. Baiquni, Phys. Rev. **137**, B 1009 (1965).

[62] L. Van Hove, Rev. Mod. Phys. **36**, 655 (1964). See this reference for previous work.

[63] Allan D. Krish, Phys. Rev. Letters **11**, 217 (1963).

[64] There is a dilemma involved in Baiquni's impressive fit to the data (see Ref. 61). He finds it a necessity to use a "negative threshold" in order to explain expansion. See also L. Van Hove, Nuovo Cimento **28**, 798 (1963) for expanding and stationary peaks.

[65] K. J. Foley *et al.*, Phys. Rev. Letters **15**, 45 (1965). See also S. J. Lindenbaum, *Symmetry Principles at High Energy* (W. H. Freeman and Company, San Francisco, 1965), p. 287. The effect of expansion is significantly larger than the experimental error. In the earlier report of K. J. Foley *et al.* Phys. Rev. Letters. **11**, 503 (1963), the effect was only slightly larger than the error.

though it could describe the nonshrinking πp peak. This behavior is strictly prohibited in a pure Regge-pole theory.[66] Even a simple branch-cut contribution like $s^{\alpha(t)} \ln^{\beta(t)} s$ with $\beta(t)>0$ will not do, although it could describe a ln ln$s$ expansion, since crossing symmetry would require an expansion of pp also which is contrary to fact.

C. Asymptotic γ_5 Invariance

So far in this section we have largely ignored the question of the importance of spin in high-energy scattering. Such an attitude is often taken without apology in discussions of high-energy scattering (both experimental and theoretical) for no other reason than the simplification of the analysis. In our case we have some reason for this since we have a model that manifests an asymptotic helicity conservation which can be traced back to the γ_5 invariance of our vertices. This *asymptotic γ_5 invariance* forms a natural basis for the usual *ad hoc* assumption of a single scalar spin-independent amplitude to describe the high-energy scattering of particles with spin. We will abstract what seem to be the physically relevant features of *asymptotic γ_5 invariance* from our models. We believe that this is of interest, since, then, we shall be able to indicate the limitations of this sort of assumption.

In the case of ΠN scattering, the scattering matrix is given by

$$T=a+b\boldsymbol{\sigma}\cdot\mathbf{n}, \qquad (55)$$

where $\mathbf{n}$ is the normal to the scattering plane.[67] From this expression one may easily compute the form of the helicity-conserving and helicity flip amplitude using 2-spinors. Our perturbation-theoretic optical model indicates that $T(\text{nonflip})\sim s$ and that $T(\text{flip})\leq K(t)s^\alpha$ where $\alpha\leq\frac{1}{2}$. From this we may easily infer that

$$\text{(a)} \quad b\leq K(t)s^\alpha,$$

and

$$\text{(b)} \quad a\sim K'(t)s, \quad \text{for fixed } t \text{ as } s\to\infty. \qquad (56)$$

We should not take the condition $\alpha\leq\frac{1}{2}$ too seriously since nonoptical terms which we have not explicitly studied may give rise to significant helicity flip contributions. We shall only insist that $\alpha<1$ since otherwise the symmetry would not mean anything. There is presently no convincing theoretical reason for a more definite bound on α. We may have to learn α from experiment itself.

We may also attempt to apply this symmetry to πp using the *figurative* interpretation of our theory or else by presuming that the Yang-Mills theory can accommodate a helicity-conserving subtheory analogous to the role of our optical model in Q.E.D.(W). For example Eq. (56) may apply to $\pi^-+p\to\pi^0+n$ with $K'(t)=0$ as we have indicated above. This may be a way of finding out what α really is in one case at least, since there may be a significant helicity flip amplitude in this reaction.[68] We may already conclude from experiment[52–55] that $\alpha=0.6$ or less for the πp charge-exchange scattering amplitude and hence for the $C=-1$ part of the πp elastic scattering amplitude.

The same considerations may be applied to the scattering of two spin-$\frac{1}{2}$ objects.

$$T=A+B(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})+C(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n}+C'(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n} +D(\boldsymbol{\sigma}_1\cdot\mathbf{m})(\boldsymbol{\sigma}_2\cdot\mathbf{m})+E(\boldsymbol{\sigma}_1\cdot\mathbf{l})(\boldsymbol{\sigma}\cdot\mathbf{l}), \qquad (57)$$

$\mathbf{n}$ is defined as before.

$$\mathbf{m}=(\mathbf{p}_1-\mathbf{p}_1')/|\mathbf{p}_1-\mathbf{p}_1'|. \quad \mathbf{l}=(\mathbf{p}_1+\mathbf{p}_1')/|\mathbf{p}_1+\mathbf{p}_1'|.$$ [69]

In this case the condition that the helicity-conserving amplitudes dominate leads to

$$\begin{aligned}&\text{(a)} \quad B, C, C', D\leq K(t)s^\alpha,\\ &\text{(b)} \quad A \text{ or } E\sim K'(t)s.\end{aligned} \qquad (58)$$

However the asymptotic dependence that we found in our model also shows that as $s\to\infty$

$$\langle 1,1|T|1,1\rangle\to\langle -1, 1|T|-1, 1\rangle.$$

This is more than mere helicity conservation, it is helicity independence. To ensure this result we must insist that E is negligible as well. That is, finally

$$\begin{aligned}&\text{(a)} \quad B, C, C', D, E\leq K(t)s^\alpha,\\ &\text{(b)} \quad A\sim K'(t)s.\end{aligned} \qquad (59)$$

The situation described by (59) might be called *strong γ_5 invariance,* whereas the more general situation described by (58) could be called *weak γ_5 invariance.* [This distinction disappears in the ΠN case.]

We wish to point out that we have merely displayed a form of asymptotic γ_5 invariance connected with an optical model of a "vector type," corresponding to the neglect of the anomalous magnetic moment. It is impossible, presently, to decide theoretically whether asymptotic γ_5 invariance should be obeyed in nature, or not. Even the considerations of Sec. VI give no clue

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[66] There are many reviews of Regge theory. The Benjamin series has many books on the subject. One such is E. J. Squires, *Complex Angular Momentum and Particle Physics*, (W. A. Benjamin, Inc., New York, 1963). Other reviews are found in R. Oehme, *Strong Interactions and High Energy Physics*, (Oliver and Boyd, London, 1964), pp. 129–222; P. G. O. Freund, *Lectures in Theoretical Physics* (University of Colorado Press, Boulder, Colorado, 1964), Vol. 6, pp. 459–512.

[67] Please note that we use an unusual normalization for $a+b\boldsymbol{\sigma}\cdot\mathbf{n}$ (from the phenomenological point of view). The T matrix normalization corresponds to the amplitude T that we have used consistently throughout the paper. We will do the same in the NN case.

[68] The presence of a strong helicity flip term could explain the dip in the $\pi^-+p\to\pi^0+n$ differential cross section near $t=0$, although that is not the only way. Such a strong helicity flip term is required by the Regge-pole model of Phillips and Rarita (see Ref. 76) to explain "crossover" in the π^-p, π^+p diffraction peaks. See also Robert K. Logan, Phys. Rev. **14**, 921 (1965).

[69] L. Puzikov, R. Ryndin, and Ia. Smordinskii, Zh. Eksperim. i Teor. Fiz. **31**, 592 (1957) [English transl. Soviet Phys.—JETP, **5**, 489 (1957)]. See also L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

since one may be able to eventually derive a covariant optical-model formalism with helicity flip included. (See also Sec. VI).

It is clear for example, that even the success of γ_5 invariance as exemplified by the conserved-vector-current hypothesis[70] and the almost conserved-axial-vector-current hypothesis[71] is not necessarily an indication of the γ_5 invariance of the asymptotic scattering amplitude. Owing to the fact that it is a symmetry concerning vector or axial-vector currents, it comes as no surprise that it is apparently impossible to derive such a result, with respect to the scattering amplitude, on general grounds. In order to draw such conclusions one must have at hand a rather specific model. The conclusion is obviously extremely model-dependent, since if we allowed for vertex renormalization effects one may arrive at significant helicity flip amplitudes via the anomalous magnetic moment. (See Secs. IV and VI).

We think it is very important to decide the question of spin dependence. There is no other way except by experiment. Perhaps the most fruitful beginning would be to study the elastic scattering of pions off of polarized protons at say 10, 12, 14, 16, $\cdots$ BeV/c. The energy dependence of the spin dependence should either vindicate or discredit the above conjecture of asymptotic γ_5 invariance. If no spin dependence is found within experimental errors, we suggest that the charge-exchange reaction: $\pi^- + p \to \pi^0 + n$ (same energies and with polarized protons), has the best chance of showing the presence of spin dependence.[72]

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APPENDIX A: FEYNMAN INTEGRALS WITH NONTRIVIAL NUMERATORS

Let us first discuss a well-known special case which will serve to describe the notation and the general technique. We define a standard Feynman integral by

$$\mathbf{I}(n,l,0) = \frac{1}{(2\pi)^{4l}} \times \int d^4q_1 \cdots d^4q_l \bigg/ \prod_{i=1}^{n} [\bar{q}_i^2 + m_i^2 - i\epsilon_i]. \quad \text{(A1)}$$

$\bar{q}_i$ is the energy-momentum flowing through the ith internal line (m_i is the mass of that particle). q_i is the "loop momentum" of the jth loop. [See, for example, Fig. 3.] We have n internal lines and l loops. Using the well-known formula

$$\frac{-i}{\bar{q}^2 + m^2 - i\epsilon} = \int_0^\infty dx \exp[-ix(\bar{q}^2 + m^2 - i\epsilon)],$$

we obtain

$$\mathbf{I}(n,l,0) = \frac{i^n}{(2\pi)^{4l}} \int_0^\infty dx_1 \cdots dx_n \int d^4q_1 \cdots d^4q_l \, e^{-i\psi} \quad \text{(A2)}$$

where

$$\begin{aligned}\psi &= \sum_{i=1}^{n} x_i(\bar{q}_i^2 + m_i^2 - i\epsilon_i) \\ &= \sum_1^l \sum_1^l A_{ij} q_i q_j + 2 \sum_1^l b_i q_j + K - i \sum_1^n x_i \epsilon_i \\ &= \tilde{q} A q + 2\tilde{b} q + K - i \sum x_i \epsilon_i .\end{aligned}$$

Let R be the real orthogonal matrix that diagonalizes A: $\tilde{R} A R = A'$. Then the substitution

$$q = Rz - A^{-1} b \quad \text{(A3)}$$

reduces ψ to

$$\psi = \sum_{j=1}^{l} A_j' z_j^2 + Q,$$

where $Q = K - \tilde{b} A^{-1} b$. Now using $C = \prod_i A_i = \det A$ and[73]

$$\int d^4q \, e^{-iaq^2} = -i\pi^2/a^2, \quad \text{(A4)}$$

we get

$$\mathbf{I}(n,l,0) = \frac{(i)^{n-l}}{(4\pi)^{2l}} \int dx \frac{e^{-\Sigma x_i \epsilon_i - iQ}}{C^2}. \quad \text{(A5)}$$

C is recognized as one of the Chisholm discriminants; the other one, D, is CQ.[8,15,16] It is well known that C and D are homogeneous of degree l and $l+1$, respectively, in the x's.[8,15,16] Thus Q is homogeneous of degree 1. Therefore, the substitution: $x_i = \rho x_i'$; $\sum_i x_i' = 1$ leads

[70] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); C. S. Wu, Rev. Mod. Phys. **36**, 618 (1964).

[71] Y. Nambu, Phys. Rev. Letters **4**, 380 (1960); M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); W. I. Weisberger, Phys. Rev. Letters. **14**, 1047 (1965); S. L. Adler, *ibid.* **14**, 1051 (1965). See the last two references for work since 1960.

[72] If no helicity flip amplitude is found this apparently will be another serious difficulty with Regge-pole models. See Ref. 68. Note that the Brookhaven group intends to do such experiments; see Ref. 65.

[73] See Bogoliubov and Shirkov, Ref. 35, p. 288, Eq. (24.13).

one easily to

$$\mathbf{I}(n,l,0)=(i)^l\frac{\Gamma(n-2l)}{(4\pi)^{2l}}\int_0^1 dx'\frac{\delta(1-\sum x')}{D^{n-2l}}C^{n-2l-2}, \quad \text{(A6)}$$

the usual Chisholm form.

The general Feynman integral differs from (A1) by the presence of a numerator $\mathfrak{N}$ which is a multinominal in the external momenta and loop momenta q_i. Equations (A5) and (A6) are of use only for those terms in $\mathfrak{N}$ which are independent of the q_i's (trivial terms).

We now study the nontrivial case. Let $Rz=r$. $\mathfrak{N}$ can be written as a multinominal in the r_i whose coefficients are multinominals involving $A^{-1}b$'s and external momenta. The odd-order multinominals in r_i will vanish upon d^4q integration. The even-order terms in general may be studied using the expressions

$$\mathbf{I}(n,l,d)=\frac{i^n}{(2\pi)^{4l}}\int_0^\infty dx_1\cdots dx_n d^4q_1\cdots d^4q_l \times r_{\alpha_1}{}^{a_1}r_{\alpha_2}{}^{a_2}\cdots r_{\alpha_{2d}}{}^{a_{2d}}e^{-i\psi}. \quad \text{(A7)}$$

The subscripts in the r's are the four-momentum indices, the superscripts are matrix indices.

Let $d=1$. We have

$$r_{\alpha_1}{}^{a_1}r_{\alpha_2}{}^{a_2}=R^{a_1b_1}z_{\alpha_1}{}^{b_1}R^{a_2b_2}z_{\alpha_2}{}^{b_2}$$

(summing over repeated b's). If $b_1\neq b_2$, then we get zero; if $b_1=b_2$ one gets the result (A5) with an extra factor in the integrand due to the presence of the $z_{\alpha_1}z_{\alpha_2}$, namely,

$$\begin{aligned}-\sum_{b_1}\frac{2i}{A_{b_1}{}'}R^{a_1b_1}R^{a_2b_1}\tfrac{1}{4}\delta_{\alpha_1\alpha_2} &=-i[R(A')^{-1}\tilde{R}]^{a_1a_2}\tfrac{1}{2}\delta_{\alpha_1\alpha_2}\\ &=(\tfrac{1}{2})(-i)A^{-1}{}_{a_1a_2}\delta_{\alpha_1\alpha_2}. \quad \text{(A8)}\end{aligned}$$

For $d=2$ we have either terms like $z_{\alpha_1}{}^1z_{\alpha_2}{}^1z_{\alpha_3}{}^2z_{\alpha_4}{}^2$, which yield

$$(-i)^2(\tfrac{1}{4})\frac{\delta_{\alpha_1\alpha_2}}{A_1{}'}\frac{\delta_{\alpha_2\alpha_3}}{A_2{}'},$$

or else like $z_{\alpha_1}{}^1z_{\alpha_2}{}^1z_{\alpha_3}{}^1z_{\alpha_4}{}^1$. The average of the latter over 4 space is

$$\frac{1}{24}[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}+\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}](z^2)^2$$

giving after integration the factor

$$\tfrac{1}{4}(-i)^2[\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}+\delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}+\delta_{\alpha_1\alpha_4}\delta_{\alpha_2\alpha_3}]\frac{1}{(A_1{}')^2}.$$

Thus we conclude that

$$R^{a_1b_1}z_{\alpha_1}{}^{b_1}R^{a_2b_2}z_{\alpha_2}{}^{b_2}R^{a_3b_3}z_{\alpha_3}{}^{b_3}R^{a_4b_4}z_{\alpha_4}{}^{b_4}$$

leads to the extra factor

$$\frac{(-i)^2}{2^2}\sum_{\substack{\text{distinct}\\ \text{permutations}\\ \text{of } 1,2,3,4}}A^{-1}{}_{a_1a_2}A^{-1}{}_{a_3a_4}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}. \quad \text{(A9)}$$

The pattern now is obvious. The term

$$r_{\alpha_1}{}^{a_1}\cdots r_{\alpha_{2d}}{}^{a_{2d}}$$

leads to the extra factor

$$\frac{(-i)^d}{2^d}\sum_{\substack{\text{distinct}\\ \text{permutations}\\ \text{of } (1,2,\cdots,2d)}}A^{-1}{}_{a_1a_2}\delta_{\alpha_1\alpha_2}\cdots \times A^{-1}{}_{a_{2d-1}a_{2d}}\delta_{\alpha_{2d-1}\alpha_{2d}}, \quad \text{(A10)}$$

in (A5) or (A6).

For $d=2$ we had two different kinds of terms, those in which four z's were equal and those which had two z^1's and two z^2's, where $z^1\neq z^2$. These cases can be characterized by the partitions of $d=2$, i.e., [2,0] and [1,1], respectively. It turned out that [2,0] terms and [1,1] terms had the same numerical coefficient: $\frac{1}{4}(-i)^2$. In the [1,1] case, the factor from averaging over all four directions, (1/24), was cancelled partially by the factor due to the radial integration, $(-i)^2 6$, to give $(-i)^2(\frac{1}{4})$. If for larger d this kind of cancellation persists, so that the numerical factor is the same for all partitions of d, then the variety of R terms are just appropriate to make up $A^{-1}{}_{a_1a_2}$ product terms. The general proof of (A10) consists, then, in showing that for each d that the numerical factor is independent of the partition, and is in fact $(-i)^d/2^d$. The factor due to the radial integration is $(-i)^d\prod_{i=1}{}^d\Gamma(2+\eta_i)$, where $2\eta_i$ is the number of times z^i occurs in the product of the z's. Let $F(a,\mathbf{p})$ be the factor due to the averaging over the directions of a term with $2d$ z's in the partition $\mathbf{p}=[\eta_1\eta_2\cdots\eta_{2d}]$. (In the case of $d=2$, $\mathbf{p}=[2,0]$, this factor was 1/24.) We notice that $F(d,\mathbf{p})=\prod_{i=1}{}^d F(\eta_i)$, where $F(\eta_i)$ is an abbreviation for the averaging factor when there are η_i identical z's and no others. Our problem simplifies to finding $F(\eta_i)$. In general, the average is given by

$$\langle z_{\alpha_1}\cdots z_{\alpha_{2\eta}}\rangle =F(\eta)(z^2)^\eta\sum_{\substack{\text{all distinct}\\ \text{permutations}\\ \text{of } 1,2,\cdots,2\eta}}\delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}\cdots\delta_{\alpha_{2\eta-1}\alpha_{2\eta}}. \quad \text{(A11)}$$

To get $F(\eta)$ we use four-dimensional spherical coordinates:

$$z_1=r\sin\theta\cos\phi\sin\chi,$$

$$z_2=r\sin\theta\sin\phi\sin\chi,$$

$$z_3=r\cos\theta\sin\chi,$$

$$z_4=r\cos\chi,$$

$$\int d^4z(\ )=\int_0^\infty r^3dr\int_0^{2\pi}d\phi\int_0^\pi \sin\theta\, d\theta\int_0^\pi \sin^2\chi\, d\chi(\ ).$$

The simplest thing, then, is to let $\alpha_1, \alpha_2,\cdots,\alpha_{2\eta}$ all be 4. From (A11) we get

$$F(\eta)\frac{(2\eta)!}{2^\eta\eta!}=\int_0^\pi \cos^{2\eta}\chi\sin^2\chi d\chi\Big/\int_0^\pi \sin^2\chi d\chi,$$

where $(2\eta)!/2^\eta\eta!$ is the number of terms in (A11). We find

$$F(\eta)=\frac{1}{2^\eta}\frac{1}{\Gamma(2+\eta)}.$$

Thus the over-all numerical factor is

$$(-i)^d\prod_{i=1}^d\Gamma(2+\eta_i)\left[\frac{1}{2^{\eta_i}}\frac{1}{\Gamma(2+\eta_i)}\right]=(-i)^d\frac{1}{2^d}$$

just as needed! Therefore (A7) becomes

$$\mathbf{I}(n,l,d)=\frac{i^{n-d-l}}{(4\pi)^{2l}}\int_0^\infty dx\frac{e^{-\Sigma x_i\epsilon_i-iQ}}{2^dC^2}\mathbf{T}^{(d)}, \quad \text{(A12)}$$

where

$$T_{\alpha_1\cdots\alpha_{2d}}{}^{a_1\cdots a_{2d}}=\sum_{\substack{\text{all distinct}\\\text{permutations}\\\text{of }1,2,\cdots,2d}} A^{-1}{}_{a_1a_2}\delta_{\alpha_1\alpha_2}A^{-1}{}_{a_3a_4}\cdots \times A^{-1}{}_{a_{2d-1}a_{2d}}\delta_{\alpha_{2d-1}\alpha_{2d}}.$$

We use the λ transformation $x=\rho x'$ as before in (A6). Since A^{-1} is homogeneous of degree -1, $\mathbf{T}^{(d)}$ is of degree $-d$; we get

$$\mathbf{I}(n,l,d)=\frac{(i)^l\Gamma(n-2l-d)}{(4\pi)^{2l}2^d}\times\int_0^1 dx'\frac{\delta(1-\sum x')}{D^{n-2l-d}}C^{n-2l-d-2}\mathbf{T}^{(d)}. \quad \text{(A13)}$$

Equation (A12) or (A13) somewhat reduces the labor in calculating a Feynman integral with a nontrivial numerator. We note that (A13) can be deduced from (A12) even in the case where the $r_1\cdots r_{2d}$ in (A7) is multiplied by products of $A^{-1}b$'s. This is because b is linear homogeneous in the x's. Since A^{-1} is of degree -1, $A^{-1}b$ is of degree zero. Thus the presence of $A^{-1}b$ factors do not disturb the transition from (A12) to (A13).

APPENDIX B: THE NEGLIGIBILITY OF THE NONTRIVIAL NUMERATORS

We have found it necessary to use the Mellin-transformation method of Bjorken and Wu in finding the asymptotic behaviors of the various contributions to the full Feynman integral containing the exact $\mathfrak{N}$.[10] [See Eq. (10).] It is necessary, however, to generalize their considerations somewhat. That is to say, we wish to study (A12) in the Mellin-transformed α plane instead of (A5).

The Mellin transform is defined by

$$\int_0^\infty F(\sigma)\sigma^{-\alpha-1}d\sigma=\mathfrak{F}(\alpha) \quad \text{(B1)}$$

in conformity with the usage in Ref. (10). Inspection of Eq. (A12) reveals that we shall need to know

$$\int_0^\infty e^{-i\zeta\sigma}\sigma^{-\alpha-1}d\sigma=(i\zeta)^\alpha\Gamma(-\alpha),\ [\mathrm{Re}\alpha<0], \quad \text{(B2)}$$

where $\zeta=(-f/C)-i\delta$, $\sigma=-s$.

Thus in taking the Mellin transform of (A12) we get

$$\mathfrak{g}(n,l,d)=\frac{i^{n-d-l+\alpha}\Gamma(-\alpha)}{(4\pi)^{2l}2^d}\times\int_0^\infty dx\frac{e^{-iJ-\Sigma x_i\epsilon_i}}{C^2}\mathbf{T}^{(d)}\left[\frac{-f}{C}-i\delta\right]^\alpha, \quad \text{(B3)}$$

where $J=(gt+h)/C$.

The Mellin-transform method consists of finding poles in α of a Mellin-transformed Feynman integral $\mathfrak{g}(n,l,d)$. The poles farthest to the right in the half plane $\mathrm{Re}\alpha<0$ lead to the dominant asymptotic behavior in the s plane via the inversion[10]

$$\frac{1}{\Gamma(b+1)}\frac{\ln^b s}{s^a}=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}d\alpha\frac{s^\alpha}{(\alpha+a)^{b+1}}. \quad \text{(B4)}$$

We search, then, for singularities of (B3) in the α plane. For planar graphs the singularities of the integral (B3) have their origin in "V sets"[8] of f, C, and A. Nonplanar graphs can also have singularities due to the vanishing of f on manifolds interior to the region R: $0\le x_i\le\infty$. One class of singularities is easy to get, that is, the singularities due to the vanishing of all the Feynman parameters (i.e., the V set is the entire graph). We can make a λ transformation: $x=\lambda x'$ in (B3) to get

$$\mathfrak{g}(n,l,d)=\frac{i^l\Gamma(n-2l-d+\alpha)\Gamma(-\alpha)}{(4\pi)^{2l}2^d}\int_0^1 dx'\times\frac{1}{C^2}\left(\frac{C}{gt+h}\right)^{n-2l-d+\alpha}\left(\frac{-f}{C}\right)^\alpha\mathbf{T}^{(d)}\delta(1-\sum x'). \quad \text{(B5)}$$

We will call the poles of $\Gamma(n-2l-d+\alpha)$, thus uncovered, Γ poles.

Let us now consider the ladder and crossed ladder graphs in the NN and ΠN cases. The $\Pi\Pi$ case is logarithmically divergent and will be taken up later. The only V sets left in (B5) now are loops and the t paths.[8] Let us λ transform both t paths $t_x=(x_1,x_2,\cdots,x_l)$ and $t_y=(y_1,y_2,\cdots,y_l)$ (see Fig. 3). Let $x_i'=\lambda x_i$ and $y_i'=\mu y_i$ with $\sum x_i=1$, $\sum y_i=1$, and $0<\lambda$, $\mu<1$. Then for small λ, μ the only factor affected is f; thus we require the convergence of

$$\int_0^\epsilon \lambda^{-1+\alpha}d\lambda\int_0^\epsilon \mu^{-1+\alpha}d\mu=\frac{\epsilon^{2(l+\alpha)}}{(\alpha+l)^2},$$

which gives us the condition $\mathrm{Re}\alpha>-l$. This result suggests a double pole at $\alpha=-l$. In fact one can derive this rigorously by integration by parts with respect to λ and μ exposing the double pole at $\alpha=-l$.[10] We note that with regard to (B5) the double λ transformations do not fill the whole region of integration and thus are inexact. However, instead, one may take the double λ transformation in (B3) taking the range of λ and μ to be 0 to ∞ and keeping the constraints $\sum x=1$ $\sum y=1$. In this case the transformation fills the whole region of integration. We obtain an integral of the form

$$\int_0^\infty \lambda^{l-1+\alpha}d\lambda\int_0^\infty \mu^{l-1+\alpha}d\mu\int_0^1 dx\int_0^1 dy\int_0^\infty dz \times\delta(1-\sum x)\delta(1-\sum y)G(x,y,z,\lambda,\mu)e^{-\epsilon(\lambda+\mu)} \quad \text{(B6)}$$

which becomes, upon integration by parts,

$$\frac{1}{(l+\alpha)^2}\int_0^\infty \lambda^{l+\alpha}d\lambda\int_0^\infty \mu^{l+\alpha}d\mu\int dx\,dy\,dz \times\delta(1-\sum x)\delta(1-\sum y)\frac{\partial}{\partial\lambda}\frac{\partial}{\partial\mu}Ge^{-\epsilon(\lambda+\mu)}.$$

This, then, is an analytic continuation of (B3) around the pole at $\alpha=-l$. We call such poles t poles.

The only remaining V sets are the loops. If we "λ transform"[8] an arbitrary loop of length L in (B3) say, we see that since $C\to\lambda C$, $f\to\lambda f$, and $gt+h\to\lambda(gt+h)$, for small λ, we derive no restriction on α. How $\mathbf{T}$ transforms can only be found by actual construction of the tensor and specifying its component. If a certain component transforms like $T\to MT$ for a certain loop, then the existence of $\int_0^\epsilon\lambda^{L-3+M}d\lambda$ follows from the convergence of the original $\mathbf{I}(n,l,d)$.

In our case $n=3l+1$. We have Γ poles at

$$\alpha=d-l-1,\quad d-l-2,\quad d-l-3,\cdots$$

and t poles at

$$\alpha=-l,\quad -l-1,\quad -l-2,\cdots$$

by further integration by parts in (B3). At $d=0$, the poles dominate giving a

$$(1/s^l)\ln s.$$

At $d=1$, the t poles superimpose on the first Γ pole:

$$(1/s^l)\ln^2 s.$$

For $d>1$, the Γ-pole dominates, giving

$$(1/s^{l+1-d}).$$

To get the over-all behavior, consider various terms in $\mathfrak{N}$. For $d=0$ this provides s^{l+1}, giving $s\ln s$ for a result. For $d=1$ we have at most s^l from the numerator or $\ln^2 s$ in all. For $d>1$ we obtain a constant.

We have neglected $A^{-1}b$ factors in $\mathbf{I}(l,n,d)$. Since $A^{-1}b$ is homogeneous of degree zero in the x's, it cannot move any Γ poles. Since b may be written $\lambda b_1+\mu b_2$, $A^{-1}b$ shifts a t pole to the left. b_1 or b_2 can contribute as much as a $\sqrt{s}$. Thus the effect of an $A^{-1}b$ is less than that of a plain $\sqrt{s}$ factor. Therefore we may neglect $A^{-1}b$ factors.

In the $\Pi\Pi$ case, one can apply the above analysis only after removing the divergence. Consider the following factor in the Feynman integral arising from a photon line and its two vertices (see Fig. 3):

$$\frac{(2p_2+q_i+q_{i-1})\cdot(2p_1-q_i-q_{i-1})}{W^2+(q_i-q_{i-1})^2}.$$

This may be rewritten

$$-1+\frac{4p_1\cdot p_2+2(p_1-p_2)\cdot(q_i+q_{i-1})+W^2-q_i\cdot q_{i-1}}{W^2+(q_i-q_{i-1})^2} =-1+B.$$

The term consisting of only B's and no -1 factors is convergent. The others diverge. The dominant contribution as $s\to\infty$ is the term consisting only of B's. The divergent terms contribute terms like $\ln^n s$.

All this analysis would apply to our nonplanar graphs except for the fact that we did not study $f=0$ for manifolds inside the cube. As mentioned in Sec. III the regular manifold gives at most a constant. In sixth-order it was shown that the irregular manifold inside the cube does not exist. As in Sec. III we expect that the irregular manifolds (if they exist) do not give rise to important asymptotic behavior in what we call non-quasiplanar graphs.

APPENDIX C: THEOREM C, DISPERSION RELATIONS AND THE SIGNATURE FACTOR

We shall often need to refer to results concerning the asymptotic behavior of dispersion integrals in this paper. The following theorem about the asymptotic behavior of Cauchy principal-value integrals is a simple generalization of a theorem stated by Hamilton and

TABLE I. The functions $F(\alpha,y)$ and $G(\alpha,x)$ used in Theorem C.

A. $[0<\alpha<1]$	
$F(\alpha,y)$	$G(\alpha,x)$
$y^{-1+\alpha}$	$-\cot\pi\alpha\,(x^{-1+\alpha})$
$y^{-1+\alpha}\ln y$	$-\cot\pi\alpha\,(x^{-1+\alpha})\ln x+\pi\csc^2\pi\alpha\,x^{-1+\alpha}$
$y^{-1+\alpha}\ln^2 y$	$-\cot\pi\alpha\,(x^{-1+\alpha})\ln^2 x+2\pi\csc^2\pi\alpha\,(x^{-1+\alpha})\ln x$ $-2\pi\cos\pi\alpha\csc^3\pi\alpha\,x^{-1+\alpha}$
etc.	etc.
B. $[\alpha=0]$	
$F(0,y)$	$G(0,y)$
y^{-1}	$\dfrac{-1}{\pi}x^{-1}\ln x$
$y^{-1}\ln y$	$\dfrac{-1}{2\pi}x^{-1}\ln^2 x+\frac{1}{2}\pi x^{-1}$
$y^{-1}\ln^2 y$	$\dfrac{-1}{3\pi}x^{-1}\ln^3 x+\frac{1}{2}\pi x^{-1}\ln x$
etc.	etc.

Woolcock.[74] We shall call it theorem C after their usage. *Theorem C*: Let $f(y)=F(a,y)+H(y)$, where $F(\alpha,y)$ is some linear combination of functions listed in Table I. Let

$$(1)\qquad yH(y)\to 0 \quad\text{as}\quad y\to\infty\,,$$

$$(2)\qquad |yH(y)-y'H(y')|<K\left|\frac{1}{y}-\frac{1}{y'}\right|^{\delta},\quad \delta>0$$

for y, y' greater than some fixed constant. If one defines

$$g(x)=\frac{1}{\pi}P\int_c^\infty\frac{f(y)dy}{y-x}$$

then

$$g(x)\to G(\alpha,y)+B(x)/x\,,$$

where B is bounded and G is given by the linear correspondence $F\to G$ also listed in Table I.

The method of the generalization is essentially the same as that of the original theorem.[75] Note that the $(n+1)$st entry in group A may be gotten from the nth by differentiating with respect to α. This formal procedure can be easily justified. Naturally this does not apply to group B. Here one may apply the original method directly.[75]

In our applications, we have dispersion relations of two kinds depending on the crossing symmetry. We shall find it convenient to utilize the following symmetrical variables.

$$\omega=s+\tfrac{1}{2}t-(m_1^2+m_2^2)\,,$$
$$\bar\omega=u+\tfrac{1}{2}t-(m_1^2+m_2^2)\,.$$

For the scattering of two particles of mass m_1 and m_2. Note that $\omega+\bar\omega=0$. The crossing symmetry is either

$$C=+1:\ f(\omega)=f(-\omega)[\mathrm{Re}f(\omega)=\mathrm{Re}f(-\omega)\ \text{and}\ \mathrm{Im}f(\omega)=-\mathrm{Im}f(-\omega)]\,,$$

or

$$C=-1:\ f(\omega)=-f(-\omega)[\mathrm{Re}f(\omega)=-f(-\omega)\ \text{and}\ \mathrm{Im}f(\omega)=\mathrm{Im}(-\omega)]\,.$$

Let N be the number of subtractions, R the real part of f, and I the imaginary part.

Case I. $C=+1$:

(a) For $N=2n$ (N even)

$$\tilde R(\omega)=\frac{\omega^{2n}}{\pi}P\int_c^\infty\frac{I(x)d(x^2)}{x^{2n}(x^2-\omega^2)}\,,$$

$$I(\omega)+\frac{\omega^{2n}}{\pi}P\int_{-c}^{c}\frac{\tilde R(x)dx}{x^{2n}(x-\omega)}=-\frac{\omega^{2n+1}}{\pi}P\int_{-c}^{\infty}\frac{\tilde R(x)d(x^2)}{x^{2n+1}(x^2-\omega^2)}\,.\qquad\text{(C1a)}$$

(b) For $N=2n+1$

$$\tilde R(\omega)=\frac{\omega^{2n+2}}{\pi}P\int_c^\infty\frac{I(x)d(x^2)}{x^{2n+2}(x^2-\omega^2)}\,,$$

$$I(\omega)+\frac{\omega^{2n+1}}{\pi}P\int_{-c}^{c}\frac{\tilde R(x)dx}{x^{2n+1}(x-\omega)}=-\frac{\omega^{2n+1}}{\pi}P\int_{c}^{\infty}\frac{\tilde R(x)d(x^2)}{x^{2n+1}(x^2-\omega^2)}\,,\qquad\text{(C1b)}$$

where

$$\tilde R(\omega)=R(\omega)-\sum_{\text{even } i}^{N-1}\frac{f^{(i)}(0)\omega^i}{i!}\,.$$

Case II. $C=-1$:

(a) For $N=2n$

$$\tilde R(\omega)=\frac{\omega^{2n+1}}{\pi}P\int_c^\infty\frac{I(x)d(x^2)}{x^{2n+1}(x^2-\omega^2)}\,,$$

$$I(\omega)+\frac{\omega^{2n}}{\pi}P\int_{-c}^{c}\frac{\tilde R(x)dx}{x^{2n}(x-\omega)}=-\frac{\omega^{2n}}{\pi}P\int_{c}^{\infty}\frac{\tilde R(x)d(x^2)}{x^{2n}(x^2-\omega^2)}\,.\qquad\text{(C2a)}$$

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[74] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

[75] For the method see F. G. Tricomi, *Integral Equations* (Interscience Publishers, Inc., New York, 1957), Chap. 4. He works out a Hilbert problem on the interval $[-1,1]$. One simply maps this problem from $[-1,1]$ to $[c,\infty]$ by a bilinear transformation. (Ref. 74) See also N. I. Muskhelishvili, *Singular Integral Equations*, (P. Noordhoff, Ltd., Gronigen, the Netherlands, 1953), Chap. 4.

(b) For $N=2n+1$

$$\tilde{R}(\omega)=\frac{\omega^{2n+1}}{\pi}P\int_c^\infty \frac{I(x)d(x^2)}{x^{2n+1}(x^2-\omega^2)},$$

$$I(\omega)+\frac{\omega^{2n+1}}{\pi}P\int_{-c}^{c}\frac{\tilde{R}(x)dx}{x^{2n+1}(x-\omega)}=-\frac{\omega^{2n+2}}{\pi}P\int_c^\infty\frac{\tilde{R}(x)d(x^2)}{x^{2n+2}(x^2-\omega^2)}, \quad \text{(C2b)}$$

where

$$\tilde{R}(\omega)=R(\omega)-\sum_{\text{odd } i}^{N-1}\frac{f^{(i)}(0)\omega^i}{i!}.$$

Equations (C1), (C2) express the dispersion relations in a form convenient for studying asymptotic properties. In this connection, note that the asymptotic behavior of the integrals over a finite range is trivial to evaluate. For example, if $|\omega|>c$

$$\frac{\omega^{2n}}{\pi}P\int_{-c}^{c}\frac{\tilde{R}(x)dx}{x^{2n}(x-\omega)}=-\frac{1}{\pi}\omega^{2n-1}\sum_{j=0}^{\infty}\frac{1}{\omega^j}\int_{-c}^{c}\tilde{R}(x)x^{j-2n}dx.$$

If $C=+1$, the odd terms in j vanish; for $C=-1$, the even terms vanish. Thus these finite-range integrals behave like subtraction terms except for terms of order ω^{-m}, $m\geq 1$. We shall call these integrals pseudosubtraction terms.

The point of Eqs. (C1) and (C2) is that if we know the asymptotic behavior of I (or R) then we may evaluate the asymptotic behavior of R (or I). In simple cases one may use Theorem C for this. In general, one has to worry about possible competition from subtraction or pseudosubtraction terms. This is actually no problem in our paper since the number of subtractions will be known in advance.

Suppose, for example, that $I(\omega)\rightarrow\omega^\alpha$ for a $C=+1$ amplitude. If we insert this into Eq. (C1), subtracted appropriately, one obtains $-\omega^\alpha\cot(\pi\alpha/2)$ for the asymptotic value of $\tilde{R}(\omega)$. Similarly if $C=-1$, one obtains $\omega^\alpha\tan(\pi\alpha/2)$ for $\tilde{R}(\omega)$. Thus, as a mnemonic, one should keep in mind the factors

$$(i-\cot\tfrac{1}{2}\pi\alpha)\omega^\alpha \quad \text{for} \quad C=+1, \quad \text{(C3a)}$$

and

$$(i+\tan\tfrac{1}{2}\pi\alpha)\omega^\alpha \quad \text{for} \quad C=-1, \quad \text{(C3b)}$$

which give the relationship between the real and imaginary part apart from the question of subtraction terms. Expressions (C3) may be rewritten

$$-C\frac{(1+Ce^{-i\pi\alpha})}{\sin\pi\alpha}\omega^\alpha, \quad \text{(C3c)}$$

which is the "signature factor" of Regge theory. Although this is not a new result[76] we have brought it out since it provides a convenient mnemonic for our purposes. Besides this, it helps to make one realize that the signature factor has a meaning entirely independent of the concept of Regge poles.

Exceptional behavior occurs in (C3) when α is an integer. We have either an infinite or a vanishing result for the real part. In the infinite case the actual result for either signature is

$$\left(i-\frac{2}{\pi}\ln\omega\right)\omega^\alpha. \quad \text{(C4)}$$

This happens when α takes on a "physical value" in Regge theory. Note also that (C3) is correct as it stands whenever $\cot(\pi\alpha/2)$ or $\tan(\pi\alpha/2)$ vanishes.

If one happens to have information about the real part instead of the imaginary part the useful replacements for expressions (C3) and (C4) are

$$(-i\tan\tfrac{1}{2}\pi\alpha+1)\omega^\alpha \quad \text{for} \quad C=+1, \quad \text{(C5a)}$$

$$(i\cot\tfrac{1}{2}\pi\alpha+1)\omega^\alpha \quad \text{for} \quad C=-1, \quad \text{(C5b)}$$

and

$$\left(\frac{2i}{\pi}\ln\omega+1\right)\omega^\alpha, \quad \text{(C6)}$$

when either $\tan(\pi\alpha/2)$ or $\cot(\pi\alpha/2)$ becomes infinite

[76] This result is mentioned by Jin (Ref. 49), Logan (Ref. 68), and by R. J. N. Philips and W. Rarita, Phys. Rev. **139**, B1336 (1965). I have not been able to trace the original source, however.