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Born Approximation and Large-Momentum-Transfer Processes in Potential Scattering

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Several problems of nonrelativistic scattering by analytic potentials are studied, where both the energy and the momentum transfer are large. Specifically, we find asymptotically the reflection coefficients from the one-dimensional potentials $Ve^x(1+e^x)^{-2}$, $Ve^{x/2}(1+e^x)^{-1}$, and $V\exp(-x^2)$, and the scattering amplitudes for their three-dimensional generalizations $Ve^r(1+e^r)^{-2}$, $Ve^{r/2}(1+e^r)^{-1}$, and $V\exp(-r^2)$. The results are compared with these obtained from Born approximation, and it is found that the first Born approximation gives the correct answer asymptotically in the cases $Ve^{x/2}(1+e^x)^{-1}$ and $Ve^{r/2}(1+e^r)^{-1}$, but not in the other cases. Conjectures about more general cases are also given.

1. INTRODUCTION

IN this paper, we consider problems of the following type. Suppose a plane wave is scattered nonrelativistically by a spherical potential $V(r)$, then the scattering amplitude is a function of the energy and the momentum transfer. With $V(r)$ fixed, what is the asymptotic behavior of the scattering amplitude when the energy and the momentum transfer are both large?

We shall restrict ourselves entirely to the physical region for the scattering amplitude so that the energy is necessarily large in order to get large momentum transfers. We shall also consider only those $V(r)$ which are real and integrable over the entire Euclidean space. By integrable, we mean that the integral of the absolute value of the function exists.

Since the energy is large, it is tempting to apply the Born approximation.[1] The Born approximation has been studied carefully by Hunziker,[2] who has also given rigorous bounds for the error. The bounds are of the form of a negative power of the energy. Accordingly, if the scattering amplitude and its Born approximation both approach zero rapidly in the limit of interest, they can obey Hunziker's result and yet they are not good approximations of each other. For example,

$$|\exp(-x)-\exp(-x^2)|<x^{-100}$$

for sufficiently large x, and yet asymptotically $\exp(-x)$ and $\exp(-x^2)$ are very different. The validity of the Born approximation in the sense of giving correctly the leading term of the asymptotic form is of central interest here. For forward or near-forward scattering where the momentum transfer is not large, the first Born approximation may be expected to be correct for large energies, unless there is fortuitous cancellation. We are asking, instead, whether the first Born approximation is accurate for large energies and also large momentum transfers.

No attempt is made in this paper to study this problem systematically. Instead, we merely consider a few examples to find out some of the possibilities. These examples suggest that the Born approximation is indeed valid when $V(r)$ is not analytic in $\mathbf{r}$, with the possible pathological exceptions of highly oscillatory potentials. On the other hand, when $V(r)$ is an analytic function or $\mathbf{r}$, the Born approximation often gives the wrong result.

We proceed to formulate the problem. The Schrödinger equation is

$$[\nabla^2+k^2-V(r)]\psi(\mathbf{r})=0. \tag{1.1}$$

In general, the incident wave $\psi^{\mathrm{inc}}(\mathbf{r})$ satisfies the equation without $V(r)$. Let $G(\mathbf{r}-\mathbf{r}')$ be the usual retarded Green's function, then (1.1) with suitable boundary conditions is equivalent to

$$\psi(\mathbf{r})=\psi^{\mathrm{inc}}(\mathbf{r})-\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}')V(r')\psi(\mathbf{r}'). \tag{1.2}$$

In the first Born approximation, the wave function is instead

$$\psi_1(\mathbf{r})=\psi^{\mathrm{inc}}(\mathbf{r})-\int d\mathbf{r}'G(\mathbf{r}-\mathbf{r}')V(r')\psi^{\mathrm{inc}}(\mathbf{r}'), \tag{1.3}$$

and various quantities, such as the differential cross sections, are computed from $\psi_1(\mathbf{r})$. In other words, for the various quantities, only terms linear in $V(r)$ are kept. Higher order Born approximations may be defined in a similar manner, and it is interesting to note that this Born series is always convergent for sufficiently large energy.

The motivation for this investigation comes from problems of strong interactions, and, in the following paper,[3] the methods and results developed in this paper are used to speculate about the scattering and production of strongly interacting particles. The methods can also be used directly to obtain the asymptotic behavior of the scattering amplitude in the complex angular-momentum plane.[4] A related but basically different problem has been treated by Hougardy.[5]

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[1] M. Born, Z. Physik 38, 803 (1926).
[2] W. Hunziker, Helv. Phys. Acta 36, 838 (1963).
[3] T. T. Wu, following paper, Phys. Rev. 142, 1117 (1966).
[4] H. Cheng and T. T. Wu, Phys. Rev. (to be published).
[5] R. W. Hougardy, doctoral dissertation, University of California at Los Angeles (1961). I would like to thank Professor D. S. Saxon for sending me a copy of this thesis.

2. ONE-DIMENSIONAL CASE

Three-dimensional problems being too complicated, we turn our attention to the very much simpler one-dimentional problems. After changing $\mathbf{r}$ to x, (1.1)–(1.3) remain valid. Let $\psi^{\text{inc}}(x)=e^{ikx}$, and we want to compare the reflection coefficient

$$R=\lim_{x\to-\infty}e^{ikx}[\psi(x)-\psi^{\text{inc}}(x)]$$

$$=(2ik)^{-1}\int_{-\infty}^{\infty}dxe^{ikx}V(x)\psi(x) \tag{2.1}$$

with its first Born approximation

$$R_1=(2ik)^{-1}\int_{-\infty}^{\infty}dx\, e^{2ikx}V(x). \tag{2.2}$$

We have worked out the following examples.

Example 1. If

$$V(x)=V\delta(x), \tag{2.3}$$

then

$$R=(2ik-V)^{-1}V, \tag{2.4}$$

and hence

$$R_1=(2ik)^{-1}V. \tag{2.5}$$

Indeed, more generally, the nth iteration of (1.2) gives

$$R_n=R_1{}^n, \tag{2.6}$$

and

$$R=\sum_{n=1}^{\infty}R_n, \tag{2.7}$$

provided that $|V|<2k$.

Example 2. If

$$V(x)=\begin{cases}V, & \text{for } |x|<1\\ 0, & \text{otherwise,}\end{cases} \tag{2.8}$$

then

$$R=2iVe^{-2ik}\sin 2k'/[(k-k')^2e^{2ik'}-(k+k')^2e^{-2ik'}], \tag{2.9}$$

where

$$k'=(k^2-V)^{1/2}. \tag{2.10}$$

It follows from (2.9) that

$$R_1=-\tfrac{1}{2}ik^{-2}V\sin 2k. \tag{2.11}$$

Example 3. If $V(x)$ is a special case of the Eckart potential[6]

$$V(x)=Ve^x(1+e^x)^{-2}, \tag{2.12}$$

then

$$R=-i\frac{\cos\pi\nu}{\sinh 2\pi k}\,\frac{\Gamma(-2ik+\frac{1}{2}+\nu)\Gamma(-2ik+\frac{1}{2}-\nu)}{\Gamma(-2ik+1)\Gamma(-2ik)}, \tag{2.13}$$

where

$$\nu=(\tfrac{1}{4}-V)^{1/2}. \tag{2.14}$$

[6] C. Eckart, Phys. Rev. 35, 1303 (1930).

The Born approximation gives

$$R_1=-\pi iV\,\text{csch}2\pi k. \tag{2.15}$$

On the other hand, for large k, by Stirling's formula, (2.13) gives asymptotically

$$R\sim-2i\cos\pi\nu e^{-2\pi k}. \tag{2.16}$$

This example is discussed in some detail in the next section.

Example 4. If

$$V(x)=Ve^{x/2}(1+e^x)^{-1}, \tag{2.17}$$

then, for large k, R is asymptotically

$$R\sim\pi ik^{-1}Ve^{-2\pi k}. \tag{2.18}$$

On the other hand, substitution of (2.17) into (2.2) gives

$$R_1=-\tfrac{1}{2}\pi ik^{-1}V\,\text{sech}2\pi k. \tag{2.19}$$

This example is discussed in Sec. 4.

Example 5. If $V(x)$ is the Gaussian potential

$$V(x)=V\exp(-x^2) \tag{2.20}$$

with $V>0$, then, for large k, R is roughly

$$-i\exp\{-2k[\ln(k^2/V)]^{1/2}\}, \tag{2.21}$$

while the first Born approximation gives

$$R_1=-\tfrac{1}{2}\pi^{1/2}iVk^{-1}\exp(-k^2). \tag{2.22}$$

This curious example is studied in Sec. 5.

From these five examples, together with the method of calculation used in the next three sections, we may draw the following conclusions concerning the validity of the first Born approximation for one-dimensional problems:

(a) If $V(x)$ is integrable, is not analytic on the real axis, but is analytic on each of a number of *closed* real intervals I_j such that the union of I_j is the real axis, then the first Born approximation for R is valid for large k.

(b) If $V(x)$ is analytic for $|\text{Im}x|\leqslant x_0$ and is integrable along any line parallel to the real axis in this strip, then, as $k\to\infty$,

$$R\exp(2kx_0)\to 0 \tag{2.23}$$

and

$$R_1\exp(2kx_0)\to 0. \tag{2.24}$$

(c) If $V(x)$ is analytic for $|\text{Im}x|<x_0$, has a finite number of singularities at $|\text{Im}x|=x_0$, and is integrable along any line parallel to the real axis in this strip, then, as $k\to\infty$, both R and R_1 have the exponential dependence $\exp(2kx_0)$. This is the basis of the considerations of the following paper.[3]

(d) If $V(x)$ is an entire function of x, R_1 is in general very different from R for large k.

The condition in (b) and (c) of integrability along a line parallel to the real axis is needed to exclude potentials such as

$$(x^2+1)^{-1}\sin x^2. \tag{2.25}$$

3. ECKART POTENTIAL

In this section, we study the Eckart potential[6] in some detail. Specificially, we want to be able to obtain (2.16) without using hypergeometric functions, so that a similar procedure may be applied to the potential (2.17). In the complex x plane, the Eckart potential (2.12) has double poles at $x=\pm(2n+1)\pi i$. Let

$$x=\pi i+\xi, \tag{3.1}$$

then, in the vicinity of $\xi=0$, the Schrödinger equation is approximately

$$d^2\psi/d\xi^2+(k^2+V\xi^{-2})\psi=0. \tag{3.2}$$

It follows from (3.2) that, near $\xi=0$, ψ is approximately equal to

$$\psi_0=\xi^{1/2}Z_\nu(k\xi), \tag{3.3}$$

where Z_ν is some linear combination of Bessel functions of order ν, defined in (2.14). Since the incident field ψ^{inc} is e^{ikx}, we must choose, from (3.3),

$$\psi_0(x)=(\tfrac{1}{2}\pi k\xi)^{1/2}e^{-k\pi}e^{i\pi/4}e^{i\nu\pi/2}H_\nu^{(1)}(k\xi). \tag{3.4}$$

Note that $\psi_0(x)$ is an analytic function of x except for a branch cut from πi to infinity.

The reflection coefficient R is given by (2.1). Thus the major contribution to R comes from the vicinity of $x=\pi i$, where $\psi(x)$ is approximately $\psi_0(x)$. Thus R is approximately R_0, where

$$R_0=(2ik)^{-1}\int_{-\infty}^{\infty}dx\, e^{ikx}V_0(x)\psi_0(x), \tag{3.5}$$

with

$$V_0(x)=-V\xi^{-2}. \tag{3.6}$$

It follows from (3.5) that

$$R_0=\lim_{x\to-\infty}e^{ikx}[\psi_0(x)-e^{ikx}]. \tag{3.7}$$

In deriving (3.4), we have used the asymptotic formula

$$H_\nu^{(1)}(k\xi)\sim(\tfrac{1}{2}\pi k\xi)^{-1/2}e^{-i\pi/4}e^{-i\nu\pi/2}e^{ik\xi}, \tag{3.8}$$

which hold for $k\xi\to\infty$ and $-\pi<\arg(k\xi)<2\pi$. Thus there is a Stokes' line when the argument is $-\pi$. By By (3.7), we need the asymptotic expansion on this Stokes' line. For this purpose, we make use of the identity

$$H_\nu^{(1)}(ze^{-i\pi})=2\cos\pi\nu H_\nu^{(1)}(z)+e^{-i\pi\nu}H_\nu^{(2)}(z). \tag{3.9}$$

Thus, by (3.7) and (3.9), R_0 is given by

$$R_0=\lim_{x\to-\infty}e^{ikx}(\tfrac{1}{2}\pi k\xi)^{1/2}e^{-k\pi}e^{i\pi/4}e^{i\nu\pi/2}\times 2\cos\pi\nu H_\nu^{(1)}(-k\xi). \tag{3.10}$$

Finally, the substitution of (3.8) into (3.10) gives

$$R_0=-2i\cos\pi\nu e^{-2\pi k}, \tag{3.11}$$

which is the desired answer.

4. HYPERBOLIC SECANT POTENTIAL

In this section, we apply the procedure of the preceding section to the potential (2.17), which has simple poles at $x=\pm(2n+1)\pi i$. With the variable ξ of (3.1), the comparing potential is now

$$V_0(x)=-iV\xi^{-1}, \tag{4.1}$$

which is closely related to the Coulomb potential. Therefore

$$\psi_0(x)=\mathrm{const}\,e^{ik\xi}\xi\Psi(1+\tfrac{1}{2}k^{-1}V,2;-2ik\xi), \tag{4.2}$$

where Ψ is the confluented hypergeometric function of Tricomi,[7] which is defined by

$$\Psi(a,2;x)=[\Gamma(a-1)]^{-1}\Big\{\Phi(a,2;x)\ln x+[\Gamma(a)]^{-1}x^{-1}+\sum_{r=0}^{\infty}\frac{\Gamma(a+r)}{\Gamma(a)r!(r+1)!}[\psi(a+r)-\psi(1+r)-\psi(2+r)]x^r\Big\}. \tag{4.3}$$

In (4.3), ψ is the logrithmic derivatve of the gamma function Γ, and Φ is the ordinary confluented hypergeometric function, which is regular at the origin. The constant that appears on the right-hand side of (4.2) is not defined, since it contains the well-known logarithmic phase factor of the Coulomb potential. However, if we set $(k\xi)^{V/k}$ equal to 1, which is a good approximation for large k but fixed ξ, then the constant is given by

$$-2ike^{-\pi k}. \tag{4.4}$$

The function Ψ is asymptotically, as $|x|\to\infty$,

$$\Psi(a,2;x)\sim x^{-a}, \tag{4.5}$$

provided that $|\arg x|<\tfrac{3}{2}\pi$. Thus the negative real ξ axis again coincides with the direction of a Stokes's line. By (4.3), if we always set $(k\xi)^{V/k}$ equal to 1, then the reflection coefficient R_0 is given by

$$R_0=e^{-2\pi k}\lim_{\xi\to-\infty}e^{2ik\xi}\{-2ik\xi[\Gamma(\tfrac{1}{2}k^{-1}V)]^{-1}\times(-2\pi i)\Phi(1+\tfrac{1}{2}k^{-1}V,2;-2ik\xi)-2ik\xi\Psi(1+\tfrac{1}{2}k^{-1}V,2;-2ik\xi)-1\}. \tag{4.6}$$

By these prescriptions, (4.6) reduces to

$$R_0=-2\pi ie^{-2\pi k}[\Gamma(\tfrac{1}{2}k^{-1}V)]^{-1}[\Gamma(1+\tfrac{1}{2}k^{-1}V)]^{-1}. \tag{4.7}$$

[7] See, for example, Bateman Manuscript Project, *Higher Transcendental Functions*, edited by H. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

As $k\to\infty$, this further simplifies to

$$R_0=-\pi i k^{-1} V e^{-2\pi k}. \qquad (4.8)$$

Equation (2.18) follows from (4.8) by the argument of Sec. 3.

5. GAUSSIAN POTENTIAL

In this section, we study the Gaussian potential (2.20). We treat this problem in two different ways, first by summing the Born series, and then by the WKB method.

a. Born Series

Iteration of (1.2) gives a sequence of functions defined by

$$\psi^{(n)}(x)=(2ik)^{-1}\int_{-\infty}^{\infty} dx' e^{ik|x-x'|} V(x')\psi^{(n-1)}(x') \qquad (5.1)$$

for $n\geqslant 1$ in the one-dimensional case. Here $\psi^{(0)}(x)=\psi^{\text{inc}}(x)$. Analogous to (2.1), the corresponding sequence of reflection coefficients are

$$R_n=\lim_{x\to-\infty} e^{ikx}\psi^{(n)}(x) \qquad (5.2)$$

for $n\geqslant 1$. In the series of the Gaussian potential (2.20), R_1 is given by (2.22), while R_2 is easily found to be

$$R_2=-\tfrac{1}{4}(2\pi)^{1/2}V^2k^{-2}e^{-k^2/2}\int_0^{\infty} dx\, e^{ikx}\exp(-\tfrac{1}{2}x^2). \qquad (5.3)$$

For large k, this is asymptotically

$$R_2\sim -\tfrac{1}{4}(2\pi)^{1/2} i V^2 k^{-3} e^{-k^2/2}, \qquad (5.4)$$

whose absolute value is much larger than $|R_1|$.

More generally, we attempt to compute R_n for k large. Explicitly,

$$R_n=V^n(2ik)^{-n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\cdots dx_n$$
$$\times\exp[-(x_1^2+x_2^2+\cdots+x_n^2)]\exp[ik(x_1+|x_1-x_2|$$
$$+|x_2-x_3|+\cdots+|x_{n-1}-x_n|+x_n)]. \qquad (5.5)$$

If we let

$$z_1=x_1$$

and

$$z_i=x_i-x_{i-1} \qquad (5.6)$$

for $2\leqslant i\leqslant n$, then

$$R_n=V^n(2ik)^{-n}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dz_1\cdots dz_n e^{ikF}$$
$$\times\exp(-Z^T M_n Z), \qquad (5.7)$$

where

$$F=2z_1+\sum_{i=2}^{n}(z_i+|z_i|), \qquad (5.8)$$

$$Z=\begin{bmatrix} z_1\\ Z'\end{bmatrix}, \qquad (5.9)$$

$$Z'=\begin{bmatrix} z_2\\ z_3\\ \vdots\\ z_n\end{bmatrix}, \qquad (5.10)$$

Z^T is the transpose of Z, and M_n is the $n\times n$ matrix

$$M_n=\begin{bmatrix} n & n-1 & n-2 & \cdots & 1\\ n-1 & n-1 & n-2 & \cdots & 1\\ n-2 & n-2 & n-2 & \cdots & 1\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & 1 & 1 & \cdots & 1\end{bmatrix}, \qquad (5.11)$$

In view of the form of F as given by (5.8), we integrate over z_1 to get

$$R_n=V^n(2ik)^{-n}\pi^{1/2}n^{-1/2}$$
$$\times\exp(-k^2/n)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dz_2\cdots dz_n$$
$$\times\exp ik[-2n^{-1}\sum_{j=2}^{n}(n+1-j)z_j+\sum_{j=2}^{n}(z_j+|z_j|)]$$
$$\times\exp\{-Z'^T M_{n-1} Z'+n^{-1}[\sum_{j=2}^{n}(n+1-j)z_j]^2\}. \qquad (5.12)$$

Because of the appearance of $|z_j|$, the major contribution to R_n comes from the vicinity of $z_j=0$ for all j, when k is large. Note that (5.4) is obtained from (5.3) by omitting the factor $\exp(-\tfrac{1}{2}x^2)$ in (5.3); more generally, we omit the last exponential factor on the right-hand side of (5.12). After this deletion, the integrations are trivial, and the result is

$$R_n\sim -\tfrac{1}{2}i\pi^{1/2}n^{-1/2}Vk^{-1}(\tfrac{1}{4}Vk^{-2})^{n-1}$$
$$\times\exp(-k^2/n)n^{2n}(n!)^{-2}. \qquad (5.13)$$

Equation (5.13) holds if

$$k\gg n. \qquad (5.14)$$

It remains to sum over n:

$$R=\sum_{n=1}^{\infty} R_n\sim -\tfrac{1}{2}i\pi^{1/2}Vk^{-1}\int_0^{\infty} dn\, n^{-1/2}(\tfrac{1}{4}Vk^{-2})^{n-1}$$
$$\times\exp\left(\frac{-k^2}{n}\right)n^{2n}[\Gamma(n+1)]^{-2}. \qquad (5.15)$$

The right-hand side of (5.15) can be easily evaluated by using Stirling's formula and the method of steep descent:

$$R\sim -i\exp\{-2k[\ln(4k^2/V)-2]^{1/2}\}. \qquad (5.16)$$

The point of steepest descent is at

$$n=k[\ln(4k^2/V)-2]^{-1/2}, \qquad (5.17)$$

which satisfies (5.14). Equation (2.21) results from (5.16) by neglecting 1 compared with $\ln(k^2/V)$.

b. WKB Procedure

We next try to adapt the WKB procedure to this Gaussian potential. Let $x=iy$, then the Schrödinger equation is

$$d^2\psi/dy^2-(k^2-V\exp y^2)\psi=0. \quad (5.18)$$

In the language of the WKB method, there is a turning point at $y=y_0$, where y_0 is defined to be

$$y_0=[\ln(k^2/V)]^{1/2}. \quad (5.19)$$

Accordingly, let

$$x=iy_0+\xi. \quad (5.20)$$

Near $x=iy_0$, the Gaussian potential is approximately

$$V_0(x)=k^2(1-2iy_0\xi). \quad (5.21)$$

It is a novel feature that V_0 depends on k. With V_0, the Schrödinger equation is

$$d^2\psi_0/d\xi^2+2ik^2y_0\xi\psi_0=0, \quad (5.22)$$

whose solution is

$$\psi_0=\text{const}\,\xi^{1/2}Z_{1/3}(\tfrac{1}{3}2^{3/2}e^{i\pi/4}ky_0^{1/2}\xi^{3/2}). \quad (5.23)$$

Note that, when $i\xi$ is positive, the argument of the variable is $-\frac{1}{2}\pi$. For Z, we again choose the Hanekl function of the first kind. Thus the Stokes' line appears when the argument of the variable is $-\pi$, which corresponds to

$$\arg\xi=-\tfrac{5}{6}\pi. \quad (5.24)$$

Thus we need to cross a Stokes' line here. The constant on the right-hand side of (5.23) is determined by applying the WKB method to the Schrödinger equation for the Gaussian potential. The WKB solution is

$$\psi(x)\sim[1-Vk^{-2}\exp(-x^2)]^{-1/4}e^{ikx}\times\exp\left[i\int_x^\infty dx'\{k-[k^2-V\exp(-x'^2)]^{1/2}\}\right], \quad (5.25)$$

and hence, for large k,

$$\psi_0\sim(\tfrac{1}{3}\pi k\xi)^{1/2}\exp(i5\pi/12)H_{1/3}^{(1)}(\tfrac{1}{3}2^{3/2}e^{i\pi/4}ky_0^{1/2}\xi^{3/2})\times\exp\left\{i\int_0^{iy_0}d\tau[k^2-V\exp(-\tau^2)]^{1/2}\right\}. \quad (5.26)$$

Finally, we again use (3.9) to get

$$R\sim-i\exp\left\{2i\int_0^{iy_0}d\tau[k^2-V\exp(-\tau^2)]^{1/2}\right\}=-i\exp\left\{-2\int_0^{y_0}d\tau[k^2-V\exp(\tau^2)]^{1/2}\right\}. \quad (5.27)$$

This is a much more accurate answer than (5.16).

In order to get (5.16) from (5.27), we expand the integral in powers of y_0^{-1}. The exponents in (5.16) and (5.27) agree up to the second term.

c. Attractive Case

Although the method of Sec. 5a cannot be used in the case $V<0$, that of Sec. 5b goes through with very little modification. In this case, let

$$y_0'=[\ln(-k^2/V)+\pi i]^{1/2}, \quad (5.28)$$

then we need to consider two turning points, one at iy_0' and the other at $iy_0'^*$. The reflection coefficient is approximately the sum of the contributions from each of the two turning points. Therefore, in this case,

$$R\sim-2i\,\text{Re}\exp\left\{2i\int_0^{iy_0'}d\tau[k^2-V\exp(-\tau^2)]^{1/2}\right\}. \quad (5.29)$$

One may raise the question whether there is any contribution from other zeros of $k^2-V\exp(-x^2)$, for example, the one at $i[\ln(-k^2/V)+3\pi i]^{1/2}$. These contributions are very small compared with that from iy_0', and are thus negligible.

More generally, consider the reflection from a potential $V(x)$, which is an entire function of x. If $V(x)$ is intergrable along every line parallel to the real axis, then, for large k,

$$R\sim-i\textstyle\sum_j\exp\left\{2i\int_0^{x_j}d\tau[k^2-V(\tau)]^{1/2}\right\}, \quad (5.30)$$

where x_j satisfies $V(x_j)=k^2$. Usually, only one or two j's contribute.

6. SPHERICALLY SYMMETRIC POTENTIALS

The above considerations can be immediately extended to treat some spherically symmetrical potentials. If the incident plane wave is e^{ikx}, then the scattering amplitude in this case is given by

$$f(\mathbf{n})=-(4\pi)^{-1}\int d\mathbf{r}\,V(r)\psi(\mathbf{r})\exp(-ik\mathbf{n}\cdot\mathbf{r}), \quad (6.1)$$

where $\mathbf{n}$ is a unit vector, and the final momentum is

$$\mathbf{k}_f=k\mathbf{n}, \quad (6.2)$$

while the initial momentum $\mathbf{k}_i$ is along the x axis. Let $\mathbf{K}=\mathbf{k}_f-\mathbf{k}_i$ be the momentum transfer, then the Born approximation gives

$$f_1(\mathbf{n})=-(4\pi)^{-1}\int d\mathbf{r}\,V(r)\exp(-i\mathbf{K}\cdot\mathbf{r}). \quad (6.3)$$

a. Hyperbolic Secant Potential

Consider first the spherical generalization of the potential (2.17):

$$V(r)=Ve^{r/2}(1+e^r)^{-1}. \quad (6.4)$$

The substitution into (6.3) gives, asymptotically for large $K=|\mathbf{K}|$,

$$f_1(\mathbf{n})\sim -\pi^2 VK^{-1}\exp(-\pi K). \tag{6.5}$$

In this case, Born approximation is valid, and hence

$$f(\mathbf{n})\sim -\pi^2 VK^{-1}\exp(-\pi K). \tag{6.6}$$

b. Eckart Potential

Next consider the spherical Eckart potential

$$V(r)=Ve^r(1+e^r)^{-2}. \tag{6.7}$$

Here the Born approximation gives

$$f_1(\mathbf{n})\sim -\pi^2 V\exp(-\pi K). \tag{6.8}$$

A comparison with (2.15) and (2.16) shows that the correct answer is

$$f(\mathbf{n})\sim -\pi\cos\pi\nu\exp-(\pi K). \tag{6.9}$$

for large K. Here, again, ν is defined by (2.14).

c. Gaussian Potential

As a final example, consider the three-dimensional Gaussian potential

$$V(r)=V\exp(-r^2), \tag{6.10}$$

with $V>0$. Before we can study this case, a better understanding is needed of the results (6.6) and (6.9). Equation (6.3) may be written as

$$f_1(\mathbf{n})=\tfrac{1}{2}iK^{-1}\int_{-\infty}^{\infty} rdr\, V(r)e^{iKr}, \tag{6.11}$$

provided that $V(r)$ is defined to be an even function of r. Equation (6.11) differs from (2.1) of the one-dimensional case mainly in the appearance of an additional r in the integrand. As $K\to\infty$, this r is evaluated at the position of the pole in the two examples of Secs. 6a and 6b; in other words, r is placed by $i\pi$. This accounts for the difference between (6.6) and (6.9) on the one hand and (2.18) and (2.16) on the other.

Applying this reasoning to the present case of the potential (6.10), we get, in view of (5.27),

$$f(\mathbf{n})\sim -\tfrac{1}{2}y_0\exp\left\{-\int_0^{y_0} d\tau[K^2-4V\exp(\tau^2)]^{1/2}\right\}, \tag{6.12}$$

where y_0 is defined as

$$y_0=[\ln(\tfrac{1}{4}K^2/V]^{1/2}, \tag{6.13}$$

similar to (5.19).

Note added in proof. Dr. N. Dombey kindly informed the author that there is a discussion of this Gaussian potential in L. D. Landau and E. M. Lifshitz, *Quantum Mechanics, Non-relativistic Theory* (Pergamon Press, Ltd., Oxford, England, 1965), 2nd. ed., pp. 491–2.

7. DISCUSSIONS

(a) Geometrical optics, as a theory of the asymptotic behavior of the solutions of Helmohltz equation or Maxwell's equations, is well developed for real coordinate variables.[8] The present work is basically an extension of geometrical optics into the complex planes of the coordinate variables. This is meaningful only when the potential, and hence the resulting wave function, are analytic functions of the complex coordinate variables.

(b) The present analysis can also be applied to the asymptotic case[9] where the angle of scattering is fixed, k and V are both large, while $V(r)/k$ is held fixed.

(c) Both in the limit of (b) and within the framework of Born approximation, the question may be raised which one is more favorable, one large transfer of momentum or several smaller ones. The answer is that this depends very much on the nature of the potential. For potentials not analytic in the coordinate variables, in general the former is favored. For analytic potentials, the answer depends on the nature of the singularities of the potential. In the case of Sec. 6(a), the former is more important; in Sec. 6(c), the latter becomes more important; while in Sec. 6(b), both give comparable contributions.

(d) Finally, we study the first two examples of the previous section in more detail from the point of view of complex variables. In Sec. 6(a) and Sec. 6(b), the potential has the singularity manifold

$$r^2=-\pi^2, \tag{7.1}$$

other singularity manifolds of the potential being irrelevant. Let us pose the problem of finding the value of

$$\max i\mathbf{K}\cdot\mathbf{r} \tag{7.2}$$

on the manifold (7.1). The answer is clearly πK. This is indeed the exponent in (6.6) and (6.9).

More generally, in the nonspherical cases where the singularity manifold is real in the variable $i\mathbf{r}$, the following procedure is suggested under the simplifying assumptions stated below.

(i) Let $V(\mathbf{r})$ be an analytic function of $\mathbf{r}$ in the region

$$\mathrm{Im}\mathbf{r}\in\mathcal{S} \tag{7.3}$$

and all values of $\mathrm{Re}\mathbf{r}$. $\mathcal{S}$ is required to contain the origin, and, without loss of generality, we take $\mathcal{S}$ to be the largest possible connected region. We assume that $\mathcal{S}$ is bounded, and, given $\mathrm{Im}\mathbf{r}$ in $\mathcal{S}$, $V(\mathbf{r})$ is integrable over $\mathrm{Re}\mathbf{r}$.

(ii) Over $\mathcal{S}$, let M be the least upper bound of $\mathrm{Im}\mathbf{r}\cdot\mathbf{K}$. Since $\mathrm{Im}\mathbf{r}\cdot\mathbf{K}$ is a linear form of $\mathbf{r}$, M is attained at some point $i\mathbf{r}_0$ on the boundary of $\mathcal{S}$. We assume that there is

[8] See, for example, R. W. P. King and T. T. Wu, *Scattering and Diffraction of Waves*, (Harvard University Press, Cambridge, Massachusetts, 1959).

[9] L. I. Schiff, Phys. Rev. **103**, 443 (1956); T. T. Wu, *ibid.* **108**, 466 (1957); and D. S. Saxon and L. I. Schiff, Nuovo Cimento **6**, 614 (1957).

only one such point. Since $\mathcal{S}$ is necessarily convex,[10] the Gaussian curvature g of the boundary of $\mathcal{S}$ at this point cannot be negative, and is assumed finite and nonzero.

(iii) Let $\boldsymbol{\tau}$ be the unit vector in the direction $\mathbf{r}_0$. Solve the one-dimensional problem of scattering the incident wave $\exp(\frac{1}{2}iKx)$ by the potential $V(x\boldsymbol{\tau})$. Let R be the reflection coefficient, then

$$f(\mathbf{n})=-\tfrac{1}{2}iRg^{-1/2}. \qquad (7.4)$$

Procedures of this variety are useful in the following paper.[3]

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APPENDIX

In this Appendix, we describe in some more detail the steps taken in Sec. 6. We follow the standard procedure of geometrical optics, as given for example by Born and Wolf.[11]

As is well known, for small wavelengths, the scattering from three-dimensional objects can be calculated with a knowledge of the solutions of one-dimensional problems together with the amplitude factors obtainable by geometrical considerations.[11] Consider, for example, the optical scattering from a polished sphere. For every point $\mathbf{r}$ on the sphere, we have a tangent plane with the normal $\mathbf{n}(\mathbf{r})$. For given $\mathbf{k}_i$ and $\mathbf{k}_f$, it is sufficient to consider the tangent plane with $\mathbf{n}(\mathbf{r})$ parallel to the vector $\mathbf{K}$. (More precisely, there are two such planes, but only one is relevant.) We then replace the sphere by this tangent plane. After solving this one-dimensional problem, we multiply it by the geometrical amplitude factor to obtain the desired answer for the scattering from the sphere. In this connection, a particularly illuminating example has been worked out by Kodis.[12]

We follow precisely this procedure, after noticing that the justifications for geometrical optics holds equally well for real or for complex variables. Consider first the hyperbolic secant potential (6.4). The complex singularity manifold of the potential plays the role of the surface of the polished sphere. We thus study the complex surface

$$x^2+y^2+z^2=-(2n+1)^2\pi^2. \qquad (A1)$$

Since only the nearest singularities are of importance, we take $n=0$ without loss of generality. Since this surface is a sphere, the normal $\mathbf{n}(\mathbf{r})$ is parallel to $\mathbf{r}$. We therefore look for the points on the surface (A1) such that $\mathbf{r}$ is parallel to $\mathbf{K}$. There are two such points, namely,

$$\mathbf{r}=\pm i\pi\mathbf{K}/K \qquad (A2)$$

for $n=0$. Let $\mathbf{x}'$ be the unit vector in the direction of $\mathbf{K}$, then, on the x' axis, the value of the potential (6.4) is simply

$$Ve^{x'/2}(1+e^{x'})^{-1}. \qquad (A3)$$

In the case of the polished sphere, we replace the sphere by its tangent plane; similarly, here we replace the hyperbolic secant potential by (A3). But this is precisely the one-dimensional problem studied in Sec. 4. It therefore remains only to find the geometrical amplitude factor.

This factor can be found in various ways. A particularly simple procedure is to note that the same factor appears in the Born approximation. Thus a comparison of (6.5) with (4.8) shows that the required factor is

$$-2i/\pi. \qquad (A4)$$

Equation (6.6) then follows immediately.

Precisely the same procedure is followed in Secs. 6b and c.

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[10] S. Bochner and W. T. Martin, *Several Complex Variables*, (Princeton University Press, Princeton, New Jersey, 1948).

[11] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Ltd., Oxford, England, 1964), 2nd ed., Chap. 3.

[12] R. D. Kodis, IEEE Trans. Antennas Propagation **AP-11**, 86 (1963).