# Wave Functions and Production Amplitudes for Any Spin

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(Received 15 November 1965)

Closed formulas are given for the relativistic helicity eigenstates of a wave function for any spin. Helicity amplitudes for the production of higher spin particles are then computed and brought to a simple form for pion and vector-meson exchange.

## I. INTRODUCTION

MOST theoretical calculations involving production of higher spin resonances require expressions for the higher spin wave functions; e.g., both in polarization and in partial-wave analysis one considers definite helicity states.[1]

In a previous paper[2] a general method was outlined to compute the relativistic wave functions for any spin $S$, and examples were given of helicity amplitudes for the production of $\frac{3}{2}^+$ and $\frac{5}{2}^+$ isobars in $pp$ scattering. In this paper we obtain explicit formulas for the wave functions of all helicity states for any spin $S$. We then use these to compute the helicity amplitudes for the production of higher spin particles, where simple formulas are obtained for pion and vector-meson exchange. Through these examples a method is outlined for treating similar cases.

## II. WAVE FUNCTIONS FOR ANY SPIN

A particle of integer spin $S$ and momentum energy $p$, $p^2=-m^2\neq 0$, can be described by a tensor wave function $U_{\mu_1\cdots\mu_S}(p)$, symmetric in $\mu_1\cdots\mu_S$ and satisfying[3]

$$\begin{aligned} p^{\mu_1}U_{\mu_1\cdots\mu_S}(p)&=0\,,\\ U_{\mu\mu_3\cdots\mu_S}{}^{\mu}(p)&=0\,. \end{aligned} \tag{2.1}$$

Let $U_{\mu_1\cdots\mu_S}{}^{(M)}(p)$ be a helicity eigenstate with eigenvalue $M$.[2] This state can be constructed as an outer product of spin-one helicity eigenstates,

$$U_{\mu_1\cdots\mu_S}{}^{(M)}(p)=\sum_{\Sigma m_i=M}\Big[\prod_{j=1}^{S-1}\langle jM_j1m_{j+1}|j1j+1M_{j+1}\rangle\Big]\times\Big[\prod_{r=1}^{S}U_{\mu_r}{}^{(m_r)}(p)\Big], \tag{2.2}$$

where

$$M_j=\sum_{r=1}^{j}m_r\,.$$

In a spherical basis,

$$U_+=(U_x-iU_y)/\sqrt{2}\,,\quad U_-=-(U_x+iU_y)/\sqrt{2}\,; \tag{2.3}$$

and in the case where $\mathbf{p}$ is along the $z$ axis, the summation in (2.1) gives (see Appendix A),

$$U_{\mu_1\cdots\mu_S}{}^{(M)}(p)=(-)^{n_3+n_0}2^{(n_3+n_0)/2}\times\binom{2S}{S+M}^{-1/2}\left(\frac{E}{m}\right)^{n_3}\left(\frac{p}{m}\right)^{n_0}\delta_{n_+-n_-,M}\,. \tag{2.4}$$

$n_+$, $n_-$, $n_3$, and $n_0$ are the numbers of the $(+)$, $(-)$, $z$, and $t$ components among $\mu_1\cdots\mu_S$, respectively.

For half-integer spin case, $S=k+\frac{1}{2}$, we describe the wave functions in the Rarita-Schwinger representation.[3] These four-spinors $\psi_{\mu_1\cdots\mu_k}(p)$, symmetric in $\mu_1\cdots\mu_k$, satisfy

$$(i\not{p}-m)\psi_{\mu_1\cdots\mu_k}(p)=0\,,\quad \gamma^{\mu}\psi_{\mu\mu_1\cdots\mu_k}(p)=0\,. \tag{2.5}$$

The helicity eigenstates can be calculated from

$$\psi_{\mu_1\cdots\mu_k}{}^{(M)}(p)=\sum_{\lambda}\psi_\lambda(p)U_{\mu_1\cdots\mu_k}{}^{(M-\lambda)}(p)\times\langle kM-\lambda,\tfrac{1}{2}\lambda|k\tfrac{1}{2}SM\rangle\,. \tag{2.6}$$

For the case where $\mathbf{p}$ is along the $z$ axis,

$$\psi_{\mu_1\cdots\mu_k}{}^{(M)}(p)=(-)^{n_3+n_0}2^{(n_3+n_0)/2}\times\binom{2S}{S+M}^{-1/2}\left(\frac{E}{m}\right)^{n_3}\left(\frac{p}{m}\right)^{n_0}\psi_\lambda(p)\delta_{n_+-n_-+\lambda,M}\,. \tag{2.7}$$

In $\psi_\lambda(p)$, $\lambda=\pm\frac{1}{2}$ denotes the helicity eigenvalue.

Since our helicity eigenstates, for integer spin $S$, are constructed by coupling $S$ spin-one states, the coupling is unique. In particular,

$$\begin{aligned} U_{\mu_1\cdots\mu_S}{}^{(M)}(p)&=\sum_{M_r}\langle rM_rS-rM-M_r|rS-rSM\rangle\times U_{\mu_1\cdots\mu_r}{}^{(M_r)}(p)U_{\mu_{r+1}\cdots\mu_S}{}^{(M-M_r)}(p)\\ &=\binom{2S}{S+M}^{-1/2}\sum_{M_r}\binom{2r}{2+M_r}^{1/2}\binom{2(S-r)}{S-r+M-M_r}^{1/2}\times U_{\mu_1\cdots\mu_r}{}^{(M_r)}(p)U_{\mu_{r+1}\cdots\mu_S}{}^{(M-M_r)}(p)\,; \end{aligned} \tag{2.8}$$

[1] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) 18, 198 (1962).
[2] Y. Frishman and E. Gotsman, Phys. Rev. 140, B1151 (1965).
[3] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956), Chap. 4; E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave Equations* (Hafner Publishing Company, New York, 1953), pp. 118–121. Further references concerning relativistic wave equations can be found in these two books.

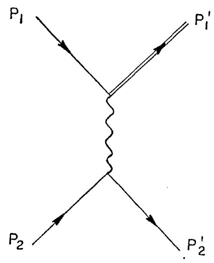


FIG. 1. Feynman diagram for isobar production.

and similarly,

$$\psi_{\mu_1\cdots\mu_k}{}^{(M)}(p)=\binom{2S}{S+M}^{-1/2}\sum_{M_r}\binom{2r+1}{r+\frac{1}{2}+M_r}^{1/2}\binom{2(k-r)}{k-r+M-M_r}^{1/2}\times\psi_{\mu_1\cdots\mu_r}{}^{(M_r)}(p)U_{\mu_{r+1}\cdots\mu_k}{}^{(M-M_r)}(p). \quad (2.9)$$

These relations will be used to simplify helicity amplitudes for higher spin particle production.

## III. HELICITY AMPLITUDES FOR PRODUCTION OF HIGHER SPIN PARTICLES

As an example we consider $NN\to NN^*$ scattering, thereby outlining the general method for other cases. Let the $N^*$ have spin $S=k+\frac{1}{2}$ and parity $\eta$. The exchanged particle is taken to be (a) a pion or (b) a vector meson (Fig. 1).

In the c.m. system we choose the positive $z$ direction in the direction of motion of the $N^*$, and confine ourselves to the $z$-$x$ plane (Fig. 2). In this system we use

$$p_1=(\mathbf{p}E_1),\qquad p_2=(-\mathbf{p},E_2),\qquad p_1'=(\mathbf{p}',E_1'),\qquad p_2'=(-\mathbf{p}',E_2'). \quad (3.1)$$

$\lambda_1$, $\lambda_2$, $\lambda_2'$ are the helicities of the nucleons; $\rho$ is the helicity of the $N^*$. The masses are $m$ for the nucleon and $M$ for the $N^*$.

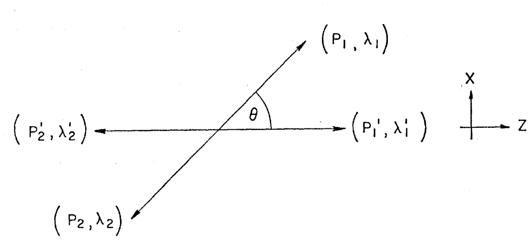


FIG. 2. Four-momenta and helicities in the c.m. system.

### (a) Pion Exchange

In this case, the two vertices of Fig. 1 can be treated separately. For the lower vertex we take

$$L_I=g\bar{\psi}_{\lambda_2'}(p_2')\gamma_5\psi_{\lambda_2}(p_2)\phi(p_2-p_2'), \quad (3.2)$$

where (we use $\gamma^0$ diagonal and $\gamma_5$ off-diagonal)

$$\psi_{\lambda_2}(p_2)=N(p_2)\begin{pmatrix}E_2+m\\ 2\lambda_2 p\end{pmatrix}e^{i(\theta/2)\sigma_y}i\sigma_y x_{\lambda_2},$$

$$\psi_{\lambda_2'}(p_2')=N(p_2')\begin{pmatrix}E_2'+m\\ 2\lambda_2' p'\end{pmatrix}i\sigma_y x_{\lambda_2'}, \quad (3.3)$$

$$N(p)=1/[2m(E+m)]^{1/2}.$$

$\theta$ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_1'$, and $x_\lambda$ are the usual Pauli spinors, referring to the $z$ axis. The lower vertex contributes

$$\langle\lambda_2|\lambda_2'\rangle=gN(p_2)N(p_2')[(E_2+m)2\lambda_2'p'-(E_2'+m)2\lambda_2 p]d_{\lambda_2'\lambda_2}{}^{(1/2)}(\theta), \quad (3.4)$$

For the upper vertex we take an interaction of the form

$$L_I=\frac{G}{m_\pi{}^k}\bar{\psi}_{\mu_1\cdots\mu_k}{}^{(\rho)}(p_1')O\psi_{\lambda_1}(p)\times p_1{}^{\mu_1}\cdots p_1{}^{\mu_k}\phi(p_1'-p_1), \quad (3.5)$$

where $O=\gamma_5$ for $\eta(-)^k=1$ and $O=1$ for $\eta(-)^k=-1$, and $m_\pi$ is the pion mass.

Using (2.6), we get for the upper vertex

$$\langle\rho|\lambda_1\rangle=\frac{G}{m_\pi{}^k}\binom{2S}{S+\rho}^{-1/2}\sum_\lambda F(k\rho-\lambda|p_1'p_1)\times(\bar{\psi}_\lambda(p_1')O\psi_{\lambda_1}(p_1)), \quad (3.6)$$

where

$$\bar{\psi}_\lambda(p_1')O\psi_{\lambda_1}(p_1)=N(p_1)N(p_1')d_{\lambda\lambda_1}{}^{(1/2)}(\theta)\times\begin{cases}[2\lambda_1 p_1(E_1'+M)-2\lambda p_1'(E_1+m)], & O=\gamma_5\\ [(E_1'+M)(E_1+m)-4\lambda\lambda_1 p_1 p_1'], & O=1\end{cases} \quad (3.7)$$

and

$$F(k\rho-\lambda|p_1',p_1)=\binom{2k}{k+\rho-\lambda}^{1/2}U_{\mu_1\cdots\mu_k}{}^{(\rho-\lambda)*}(p_1')p_1{}^{\mu_1}\cdots p_1{}^{\mu_k}. \quad (3.8)$$

This can be evaluated (Appendix B) to give

$$F(k,\rho-\lambda|p_1',p_1)=\sum_{\substack{r=0\\ [(k-r)+(\rho-\lambda)]\ \text{even}}}^{k-|\rho-\lambda|}\frac{k!}{r![\frac{1}{2}(k-r)+\frac{1}{2}(\rho-\lambda)]![\frac{1}{2}(k-r)-\frac{1}{2}(\rho-\lambda)]!}\times(-)^{[(k-r)-(\rho-\lambda)]/2}2^{r-k/2}\left(\frac{p_1'E_1-p_1E_1'\cos\theta}{M}\right)^r(p_1\sin\theta)^{k-r}. \quad (3.9)$$

Separating a factor $(\sin\theta)^{|\rho-\lambda|}$, we are left with a polynom in $\cos\theta$ of degree $k-|\rho-\lambda|$. Another expression for $F$ is (see Appendix B),

$$F(k,\rho-\lambda|p_1',p_1)=\binom{2k}{k+\rho-\lambda}^{1/2}\binom{2k}{k}^{-1/2}\times(-)^{kk/2}\left(\frac{p_1\sin\theta}{\sin\tilde{\theta}}\right)^k d_{0,\rho-\lambda}^{(k)}(\tilde{\theta}),\quad(3.10)$$

where $\tilde{\theta}$ is the angle corresponding to $\theta$ in the rest frame of the $N^*$, namely

$$\cot\tilde{\theta}=(p_1E_1'\cos\theta-p_1'E_1)/Mp_1\sin\theta.\quad(3.11)$$

The helicity amplitude for the process is constructed by forming the product of the proper terms from the two vertices, introducing the pion propagator, and antisymmetrizing with respect to $p_1$, $p_2$.

### (b) Vector-Meson Exchange

For the lower vertex we take

$$\begin{aligned}&1.\quad g_1\bar{\psi}(p_2')\gamma_\mu\psi(p_2)V^\mu(p_2'-p_2),\\&2.\quad g_2\bar{\psi}(p_2')\sigma_{\mu\nu}\psi(p_2)(p_2'-p_2)^\mu V^\nu(p_2'-p_2),\quad(3.12)\\&3.\quad g_3\bar{\psi}(p_2')\psi(p_2)(p_2'-p_2)_\mu V^\mu(p_2'-p_2).\end{aligned}$$

And for the upper,

$$\begin{aligned}&\text{I.}\quad G_1\bar{\psi}_{\mu_1\cdots\mu_k}(p_1')O\psi(p_1)p_1^{\mu_1}\cdots p_1^{\mu_{k-1}}V^{\mu_k}(p_1'-p_1),\\&\text{II.}\quad G_2\bar{\psi}_{\mu_1\cdots\mu_k}(p_1')O\gamma_\mu\psi(p_1)p_1^{\mu_1}\cdots p_1^{\mu_k}V^\mu(p_1'-p_1),\\&\text{III.}\quad G_3\bar{\psi}_{\mu_1\cdots\mu_k}(p_1')O\psi(p_1)p_1^{\mu_1}\cdots\times p_1^{\mu_k}(p_1'-p_1)_\mu V^\mu(p_1'-p_1),\quad(3.13)\end{aligned}$$

where $O=\gamma_5$ when $\eta(-)^k=-1$, and $O=1$ when $\eta(-)^k=1$. The case 3 in (3.12) is absent when $p_2'^2=p_2^2=-m^2$ and the current coupled to the vector meson is conserved. When constructing the helicity amplitudes for the process, we remember that the vector-meson propagator is of the form

$$D_{\mu\nu}(k)=A(k^2)g_{\mu\nu}+B(k^2)(k_\mu k_\nu/m_v^2),\quad(3.14)$$

$m_v$ being the mass of the vector meson.

In the Born approximation

$$A(k^2)=B(k^2)=1/(k^2+m_v^2).\quad(3.15)$$

Let $M(S\rho,\lambda_1|\lambda_2'\lambda_2)$ be the matrix element for this process. Then, when dealing with interactions II–III for the upper vertex, the matrix element can be simplified to

$$M(S\rho,\lambda_1|\lambda_2'\lambda_2)=\binom{2S}{S+\rho}^{-1/2}\times\sum_{\lambda_1'}F(k,\rho-\lambda_1'|p_1'p_1)M(\tfrac{1}{2}\lambda_1'\lambda_1|\lambda_2'\lambda_2).\quad(3.16)$$

And when dealing with I,

$$M(S\rho,\lambda_1|\lambda_2'\lambda_2)=\binom{2S}{S+\rho}^{-1/2}\sum_m\binom{3}{\tfrac{3}{2}+m}^{1/2}\times F(k-1,\rho-m|p_1'p_1)M(\tfrac{3}{2}m\lambda_1|\lambda_2'\lambda_2),\quad(3.17)$$

where (2.9) has been used. We thus see that the case of vector-meson exchange is reduced to the cases of spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ isobars only.


### ACKNOWLEDGMENTS

It is a pleasure to thank Professor G. Yekutieli and other members of the Experimental High Energy group at the Weizmann Institute for suggesting the problem which arose from their recent experimental findings, Dr. E. Gotsman for a stimulating correspondence and Dr. A. Katz for helpful suggestions concerning the manuscript.


### APPENDIX A

Since

$$\langle jM_j1m_{j+1}|j1j+1M_{j+1}\rangle=\left[\binom{2j}{j+M_j}\binom{2}{1+m_{j+1}}\Big/\binom{2j+2}{j+1+M_{j+1}}\right]^{1/2},\quad(\text{A1})$$

we have, with $M_j=\sum_{r=1}^{j}m_r$,

$$\prod_{j=1}^{S-1}\langle jM_j1m_{j+1}|j1j+1M_{j+1}\rangle=\binom{2S}{S+M}^{-1/2}\prod_{j=1}^{S}\binom{2}{1+m_j}^{1/2}.\quad(\text{A2})$$

For the case where $\mathbf{p}$ is along the $z$ axis, the $U_\mu^{(m)}(p)$ are given by[2]

$$\begin{aligned}&m=1:\quad U_+=1,\quad U_-=U_3=U_0=0,\\&m=0:\quad U_+=U_-=0,\quad U_3=-\frac{E}{m},\quad U_0=-\frac{p}{m},\quad(\text{A3})\\&m=-1:\quad U_+=0,\quad U_-=1,\quad U_3=U_0=0.\end{aligned}$$

Hence, from (2.2)

$$\begin{aligned}U_{\mu_1\cdots\mu_S}^{(M)}(p)&=\binom{2S}{S+M}^{-1/2}\sum_{\Sigma m_j=M}\left[\prod_{j=1}^{S}\binom{1}{1+m_j}^{1/2}U_{\mu_j}^{(m_j)}(p)\right]\\&=\binom{2S}{S+M}^{-1/2}(-)^{n_3+n_0}2^{(n_3+n_0)/2}\left(\frac{E}{m}\right)^{n_3}\left(\frac{p}{m}\right)^{n_0},\quad(\text{A4})\end{aligned}$$

as stated in (2.4).

**APPENDIX B**

Using (2.4), we get for (3.8)

$$F(k,\rho-\lambda|p_1'p_1) = \sum_{\substack{n_+-n_-=\rho-\lambda\\ n_++n_-+n_3+n_0=k}} \frac{k!}{n_+!n_-!n_3!n_0!}(-)^{n_3+n_0}2^{(n_3+n_0)/2} \times\left(\frac{E_1'}{M}\right)^{n_3}\left(\frac{p_1'}{M}\right)^{n_0}\left(\frac{p_1\sin\theta}{\sqrt{2}}\right)^{n_+}\left(-\frac{p_1\sin\theta}{\sqrt{2}}\right)^{n_-} \times(p_1\cos\theta)^{n_3}(-E_1)^{n_0}$$

$$= \sum_{\substack{n_+-n_-=\rho-\lambda\\ n_++n_-+r=k}} \frac{k!}{n_+!n_-!r!}(-)^{n_-}2^{r-k/2} \times(p_1\sin\theta)^{k-r}\left(\frac{p_1'E_1-p_1E_1'\cos\theta}{M}\right)^r, \quad \text{(B1)}$$

which is exactly (3.9).

The sum of $r$ can be worked out, but the calculation can be avoided. For, since $F$ of (3.8) is a scalar product, it can be calculated in the rest frame of the $N^*$, i.e.,

$$F(k,\rho-\lambda|p_1',p_1) = \left(\begin{matrix}2k\\ k+\rho-\lambda\end{matrix}\right)^{1/2} U_{r_1\cdots r_k}{}^{(\rho-\lambda)*}(\mathbf{o})\tilde{p}_1{}^{r_1}\cdots\tilde{p}_1{}^{r_k}, \quad \text{(B2)}$$

where $\tilde{p}_1$ is the $N$ energy-momentum in this rest frame. If $\tilde{\theta}$ is the angle of $\tilde{p}_1$ to the $z$ axis, we have

$$U_{r_1\cdots r_k}{}^{(\rho-\lambda)*}(\mathbf{o})\tilde{p}_1{}^{r_1}\cdots\tilde{p}_1{}^{r_k} = U_{3\cdots 3}{}^{(M')*}(\mathbf{o})|\tilde{\mathbf{p}}_1|^k d_{M',\rho-\lambda}{}^{(k)*}(\tilde{\theta}). \quad \text{(B3)}$$

This follows by rotating the coordinate system so that $\tilde{\mathbf{p}}_1$ coincides with the $z$ axis. Using (2.4), we find that only $M'=0$ contributes to (B3), so that we finally get

$$F(k,\rho-\lambda|p_1',p_1) = \left(\begin{matrix}2k\\ k+\rho-\lambda\end{matrix}\right)^{1/2}\left(\begin{matrix}2k\\ k\end{matrix}\right)^{-1/2} \times(-)^{kk/2}\left(\frac{p_1\sin\theta}{}\right)^k d_{0,\rho-\lambda}{}^{(k)}(\tilde{\theta}), \quad \text{(B4)}$$

which is (3.10).

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PHYSICAL REVIEW VOLUME 143, NUMBER 4 MARCH 1966

# Current Algebras and Approximate Symmetries*

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The components of currents involved in weak and electromagnetic interactions are assumed to generate the algebra of chiral $U(6)\otimes U(6)$. For the hadrons, the spurion scheme of broken $U(6,6)$ symmetry is considered, which implies nonchiral $U(6)\otimes U(6)$ symmetry for hadrons at rest, and invariance with respect to collinear $U(6)_p$ for particles moving in a given direction. The chiral $U(6)\otimes U(6)$ commutation relations are evaluated in a single-particle approximation with $U(6)_p$ supermultiplets. It is found that one can obtain $U(6)$ results, even though currents are involved which are not elements of the algebra of $U(6)_p$ or even of nonchiral $U(6)\otimes U(6)$. There appear, however, factors $v/c$ or $1-v/c$, which either cancel or can be eliminated by taking the appropriate limits. The effect of mass splitting in the single-particle approximation of the commutation relations is evaluated. The relation of these results to the sum rules of Adler and Weisberger is discussed, as well as the relevance of these sum rules for the problem of the damping of semileptonic $\Delta S=1$ transitions.

## I. INTRODUCTION

THE spurion scheme of broken $U(6,6)$ symmetry[1,2] leads naturally to the nonchiral $U(6)\otimes U(6)$ group for hadrons at rest and to a collinear $U(6)_p$ group for particles moving in a given direction. On the other hand, the components of the currents involved in weak and electromagnetic interactions can generate the algebra of chiral $U(6)\otimes U(6)$.[3,4] In this article, we consider single-particle matrix elements of the current commutation relations. Restricting the intermediate states to a $U(6)$ supermultiplet, we find that in most cases the chiral $U(6)\otimes U(6)$ commutation relations give the familiar $U(6)$ results. This is the case even though the commutation relation may involve currents which are not elements of $U(6)_p$. There appear factors $v/c$ or $1-v/c$. Either these factors cancel on both sides of the equation, or they must be eliminated by taking the

* This work supported in part by the U. S. Atomic Energy Commission.

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