Photon Splitting in a Nuclear Electric Field*

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The cross section for photon splitting in a nuclear electric field is derived, using the Feynman formulation of quantum electrodynamics. A closed form for the cross section is obtained and numerical values for interesting cases are calculated.

I. INTRODUCTION

THE purpose of this paper is the calculation of the cross section for photon splitting in a nuclear electric field. This process occurs as a result of the interaction of electromagnetic fields with the electron-positron pair field. It is similar to two other processes, the photon-photon scattering, and the elastic scattering of photons in a nuclear electric field (Delbrück scattering). In all these three processes electron-positron pairs appear in the intermediate but not in the initial and final states.

Karplus and Neuman[1] have developed a method for calculating the cross sections of such processes. Their method is based on the Feynman formulation of quantum electrodynamics,[2] and they applied it[3] to the simple case of photon-photon scattering. The calculation of the cross section for Delbrück scattering was carried out[4] only for a very special case-forward scattering.

II. THE MATRIX ELEMENT

The incoming photon is split, under the action of the electric field of the nucleus, into two photons, and the momentum difference is taken by the recoiling nucleus. We shall denote by[5] $-k^{(1)}$ the (four-dimensional) momentum vector of the incoming photon (the minus sign is used for symmetry reasons), and by $k^{(2)}$ and $k^{(3)}$ the momentum vectors of the two outgoing photons. The nucleus receives a momentum q and the law of conservation of momentum is expressed in the symmetrical form

$$k^{(1)}+k^{(2)}+k^{(3)}+q=0. \tag{1}$$

We shall further assume that the rest energy of the nucleus is much greater than the energies of the photons involved in the process. The recoil energy of the nucleus can then be neglected and the electric field of the nucleus is then regarded as static. It receives a (three-dimensional) momentum but not energy. It follows that the time component of q vanishes. Equation (1) may be decomposed into space and time components

$$\mathbf{k}^{(1)}+\mathbf{k}^{(2)}+\mathbf{k}^{(3)}+\mathbf{q}=0, \tag{2}$$

$$k^{(1)0}+k^{(2)0}+k^{(3)0}=0. \tag{3}$$

Equation (3) may also be written in the form

$$\omega_1=\omega_2+\omega_3, \tag{4}$$

where ω_1 is the energy of the incoming photon ($\omega_1=-k^{(1)0}$) and ω_2 and ω_3 are the energies of the outgoing photons ($\omega_2=k^{(2)0}$, $\omega_3=k^{(3)0}$).

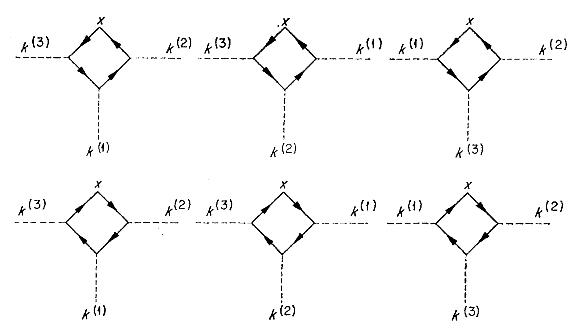

FIG. 1. Feynman diagrams for photon splitting.

The first-order splitting process is described by six Feynman diagrams (Fig. 1).

Following Feynman rules, the corresponding matrix element $(f|M|i)$ is expressed in the following form

$$(f|M|i)=2\frac{Ze^5}{(2\pi)^{15/2}}\frac{1}{(2\omega_1)^{1/2}}\frac{1}{(2\omega_2)^{1/2}}\frac{1}{(2\omega_3)^{1/2}}\frac{1}{|\mathbf{k}^{(1)}+\mathbf{k}^{(2)}+\mathbf{k}^{(3)}|^2}\epsilon^\mu(k^{(1)})\epsilon^\nu(k^{(2)})\epsilon^\lambda(k^{(3)})H_{\mu\nu0\lambda}(k^{(1)}k^{(2)}k^{(3)}), \tag{5}$$

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[1] R. Karplus and M. Neuman, Phys. Rev. **80**, 380 (1950).
[2] R. P. Feynman, Phys. Rev. **76**, 749, 769 (1949).
[3] R. Karplus and M. Neuman, Phys. Rev. **83**, 776 (1951).
[4] F. Rohrlich and R. L. Gluckstern, Phys. Rev. **86**, 1 (1952).
[5] Our notation follows that of J. M. Jauch and F. Rohrlich, in *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, New York, 1955). In this system of units $\hbar=c=1$, $e^2/4\pi\sim 1/137$, $e^2/4\pi m=r_0\approx 2.8\times 10^{-13}$ cm.

where Z is the atomic number of the nucleus and $\epsilon(k^{(1)})$, $\epsilon(k^{(2)})$, and $\epsilon(k^{(3)})$ are the polarization vectors of the three photons. It should be here noted that we are treating the nuclear electric field in the first Born approximation. The next, nonvanishing, term will be described by six-cornered closed-loop Feynman diagrams (the contribution from closed-loop diagrams with an odd number of corners vanishes by Furry theorem). The expansion parameter for the matrix element will therefore be $[(e^2/4\pi)Z]^2 \approx (Z/137)^2$.

$H_{\mu\nu0\lambda}(k^{(1)}k^{(2)}k^{(3)})$ is a sum of three terms corresponding to the first three diagrams. The contribution of each of the last three diagrams is respectively equal to that of each of the first three. This lead to the factor 2 in (5).

We have

$$H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=F_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})+F_{\nu\mu\sigma\nu}(k^{(2)}k^{(1)}k^{(3)})+F_{\lambda\nu\sigma\mu}(k^{(3)}k^{(2)}k^{(1)}), \tag{6}$$

where

$$F_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)}) = \int d^4p\, \mathrm{Tr}\left\{\gamma_\mu \frac{ip\cdot\gamma-m}{p^2+m^2-i\delta}\gamma_\nu \frac{i(p-k^{(2)})\gamma-m}{(p-k^{(2)})^2+m^2-i\delta}\gamma_\sigma \frac{i(p+k^{(1)}+k^{(3)})\gamma-m}{(p+k^{(1)}+k^{(3)})^2+m^2-i\delta}\gamma_\lambda \frac{i(p+k^{(1)})\gamma-m}{(p+k^{(1)})^2+m^2-i\delta}\right\}. \tag{7}$$

It follows from (6) and (7) that $H_{\mu\nu\sigma\lambda}$ is a component of a covariant tensor, of the fourth order, which is a function of the three vectors $k^{(i)}$ and m^2. Therefore, the general expression of $H_{\mu\nu\sigma\lambda}$ must have the form

$$\begin{aligned} H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)}) = &\sum_{i,j,l,m=1}^{3} A^{ijlm}(k^{(1)}k^{(2)}k^{(3)})k_\mu{}^{(i)}k_\nu{}^{(j)}k_\lambda{}^{(l)}k_\sigma{}^{(m)} \\ &+\sum_{i,m=1}^{3} B_1{}^{im}(k^{(1)}k^{(2)}k^{(3)})k_\mu{}^{(i)}k_\sigma{}^{(m)}g_{\nu\lambda}+\sum_{j,m=1}^{3} B_2{}^{jm}(k^{(1)}k^{(2)}k^{(3)})k_\nu{}^{(j)}k_\sigma{}^{(m)}g_{\mu\lambda}+\sum_{l,m=1}^{3} B_3{}^{lm}(k^{(1)}k^{(2)}k^{(3)})k_\lambda{}^{(l)}k_\sigma{}^{(m)}g_{\mu\nu} \\ &+\sum_{i,j=1}^{3} B_4{}^{ij}(k^{(1)}k^{(2)}k^{(3)})k_\mu{}^{(i)}k_\nu{}^{(j)}g_{\lambda\sigma}+\sum_{i,l=1}^{3} B_5{}^{il}(k^{(1)}k^{(2)}k^{(3)})k_\mu{}^{(i)}k_\lambda{}^{(l)}g_{\nu\sigma}+\sum_{j,l=1}^{3} B_6{}^{jl}(k^{(1)}k^{(2)}k^{(3)})k_\nu{}^{(j)}k_\lambda{}^{(l)}g_{\mu\sigma} \\ &+C_1(k^{(1)}k^{(2)}k^{(3)})g_{\mu\nu}g_{\lambda\sigma}+C_2(k^{(1)}k^{(2)}k^{(3)})g_{\mu\lambda}g_{\nu\sigma}+C_3(k^{(1)}k^{(2)}k^{(3)})g_{\mu\sigma}g_{\nu\lambda}, \end{aligned} \tag{8}$$

where the A^{ijlm}, $B_{1,\dots,6}{}^{ij}$ and $C_{1,2,3}$ are functions of the scalar products $k^{(1)}\cdot k^{(2)}$, $k^{(1)}\cdot k^{(3)}$, $k^{(2)}\cdot k^{(3)}$, and of m^2 only [it must be remembered that $(k^{(1)})^2=(k^{(2)})^2=(k^{(3)})^2=0$].

The terms $A^{ijlm}(k^{(1)}k^{(2)}k^{(3)})k_\mu{}^{(i)}k_\nu{}^{(j)}k_\lambda{}^{(l)}k_\sigma{}^{(m)}$ will henceforth be referred to as leading terms.

Each polarization vector $\epsilon(k)$ is perpendicular to its k, $\epsilon(k)\cdot k=0$ and its time component $\epsilon^0(k)$ is zero. It follows that terms in (8) for which at least one of the following conditions

$$\mu=0,\quad \nu=0,\quad \lambda=0 \qquad i=1,\quad j=2,\quad l=3,$$

holds do not contribute to $\epsilon^\mu(k^{(1)})\epsilon^\nu(k^{(2)})\epsilon^\lambda(k^{(3)})H_{\mu\nu0\lambda}(k^{(1)}k^{(2)}k^{(3)})$ in (5). Let $\bar{H}_{rst}(k^{(1)}k^{(2)}k^{(3)})(r, s, t=1, 2, 3)$ denote that part of (8) (with $\sigma=0$) where $i=2, 3$; $j=1, 3$; $l=1, 2$; $m=1, 2, 3$. We have

$$\begin{aligned} \bar{H}_{rst}(k^{(1)}k^{(2)}k^{(3)}) = &\sum_{\substack{i=2,3\\ j=1,3\\ l=1,2\\ m=1,2,3}} A^{ijlm}(k^{(1)}k^{(2)}k^{(3)})k_r{}^{(i)}k_s{}^{(j)}k_t{}^{(l)}k_0{}^{(m)}+\sum_{\substack{i=2,3\\ m=1,2,3}} B_1{}^{im}(k^{(1)}k^{(2)}k^{(3)})k_r{}^{(i)}k_0{}^{(m)}g_{st} \\ &+\sum_{\substack{j=1,3\\ m=1,2,3}} B_2{}^{jm}(k^{(1)}k^{(2)}k^{(3)})k_s{}^{(j)}k_0{}^{(m)}g_{rt}+\sum_{\substack{l=1,2\\ m=1,2,3}} B_3{}^{lm}(k^{(1)}k^{(2)}k^{(3)})k_t{}^{(l)}k_0{}^{(m)}g_{rs} \end{aligned} \tag{9}$$

and the following equation holds:

$$\epsilon^\mu(k^{(1)})\epsilon^\nu(k^{(2)})\epsilon^\lambda(k^{(3)})H_{\mu\nu0\lambda}(k^{(1)}k^{(2)}k^{(3)})=\epsilon^r(k^{(1)})\epsilon^s(k^{(2)})\epsilon^t(k^{(3)})\bar{H}_{rst}(k^{(1)}k^{(2)}k^{(3)}). \tag{10}$$

This equation shows that for determining $(f|M|i)$ in (5), $\bar{H}_{rst}$ can replace $H_{\mu\nu0\lambda}$.

Each of the B^{ij} appearing in (9) can be written as a linear combination of some of the A^{ijlm}. This is seen by the following consideration. $H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})$ is a gauge-invariant tensor and the identities

$$k^{(1)\mu}H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=0, \tag{11a}$$

$$k^{(2)\nu}H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=0, \tag{11b}$$

$$k^{(3)\lambda}H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=0, \tag{11c}$$

hold. Inserting (8) in (11) we obtain three tensors of the third order that vanish identically. It can then be shown that their 108 coefficients, which are combinations of the A^{ijlm}, B^{ij} and C are zero. This gives 108 equations which are not independent. Nevertheless, the desired 18 B^{ij} can be obtained as linear combinations of the A^{ijlm}. The combinations are

$$
\begin{aligned}
&B_1^{21}=-k^{(1)}\cdot k^{(3)}A^{2311}-k^{(2)}\cdot k^{(3)}A^{2321}, && B_2^{11}=-k^{(2)}\cdot k^{(3)}A^{3121}-k^{(1)}\cdot k^{(3)}A^{3111}, && B_3^{11}=-k^{(2)}\cdot k^{(3)}A^{2311}-k^{(1)}\cdot k^{(2)}A^{2111},\\
&B_1^{22}=-k^{(1)}\cdot k^{(3)}A^{2312}-k^{(2)}\cdot k^{(3)}A^{2322}, && B_2^{12}=-k^{(2)}\cdot k^{(3)}A^{3122}-k^{(1)}\cdot k^{(3)}A^{3112}, && B_3^{13}=-k^{(2)}\cdot k^{(3)}A^{2313}-k^{(1)}\cdot k^{(2)}A^{2113},\\
&B_1^{31}=-k^{(1)}\cdot k^{(2)}A^{3121}-k^{(2)}\cdot k^{(3)}A^{3321}, && B_2^{32}=-k^{(1)}\cdot k^{(2)}A^{2312}-k^{(1)}\cdot k^{(3)}A^{3312}, && B_3^{22}=-k^{(1)}\cdot k^{(3)}A^{3122}-k^{(1)}\cdot k^{(2)}A^{2122},\\
&B_1^{33}=-k^{(1)}\cdot k^{(2)}A^{3123}-k^{(2)}\cdot k^{(3)}A^{3323}, && B_2^{33}=-k^{(1)}\cdot k^{(2)}A^{2313}-k^{(1)}\cdot k^{(3)}A^{3313}, && B_3^{23}=-k^{(1)}\cdot k^{(3)}A^{3123}-k^{(1)}\cdot k^{(2)}A^{2123},\\
&B_1^{23}=k^{(1)}\cdot k^{(3)}A^{3123}-k^{(2)}\cdot k^{(3)}A^{2323}, && B_2^{13}=k^{(2)}\cdot k^{(3)}A^{2313}-k^{(1)}\cdot k^{(3)}A^{3113}, && B_3^{12}=k^{(2)}\cdot k^{(3)}A^{3122}-k^{(1)}\cdot k^{(2)}A^{2112},\\
&B_1^{32}=k^{(1)}\cdot k^{(2)}A^{2312}-k^{(2)}\cdot k^{(3)}A^{3322}, && B_2^{31}=k^{(1)}\cdot k^{(2)}A^{3121}-k^{(1)}\cdot k^{(3)}A^{3311}, && B_3^{21}=k^{(1)}\cdot k^{(3)}A^{2311}-k^{(1)}\cdot k^{(2)}A^{2121}.
\end{aligned}
\tag{12}
$$

In addition, six relations between 12 of the A^{ijlm} of (9) are obtained;

$$
\begin{aligned}
&k^{(1)}\cdot k^{(2)}A^{3113}+k^{(2)}\cdot k^{(3)}A^{3313}=0, \quad k^{(1)}\cdot k^{(2)}A^{2323}+k^{(1)}\cdot k^{(3)}A^{3323}=0, \quad k^{(2)}\cdot k^{(3)}A^{3311}+k^{(1)}\cdot k^{(2)}A^{3111}=0,\\
&k^{(1)}\cdot k^{(3)}A^{2112}+k^{(2)}\cdot k^{(3)}A^{2122}=0, \quad k^{(2)}\cdot k^{(3)}A^{2121}+k^{(1)}\cdot k^{(3)}A^{2111}=0, \quad k^{(1)}\cdot k^{(3)}A^{3322}+k^{(1)}\cdot k^{(2)}A^{2322}=0.
\end{aligned}
\tag{13}
$$

Inserting (12) and (13) in (9) we get

$$
\begin{aligned}
\bar{H}_{rst}(k^{(1)}k^{(2)}k^{(3)})=&A^{2123}(k_r^{(2)}k_s^{(1)}k_t^{(2)}k_0^{(3)}-k^{(1)}\cdot k^{(2)}k_t^{(2)}k_0^{(3)}g_{rs})+A^{2113}(k_r^{(2)}k_s^{(1)}k_t^{(1)}k_0^{(3)}-k^{(1)}\cdot k^{(2)}k_t^{(1)}k_0^{(3)}g_{rs})\\
&+A^{2321}(k_r^{(2)}k_s^{(3)}k_t^{(2)}k_0^{(1)}-k^{(2)}\cdot k^{(3)}k_r^{(2)}k_0^{(1)}g_{st})+A^{3312}(k_r^{(3)}k_s^{(3)}k_t^{(1)}k_0^{(2)}-k^{(1)}\cdot k^{(3)}k_s^{(3)}k_0^{(2)}g_{rt})\\
&+A^{3321}(k_r^{(3)}k_s^{(3)}k_t^{(2)}k_0^{(1)}-k^{(2)}\cdot k^{(3)}k_r^{(3)}k_0^{(1)}g_{st})+A^{3112}(k_r^{(3)}k_s^{(1)}k_t^{(1)}k_0^{(2)}-k^{(1)}\cdot k^{(3)}k_s^{(1)}k_0^{(2)}g_{rt})\\
&+A^{2313}(k_r^{(2)}k_s^{(3)}k_t^{(1)}k_0^{(3)}-k^{(2)}\cdot k^{(3)}k_t^{(1)}k_0^{(3)}g_{rs}+k^{(2)}\cdot k^{(3)}k_s^{(1)}k_0^{(3)}g_{rt}-k^{(1)}\cdot k^{(2)}k_s^{(3)}k_0^{(3)}g_{rt})\\
&+A^{3123}(k_r^{(3)}k_s^{(1)}k_t^{(2)}k_0^{(3)}-k^{(1)}\cdot k^{(3)}k_t^{(2)}k_0^{(3)}g_{rs}+k^{(1)}\cdot k^{(3)}k_r^{(2)}k_0^{(3)}g_{st}-k^{(1)}\cdot k^{(2)}k_r^{(3)}k_0^{(3)}g_{st})\\
&+A^{3121}(k_r^{(3)}k_s^{(1)}k_t^{(2)}k_0^{(1)}-k^{(1)}\cdot k^{(2)}k_r^{(3)}k_0^{(1)}g_{st}+k^{(1)}\cdot k^{(2)}k_s^{(3)}k_0^{(1)}g_{rt}-k^{(2)}\cdot k^{(3)}k_s^{(1)}k_0^{(1)}g_{rt})\\
&+A^{3122}(k_r^{(3)}k_s^{(1)}k_t^{(2)}k_0^{(2)}-k^{(2)}\cdot k^{(3)}k_s^{(1)}k_0^{(2)}g_{rt}+k^{(2)}\cdot k^{(3)}k_t^{(1)}k_0^{(2)}g_{rs}-k^{(1)}\cdot k^{(3)}k_t^{(2)}k_0^{(2)}g_{rs})\\
&+A^{2311}(k_r^{(2)}k_s^{(3)}k_t^{(1)}k_0^{(1)}-k^{(1)}\cdot k^{(3)}k_r^{(2)}k_0^{(1)}g_{st}+k^{(1)}\cdot k^{(3)}k_t^{(2)}k_0^{(1)}g_{rs}-k^{(2)}\cdot k^{(3)}k_t^{(1)}k_0^{(1)}g_{rs})\\
&+A^{2312}(k_r^{(2)}k_s^{(3)}k_t^{(1)}k_0^{(2)}-k^{(1)}\cdot k^{(2)}k_s^{(3)}k_0^{(2)}g_{rt}+k^{(1)}\cdot k^{(2)}k_r^{(3)}k_0^{(2)}g_{st}-k^{(1)}\cdot k^{(3)}k_r^{(2)}k_0^{(2)}g_{st})\\
&+A^{3113}\left(k_r^{(3)}k_s^{(1)}k_t^{(1)}k_0^{(3)}-\frac{k^{(1)}\cdot k^{(2)}}{k^{(2)}\cdot k^{(3)}}k_r^{(3)}k_s^{(3)}k_t^{(1)}k_0^{(3)}-k^{(1)}\cdot k^{(3)}k_s^{(1)}k_0^{(3)}g_{rt}+\frac{k^{(1)}\cdot k^{(3)}k^{(1)}\cdot k^{(2)}}{k^{(2)}\cdot k^{(3)}}k_s^{(3)}k_0^{(3)}g_{rt}\right)\\
&+A^{2323}\left(k_r^{(2)}k_s^{(3)}k_t^{(2)}k_0^{(3)}-\frac{k^{(1)}\cdot k^{(2)}}{k^{(1)}\cdot k^{(3)}}k_r^{(3)}k_s^{(3)}k_t^{(2)}k_0^{(3)}-k^{(2)}\cdot k^{(3)}k_r^{(2)}k_0^{(3)}g_{st}+\frac{k^{(2)}\cdot k^{(3)}k^{(1)}\cdot k^{(2)}}{k^{(1)}\cdot k^{(3)}}k_r^{(3)}k_0^{(3)}g_{st}\right)\\
&+A^{3311}\left(k_r^{(3)}k_s^{(3)}k_t^{(1)}k_0^{(1)}-\frac{k^{(2)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(2)}}k_r^{(3)}k_s^{(1)}k_t^{(1)}k_0^{(1)}-k^{(1)}\cdot k^{(3)}k_s^{(3)}k_0^{(1)}g_{rt}+\frac{k^{(1)}\cdot k^{(3)}k^{(2)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(2)}}k_s^{(1)}k_0^{(1)}g_{rt}\right)\\
&+A^{2112}\left(k_r^{(2)}k_s^{(1)}k_t^{(1)}k_0^{(2)}-\frac{k^{(1)}\cdot k^{(3)}}{k^{(2)}\cdot k^{(3)}}k_r^{(2)}k_s^{(1)}k_t^{(2)}k_0^{(2)}-k^{(1)}\cdot k^{(2)}k_t^{(1)}k_0^{(2)}g_{rs}+\frac{k^{(1)}\cdot k^{(2)}k^{(1)}\cdot k^{(3)}}{k^{(2)}\cdot k^{(3)}}k_t^{(2)}k_0^{(2)}g_{rs}\right)\\
&+A^{2121}\left(k_r^{(2)}k_s^{(1)}k_t^{(2)}k_0^{(1)}-\frac{k^{(2)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(3)}}k_r^{(2)}k_s^{(1)}k_t^{(1)}k_0^{(1)}-k^{(1)}\cdot k^{(2)}k_t^{(2)}k_0^{(1)}g_{rs}+\frac{k^{(1)}\cdot k^{(2)}k^{(2)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(3)}}k_t^{(1)}k_0^{(1)}g_{rs}\right)\\
&+A^{3322}\left(k_r^{(3)}k_s^{(3)}k_t^{(2)}k_0^{(2)}-\frac{k^{(1)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(2)}}k_r^{(2)}k_s^{(3)}k_t^{(2)}k_0^{(2)}-k^{(2)}\cdot k^{(3)}k_r^{(3)}k_0^{(2)}g_{st}+\frac{k^{(2)}\cdot k^{(3)}k^{(1)}\cdot k^{(3)}}{k^{(1)}\cdot k^{(2)}}k_r^{(2)}k_0^{(2)}g_{st}\right).
\end{aligned}
\tag{14}
$$

Since $H_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})$ is symmetric with respect to the simultaneous permutation of μ, ν, λ and $k^{(1)}$, $k^{(2)}$, $k^{(3)}$ the following relations between 18 A^{ijlm}, whose indices are identical to those which appear in (14), hold:

$$
\begin{aligned}
&A^{2123}(k^{(1)}k^{(2)}k^{(3)})=A^{2113}(k^{(2)}k^{(1)}k^{(3)})=A^{2321}(k^{(3)}k^{(2)}k^{(1)})=A^{3312}(k^{(1)}k^{(3)}k^{(2)})=A^{3321}(k^{(2)}k^{(3)}k^{(1)})=A^{3112}(k^{(3)}k^{(1)}k^{(2)}),\\
&A^{2313}(k^{(1)}k^{(2)}k^{(3)})=A^{3123}(k^{(2)}k^{(1)}k^{(3)})=A^{3121}(k^{(3)}k^{(2)}k^{(1)})=A^{3122}(k^{(1)}k^{(3)}k^{(2)})=A^{2311}(k^{(2)}k^{(3)}k^{(1)})=A^{2312}(k^{(3)}k^{(1)}k^{(2)}),\\
&A^{3113}(k^{(1)}k^{(2)}k^{(3)})=A^{2323}(k^{(2)}k^{(1)}k^{(3)})=A^{3311}(k^{(3)}k^{(2)}k^{(1)})=A^{2112}(k^{(1)}k^{(3)}k^{(2)})=A^{2121}(k^{(2)}k^{(3)}k^{(1)})=A^{3322}(k^{(3)}k^{(1)}k^{(2)}).
\end{aligned}
\tag{15}
$$

It is sufficient therefore to calculate $A^{2123}(k^{(1)}k^{(2)}k^{(3)})$, $A^{2313}(k^{(1)}k^{(2)}k^{(3)})$ and $A^{3113}(k^{(1)}k^{(2)}k^{(3)})$. The other 15 A^{ijlm} are obtained by appropriate permutations of $k^{(1)}$, $k^{(2)}$, $k^{(3)}$ in the three A^{ijlm}.

III. CALCULATION OF THE A^{ijlm}

In order to know the A^{ijlm} explicitly, it is necessary to perform the integration in (7). The method for performing such an integration was given by Feynman.[2] We use the identity

$$\frac{1}{a_1a_2a_3a_4}=3!\int_0^1 dx\int_0^x dy\int_0^y dz\frac{1}{[a_1z+a_2(y-z)+a_3(x-y)+a_4(1-x)]^4}. \tag{16}$$

For the a_i ($i=1, 2, 3, 4$) we take the four factors of the denominator in (7)

$$a_1=p^2+m^2-i\delta,\quad a_2=(p-k^{(2)})^2+m^2-i\delta,\quad a_3=(p+k^{(1)}+k^{(3)})^2+m^2-i\delta,\quad a_4=(p+k^{(1)})^2+m^2-i\delta, \tag{17}$$

we have

$$a_1z+a_2(y-z)+a_3(x-y)+a_4(1-x)=(p+K)^2+a, \tag{18}$$

where

$$K=k^{(1)}(1-y)-k^{(2)}(y-z)+k^{(3)}(x-y) \tag{19}$$

and

$$a=2k^{(1)}\cdot k^{(2)}(1-y)(y-z)+2k^{(1)}\cdot k^{(3)}y(x-y)+2k^{(2)}\cdot k^{(3)}(x-y)(y-z)+m^2-i\delta. \tag{20}$$

Equation (7) now has the form

$$F_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=\int d^4p\int_0^1 dx\int_0^x dy\int_0^y dz\frac{3!}{[(p+K)^2+a]^4}$$
$$\times\mathrm{Tr}\{\gamma_\mu(ip\cdot\gamma-m)\gamma_\nu[i(p-k^{(2)})\gamma-m]\gamma_\sigma[i(p+k^{(1)}+k^{(3)})\gamma-m]\gamma_\lambda[i(p+k^{(1)})\gamma-m]\}. \tag{21}$$

Making the substitution $p\rightarrow p-K$ we get

$$F_{\mu\nu\sigma\lambda}(k^{(1)}k^{(2)}k^{(3)})=\int d^4p\int_0^1 dx\int_0^x dy\int_0^y dz\frac{3!}{(p^2+a)^4}$$
$$\times\mathrm{Tr}\{\gamma_\mu[i(p-l^{(1)})\gamma-m]\gamma_\nu[i(p-l^{(2)})\gamma-m]\gamma_\sigma[i(p-l^{(3)})\gamma-m]\gamma_\lambda[i(p-l^{(4)})\gamma-m]\}, \tag{22}$$

where

$$\begin{aligned} l^{(1)}&=K &&=k^{(1)}(1-y)-k^{(2)}(y-z)+k^{(3)}(x-y),\\ l^{(2)}&=K+k^{(2)} &&=k^{(1)}(1-y)+k^{(2)}(1-y+z)+k^{(3)}(x-y),\\ l^{(3)}&=K-k^{(1)}-k^{(3)}&&=-k^{(1)}y-k^{(2)}(y-z)-k^{(3)}(1-x+y),\\ l^{(4)}&=K-k^{(1)} &&=-k^{(1)}y-k^{(2)}(y-z)+k^{(3)}(x-y). \end{aligned} \tag{23}$$

We are interested only in that part of $F_{\mu\nu\sigma\lambda}$ which contains terms which contribute to the leading terms of (8). Denoting this part by $F_{\mu\nu\sigma\lambda}'$ we have

$$F_{\mu\nu\sigma\lambda}'(k^{(1)}k^{(2)}k^{(3)})=\sum_{i,j,l,m=1}^{3}A_1{}^{ijlm}k_\mu{}^{(i)}k_\nu{}^{(j)}k_\lambda{}^{(l)}k_\sigma{}^{(m)}$$
$$=3!\int\frac{d^4p}{(p^2+a)^4}\int_0^1 dx\int_0^x dy\int_0^y dz\,\mathrm{Tr}'\{\gamma_\mu l^{(1)}\cdot\gamma\gamma_\nu l^{(2)}\cdot\gamma\gamma_\sigma l^{(3)}\cdot\gamma\gamma_\lambda l^{(4)}\cdot\gamma\}. \tag{24}$$

The symbol Tr′{ } means that in performing the trace calculations, one has to take into account only the terms which contribute to the leading terms.

Using the $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2g_{\mu\nu}$ relation, the following equation can be derived:

$$\tfrac{1}{4}\mathrm{Tr}'\{\gamma_\mu l^{(1)}\cdot\gamma\gamma_\nu l^{(2)}\cdot\gamma\gamma_\sigma l^{(3)}\cdot\gamma\gamma_\lambda l^{(4)}\cdot\gamma\}$$
$$=(l_\mu{}^{(1)}l_\nu{}^{(2)}+l_\mu{}^{(2)}l_\nu{}^{(1)})(l_\sigma{}^{(3)}l_\lambda{}^{(4)}+l_\sigma{}^{(4)}l_\lambda{}^{(3)})+(l_\mu{}^{(1)}l_\nu{}^{(3)}+l_\mu{}^{(3)}l_\nu{}^{(1)})(l_\sigma{}^{(2)}l_\lambda{}^{(4)}-l_\sigma{}^{(4)}l_\lambda{}^{(2)})$$
$$+(l_\mu{}^{(1)}l_\nu{}^{(4)}+l_\mu{}^{(4)}l_\nu{}^{(1)})(l_\sigma{}^{(2)}l_\lambda{}^{(3)}+l_\sigma{}^{(3)}l_\lambda{}^{(2)})+(l_\mu{}^{(2)}l_\nu{}^{(3)}-l_\mu{}^{(3)}l_\nu{}^{(2)})(l_\sigma{}^{(4)}l_\lambda{}^{(1)}-l_\sigma{}^{(1)}l_\lambda{}^{(4)})$$
$$+(l_\mu{}^{(4)}l_\nu{}^{(2)}-l_\mu{}^{(2)}l_\nu{}^{(4)})(l_\sigma{}^{(1)}l_\lambda{}^{(3)}+l_\sigma{}^{(3)}l_\lambda{}^{(1)})+(l_\mu{}^{(3)}l_\nu{}^{(4)}-l_\mu{}^{(4)}l_\nu{}^{(3)})(l_\sigma{}^{(1)}l_\lambda{}^{(2)}-l_\sigma{}^{(2)}l_\lambda{}^{(1)}). \tag{25}$$

The p integration gives

$$\int \frac{d^4p}{(p^2+a)^4}=\frac{i\pi^2}{6}\frac{1}{a^2}. \tag{26}$$

From (23)–(26) it follows that $A_1{}^{ijlm}(k^{(1)}k^{(2)}k^{(3)})$ is of the form

$$A_1{}^{ijlm}(k^{(1)}k^{(2)}k^{(3)})=4i\pi^2\int_0^1 dx\int_0^x dy\int_0^y dz\frac{D^{ijlm}}{a^2}, \tag{27}$$

where D^{ijlm} is a polynomial of the fourth degree in x, y, z and is the coefficient of $k_\mu{}^{(i)}k_\nu{}^{(j)}k_\lambda{}^{(l)}k_\sigma{}^{(m)}$ in the right-hand side of (25) when the l are replaced by the k according to (23).

Each of the three A^{ijlm} which are to be calculated is a sum of three $A_1{}^{ijlm}$ (contributions from three Feynman diagrams) and the following relations hold:

$$\begin{aligned}A^{2123}(k^{(1)}k^{(2)}k^{(3)})&=A_1{}^{2123}(k^{(1)}k^{(2)}k^{(3)})+A_1{}^{2113}(k^{(2)}k^{(1)}k^{(3)})+A_1{}^{2321}(k^{(3)}k^{(2)}k^{(1)}),\\ A^{2313}(k^{(1)}k^{(2)}k^{(3)})&=A_1{}^{2313}(k^{(1)}k^{(2)}k^{(3)})+A_1{}^{3123}(k^{(2)}k^{(1)}k^{(3)})+A_1{}^{3121}(k^{(3)}k^{(2)}k^{(1)}),\\ A^{3113}(k^{(1)}k^{(2)}k^{(3)})&=A_1{}^{3113}(k^{(1)}k^{(2)}k^{(3)})+A_1{}^{2323}(k^{(2)}k^{(1)}k^{(3)})+A_1{}^{3311}(k^{(3)}k^{(2)}k^{(1)}).\end{aligned} \tag{28}$$

It is necessary therefore to calculate 9 $A_1{}^{ijlm}(k^{(1)}k^{(2)}k^{(3)})$ and to perform the necessary permutations of the k in them. The corresponding 9 D^{ijlm} are derived from the right-hand side of (25);

$$\begin{aligned}D^{2123}&=2(y-z)[2(x-y)(y-z)(1-2y)-(2y-2z-1)(1-x)],\\ D^{2113}&=4y(y-z)(1-y)(2x-2y-1)-2y(x-z)+2(y-z),\\ D^{2321}&=2(y-z)[2(x-y)(y-z)(1-2y)+x-1],\\ D^{2313}&=2(x-y)(y-z)[4y(x-y)+2y-1],\\ D^{3123}&=-2(x-y)[2(y-z)(1-y)(2x-2y-1)-y],\\ D^{3121}&=2(y-z)(1-y)[4y(x-y)-2(x-y)+1],\\ D^{3113}&=-4y(x-y)(1-y)(2x-2y-1),\\ D^{2323}&=4(x-y)(y-z)^2(2x-2y-1),\\ D^{3311}&=-4y(x-y)^2(1-2y).\end{aligned} \tag{29}$$

In order to integrate it is convenient to introduce the three parameters

$$\begin{aligned}\alpha&=\frac{2}{m^2}k^{(2)}\cdot k^{(3)}=\frac{2}{m^2}(-\omega_2\omega_3+\mathbf{k}^{(2)}\cdot\mathbf{k}^{(3)}),\\ \beta&=\frac{2}{m^2}k^{(1)}\cdot k^{(2)}=\frac{2}{m^2}(\omega_1\omega_2+\mathbf{k}^{(1)}\cdot\mathbf{k}^{(2)}),\\ \gamma&=\frac{2}{m^2}k^{(1)}\cdot k^{(3)}=\frac{2}{m^2}(\omega_1\omega_3+\mathbf{k}^{(1)}\cdot\mathbf{k}^{(3)}).\end{aligned} \tag{30}$$

Because the incoming photon is $-k^{(1)}$ and the outgoing photons are $k^{(2)}$ and $k^{(3)}$, $\alpha\leqslant 0$ and β, $\gamma\geqslant 0$. It can be shown also that $\alpha+\beta+\gamma\geqslant 0$.

$$\begin{aligned}|\mathbf{k}^{(1)}+\mathbf{k}^{(2)}+\mathbf{k}^{(3)}|^2&=|\mathbf{k}^{(1)}|^2+|\mathbf{k}^{(2)}|^2+|\mathbf{k}^{(3)}|^2+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(2)}+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(3)}+2\mathbf{k}^{(2)}\cdot\mathbf{k}^{(3)}\\ &=(k_0{}^{(1)})^2+(k_0{}^{(2)})^2+(k_0{}^{(3)})^2+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(2)}+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(3)}+2\mathbf{k}^{(2)}\cdot\mathbf{k}^{(3)}\\ &=(k_0{}^{(1)}+k_0{}^{(2)}+k_0{}^{(3)})^2-2k_0{}^{(1)}k_0{}^{(2)}-2k_0{}^{(1)}k_0{}^{(3)}-2k_0{}^{(2)}k_0{}^{(3)}+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(2)}+2\mathbf{k}^{(1)}\cdot\mathbf{k}^{(3)}+2\mathbf{k}^{(2)}\cdot\mathbf{k}^{(3)}\\ &=m^2(\alpha+\beta+\gamma).\end{aligned} \tag{31}$$

The last equality holds because $k_0{}^{(1)}+k_0{}^{(2)}+k_0{}^{(3)}=-\omega_1+\omega_2+\omega_3=0$.

The results of the integration are written with the help of rational polynomials in α, β, γ and three transcendental functions $A(u)$, $B(u)$, $S(u,v,w)$ (see Appendix I).

The expressions for the three A^{ijlm} are given in Appendix II. The other 15 A^{ijlm}, which appear in (14), can be found, according to (15), by appropriate permutations of $k^{(1)}$, $k^{(2)}$, and $k^{(3)}$. The function $A^{2113}(k^{(1)}k^{(2)}k^{(3)})$, for example, is derived from $A^{2123}(k^{(1)}k^{(2)}k^{(3)})$ by the permutation

$$k^{(2)}\rightarrow k^{(1)},\quad k^{(1)}\rightarrow k^{(2)},\quad k^{(3)}\rightarrow k^{(3)},$$

which is equivalent to the permutation

$$\alpha\rightarrow\gamma,\quad \beta\rightarrow\beta,\quad \gamma\rightarrow\alpha.$$

IV. THE CROSS SECTION

The differential cross section $d\sigma$ for splitting is given by[6]

$$d\sigma=(2\pi)^2\int|(f|M|i)|^2\delta(E_f-E_i)\omega_2^2d\omega_2\omega_3^2d\omega_3d\Omega_2d\Omega_3. \tag{32}$$

E_i and E_f are, respectively, the initial and final energies of the system. The integration in (32) is to be carried out over all the states of the system upon which $d\sigma$ does not depend. If, for example, one is interested in the cross section for splitting of a photon with polarization direction $\epsilon(k^{(1)})$, into two photons, with polarization directions $\epsilon(k^{(2)})$ and $\epsilon(k^{(3)})$, which emerge into angle elements $d\Omega_2$ and $d\Omega_3$, we obtain from (32)

$$\frac{d^3\sigma}{d\Omega_2d\Omega_3d\omega_2}=(2\pi)^2|(f|M|i)|^2\omega_2^2\omega_3^2, \tag{33}$$

where $\omega_3=\omega_1-\omega_2$. Inserting (5) into (33) and using (10) and (31), we have

$$\frac{d^3\sigma}{d\Omega_2d\Omega_3d\omega_2}=\frac{1}{(2\pi)^4}Z^2\left(\frac{e^2}{4\pi}\right)^3\left(\frac{e^2}{4\pi m}\right)^2\frac{T}{(\alpha+\beta+\gamma)^2}\frac{\omega_2\omega_3d\omega_2}{m^2\omega_1}, \tag{34}$$

where

$$T=(1/\pi^4)|\epsilon^r(k^{(1)})\epsilon^s(k^{(2)})\epsilon^t(k^{(3)})H_{rst}(k^{(1)}k^{(2)}k^{(3)})|^2. \tag{35}$$

If one is interested in the cross section for splitting of unpolarized photons into photons of any directions of polarization, it is necessary to make a polarization sum. T is then replaced by $\bar{T}$, where

$$\bar{T}=\tfrac{1}{2}\sum_{\text{pol}}T. \tag{36}$$

The result of this polarization sum is

$$\bar{T}=-(8/\alpha\beta\gamma)\{|x_1|^2+|x_2|^2+|x_3|^2+|u|^2+t(|y_1|^2+|y_2|^2+|y_3|^2+|v|^2)\}. \tag{37}$$

The different functions in (37) are defined in Appendix III.

It should be noted that $\bar{T}$ is symmetric under each of the following three operations:

(1) interchange at the same time of α and γ and of ω_2 and $-\omega_1$;
(2) interchange at the same time of α and β and of ω_3 and $-\omega_1$;
(3) interchange at the same time of β and γ and of ω_2 and ω_3.

For small incoming photon energy, $\omega_1/m\ll 1$, we can use the expansion formulas of Appendices I and III. The resulting $\bar{T}$ is then

$$\bar{T}=\left(\frac{11}{90}\right)^2\left\{2(\alpha+\beta+\gamma)\left[\alpha^2\left(\frac{\omega_1}{m}\right)^2+\beta^2\left(\frac{\omega_3}{m}\right)^2+\gamma^2\left(\frac{\omega_2}{m}\right)^2+\tfrac{1}{2}\alpha\beta\gamma\right]-(\alpha\beta+\alpha\gamma+\beta\gamma)^2\right\}$$
$$+\left(\frac{1}{15}\right)^2\left\{2(\alpha+\beta+\gamma)\left[\alpha^2\left(\frac{\omega_1}{m}\right)^2+\beta^2\left(\frac{\omega_3}{m}\right)^2+\gamma^2\left(\frac{\omega_2}{m}\right)^2+\alpha\gamma\frac{\omega_1\omega_2}{m^2}+\alpha\beta\frac{\omega_1\omega_3}{m^2}-\beta\gamma\frac{\omega_2\omega_3}{m^2}\right]-\tfrac{1}{2}(\alpha\beta+\alpha\gamma+\beta\gamma)^2\right\}. \tag{38}$$

The cross section was computed, for some experimentally interesting cases, on the 1604 Control Data Computer. The results are given in Figs. 2, 3, and 4. The scatterer was taken to be lead ($Z=82$). It should be noted that the Born approximation expansion parameter here is $(Z/137)^2=(82/137)^2\approx 0.36$.

In Figs. 2 and 3, $\omega_2=\omega_3=\frac{1}{2}\omega_1$ and both of the outgoing photons are perpendicular to the incoming one. In Fig. 2

[6] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley Publishing Company, New York, 1955), pp. 163–167.

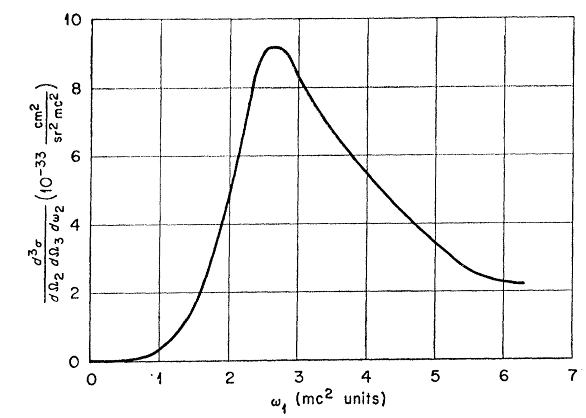

FIG. 2. Differential cross section for photons on lead (cm^2/sr^2mc^2) as a function of incoming photon energy. Both outgoing photons are perpendicular to the incoming photon and the angle between them is 180 deg. Both outgoing photons have the same energy $\omega_2=\omega_3=\frac{1}{2}\omega_1$.

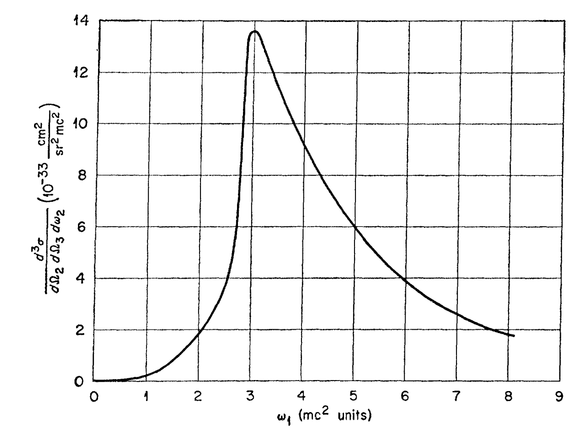

FIG. 3. Differential cross section for photons on lead (cm^2/sr^2mc^2) as a function of incoming photon energy. All three photons are mutually orthogonal. Both outgoing photons have the same energy $\omega_2=\omega_3=\frac{1}{2}\omega_1$.

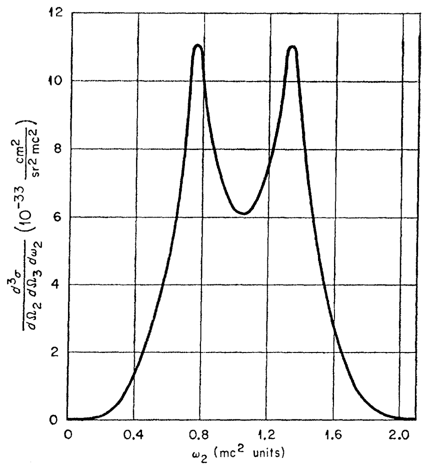

FIG. 4. Differential cross section for photons on lead (cm^2/sr^2mc^2) as a function of the energy of one of the outgoing photons. The incoming photon energy is $2.1mc^2$. Both outgoing photons are perpendicular to the incoming photon and the angle between them is 180 deg.

the angle between ω_2 and ω_3 is 180 deg. In Fig. 3 this angle is 90 deg. In both figures the cross section is shown as a function of ω_1.

In Fig. 4, $\omega_1=2.1m$, the angle between ω_2 and ω_3 is 180 deg, and the angles between ω_1 and ω_2 and between ω_1 and ω_3 are 90 deg. The cross section is shown as a function of ω_2.

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APPENDIX I

The three transcendental functions which appear in the calculations are

$$\begin{aligned} A(u)=\int_0^1 \ln[1-i\delta+ux(1-x)]dx &= 2\left\{\left(\frac{u+4}{u}\right)^{1/2}\text{arc sinh}\left(\frac{u}{4}\right)^{1/2}-1\right\}, && 0\leqslant u\\ &=2\left\{\left(-\frac{u+4}{u}\right)^{1/2}\text{arc sin}\left(-\frac{u}{4}\right)^{1/2}-1\right\}, && -4\leqslant u\leqslant 0\\ &=2\left\{\left(\frac{u+4}{u}\right)^{1/2}\text{arc cosh}\left(-\frac{u}{4}\right)^{1/2}-1-\tfrac{1}{2}i\pi\left(\frac{u+4}{u}\right)^{1/2}\right\}, && u\leqslant -4 \end{aligned} \tag{A1}$$

$$\begin{aligned} B(u)=\int_0^1 \frac{1}{x(1-x)}\ln[1-i\delta+4x(1-x)]dx &= 4\left\{\text{arc sinh}\left(\frac{u}{4}\right)^{1/2}\right\}^2, && 0\leqslant u\\ &=-4\left\{\text{arc sin}\left(-\frac{u}{4}\right)^{1/2}\right\}^2, && -4\leqslant u\leqslant 0\\ &=4\left\{\text{arc cosh}\left(-\frac{u}{4}\right)^{1/2}\right\}^2-\pi^2-4i\pi\,\text{arc cosh}\left(-\frac{u}{4}\right)^{1/2}, && u\leqslant -4 \end{aligned} \tag{A2}$$

$$S(u,v,w)=\frac{1}{2}\int_0^1 \frac{vw}{u-vwx(1-x)}\ln\frac{[1-i\delta+vx(1-x)][1-i\delta+wx(1-x)]}{1+(u+v+w)x(1-x)}dx. \tag{A3}$$

The last function $S(u,v,w)$ is symmetric in v and w. u, v, w is a permutation of α, β, γ. When $u<0$ $(u=\alpha)$, $S(u,v,w)$ is real and can be written in terms of the dilogarithmic function[7] $R(x)$.

$$R(x)=\int_0^x \frac{1}{t}\ln(1-t)dt. \tag{A4}$$

We give here the explicit form of $S(u,v,w)$ for the case $u<0$, $u+v>0$, $u+w>0$

$$\begin{aligned}\left(1-\frac{4u}{vw}\right)^{1/2} & S(u,v,w)\\
=&R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}+(1+4/v)^{1/2}}\right)-R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}+(1+4/v)^{1/2}}\right)-R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}-(1+4/v)^{1/2}}\right)\\
&+R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}-(1+4/v)^{1/2}}\right)+R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}+(1+4/w)^{1/2}}\right)-R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}+(1+4/w)^{1/2}}\right)\\
&-R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}-(1+4/w)^{1/2}}\right)+R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}-(1+4/w)^{1/2}}\right)\\
&-R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}+[1+4/(u+v+w)]^{1/2}}\right)+R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}+[1+4/(u+v+w)]^{1/2}}\right)\\
&+R\left(\frac{(1-4u/vw)^{1/2}+1}{(1-4u/vw)^{1/2}-[1+4/(u+v+w)]^{1/2}}\right)-R\left(\frac{(1-4u/vw)^{1/2}-1}{(1-4u/vw)^{1/2}-[1+4/(u+v+w)]^{1/2}}\right).\end{aligned} \tag{A5}$$

When $u>0$, $S(u,v,w)$ has an imaginary part if $\alpha<-4$. α is in this case either v or w. Let $\alpha=v$. The imaginary part is then

$$\mathrm{Im}\{S(u,\alpha,w)\}=\frac{\pi}{p}\ln\frac{p+g}{p-g},\quad p=\left(1-\frac{4u}{\alpha w}\right)^{1/2},\quad g=\left(1+\frac{4}{\alpha}\right)^{1/2},\quad \alpha<-4. \tag{A6}$$

The real part can again be written in terms of the dilogarithmic function $R(x)$.

For small values of the arguments, the three functions A, B, and S have the following expansions:

$$A(u)=\frac{1}{6}u-\frac{1}{60}u^2+\frac{1}{420}u^3-\frac{1}{2520}u^4,\quad |u|\ll 1 \tag{A7}$$

$$B(u)=u-\frac{1}{12}u^2+\frac{1}{90}u^3-\frac{1}{560}u^4,\quad |u|\ll 1 \tag{A8}$$

$$S(u,v,w)=-\tfrac{1}{2}vw\left(\frac{1}{6}-\frac{1}{60}u-\frac{1}{30}(v+w)+\frac{1}{420}u^2+\frac{1}{140}(v^2+w^2)+\frac{1}{140}u(v+w)+\frac{3}{280}vw\right),$$
$$|u|\ll 1,\quad |v|\ll 1,\quad |w|\ll 1. \tag{A9}$$

APPENDIX II

The three A^{ijlm}, from which all 18 A^{ijlm} which appear in (14) can be derived by appropriate permutations, are given here.

[7] K. Mitchell, Phil. Mag. 40, 351 (1949).

$$\frac{m^4}{4i\pi^2}A^{2123}(k^{(1)}k^{(2)}k^{(3)})$$
$$=\frac{4(\beta+\gamma)}{3\beta^2\gamma}[A(\alpha)-A(\alpha+\beta+\gamma)]-\frac{4}{3\alpha}\left(\frac{2}{\alpha}-\frac{1}{\gamma}\right)[A(\beta)-A(\alpha+\beta+\gamma)]-\frac{4\gamma}{3\alpha\beta}\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)[A(\gamma)-A(\alpha+\beta+\gamma)]$$
$$-\left(\frac{2\gamma}{3\beta^3}+\frac{1}{\beta^2}+\frac{2}{3\gamma^2}-\frac{2}{\alpha\beta^2}\right)B(\alpha)+\left(\frac{4\gamma}{3\alpha^3}+\frac{1}{\alpha^2}-\frac{2}{3\gamma^2}+\frac{2}{\alpha\beta^2}\right)B(\beta)+\left(\frac{4\gamma}{3\alpha^3}+\frac{1}{\alpha^2}-\frac{2\gamma}{3\beta^3}-\frac{1}{\beta^2}+\frac{2}{\alpha\beta^2}\right)B(\gamma)$$
$$-\left(\frac{4\gamma}{3\alpha^3}+\frac{1}{\alpha^2}-\frac{2}{3\gamma^2}-\frac{2\gamma}{3\beta^3}-\frac{1}{\beta^2}+\frac{2}{\alpha\beta^2}\right)B(\alpha+\beta+\gamma)-\frac{4}{3\alpha\beta}+\left(\frac{8\gamma}{3\alpha^3}+\frac{2}{\alpha^2}-\frac{4}{\alpha\beta\gamma}-\frac{16}{3\alpha^2\beta}+\frac{4}{\alpha\beta^2}+\frac{4}{\beta^2\gamma}-\frac{16}{3\alpha\beta^2\gamma}\right)S(\alpha,\beta,\gamma)$$
$$-\left(\frac{4\gamma}{3\beta^3}+\frac{2}{\beta^2}-\frac{20}{3\alpha\beta^2}-\frac{4}{\alpha\beta\gamma}+\frac{4}{\alpha^2\gamma}+\frac{16}{3\alpha^2\beta\gamma}\right)S(\beta,\alpha,\gamma)-\left(\frac{4}{3\gamma^2}-\frac{8}{3\alpha\beta\gamma}+\frac{4}{\alpha^2\beta}+\frac{16}{3\alpha^2\beta^2}\right)S(\gamma,\alpha,\beta).\quad\text{(A10)}$$

$$\frac{m^4}{4i\pi^2}A^{2313}(k^{(1)}k^{(2)}k^{(3)})$$
$$=\frac{2}{3\gamma}\left(\frac{8}{\gamma}-\frac{1}{\beta}\right)[A(\alpha)-A(\alpha+\beta+\gamma)]-\frac{2}{\alpha}\left(\frac{4\beta}{3\alpha\gamma}-\frac{8\beta}{3\gamma^2}-\frac{\alpha-\gamma}{\gamma(\alpha+\gamma)}+\frac{2\beta}{(\alpha+\gamma)^2}\right)[A(\beta)-A(\alpha+\beta+\gamma)]$$
$$-\frac{2}{3\alpha}\left(\frac{4}{\alpha}+\frac{1}{\beta}\right)[A(\gamma)-A(\alpha+\beta+\gamma)]-\left(\frac{8\beta}{3\gamma^3}+\frac{1}{\gamma^2}-\frac{1}{3\beta^2}-\frac{4}{\alpha\gamma^2}\right)B(\alpha)+\left(\frac{4\beta}{3\alpha^3}-\frac{8\beta}{3\gamma^3}-\frac{1}{\gamma^2}+\frac{4(\alpha+2\gamma)}{\gamma^2(\alpha+\gamma)^2}\right)B(\beta)$$
$$+\left(\frac{4\beta}{3\alpha^3}+\frac{1}{3\beta^2}\right)B(\gamma)-\left(\frac{4\beta}{3\alpha^3}-\frac{8\beta}{3\gamma^3}-\frac{1}{\gamma^2}+\frac{1}{3\beta^2}+\frac{4(\alpha+2\gamma)}{\gamma^2(\alpha+\gamma)^2}\right)B(\alpha+\beta+\gamma)+\frac{8}{3\alpha\gamma}-\frac{4}{\alpha(\alpha+\gamma)}+\frac{8}{3\alpha}\left(\frac{\beta}{\alpha^2}-\frac{2}{\alpha\gamma}-\frac{2}{\beta\gamma^2}\right)S(\alpha,\beta,\gamma)$$
$$+\left(\frac{2}{3\beta^2}-\frac{4}{3\alpha\beta\gamma}+\frac{4}{\alpha^2\gamma}-\frac{16}{3\alpha^2\gamma^2}\right)S(\beta,\alpha,\gamma)-\left(\frac{16\beta}{3\gamma^3}+\frac{2}{\gamma^2}-\frac{56}{3\alpha\gamma^2}-\frac{4}{\alpha\beta\gamma}-\frac{4}{\alpha^2\beta}+\frac{16}{3\alpha^2\beta\gamma}\right)S(\gamma,\alpha,\beta);\quad\text{(A11)}$$

$$\frac{m^4}{4i\pi^2}A^{3113}(k^{(1)}k^{(2)}k^{(3)})$$
$$=\frac{8\alpha}{\gamma^3}[A(\alpha)-A(\alpha+\beta+\gamma)]+\frac{4\alpha}{\gamma^2(\alpha+\gamma)}\left(1+\frac{2\beta}{\gamma}+\frac{\beta}{\alpha+\gamma}\right)[A(\beta)-A(\alpha+\beta+\gamma)]-\frac{2}{\gamma^2}\left(\frac{2\alpha\beta}{\gamma^2}+\frac{\alpha}{\gamma}-\frac{4}{\gamma}-\frac{1}{\beta}\right)B(\alpha)$$
$$-\left(\frac{4\alpha\beta}{\gamma^4}+\frac{2\alpha}{\gamma^3}-\frac{8}{\gamma^3}-\frac{2}{\beta\gamma^2}-\frac{4}{\alpha(\alpha+\gamma)^2}\right)[B(\beta)-B(\alpha+\beta+\gamma)]+\frac{2}{\gamma^2}\left(\frac{2}{\alpha}+\frac{1}{\beta}\right)B(\gamma)$$
$$+\frac{4\alpha}{\gamma^2(\alpha+\gamma)}+\frac{4}{\gamma^2}\left(\frac{2}{\alpha}+\frac{1}{\beta}-\frac{4}{\beta\gamma}\right)S(\alpha,\beta,\gamma)+\frac{4}{\gamma^2}\left(\frac{1}{\beta}-\frac{4}{\alpha\gamma}\right)S(\beta,\alpha,\gamma)-\frac{4}{\gamma^2}\left(\frac{2\alpha\beta}{\gamma^2}+\frac{\alpha}{\gamma}-\frac{8}{\gamma}-\frac{3}{\beta}+\frac{4}{\alpha\beta}\right)S(\gamma,\alpha,\beta).\quad\text{(A12)}$$

APPENDIX III

The functions which appear in Eq. (37) for $\bar{T}$ are listed here;

$$u=\frac{m^4}{16i\pi^2}\Big\{\alpha[-\alpha\beta(A^{2123}+A^{2321})+\alpha\gamma(A^{3312}+A^{3321})+2\beta^2A^{2121}-2\gamma^2A^{3311}+\beta\gamma(2A^{3121}-2A^{2311}+A^{3112}-A^{2113})]\frac{\omega_1}{m}$$
$$+\beta[-\alpha\beta(A^{2321}+A^{2123})+\beta\gamma(A^{3112}+A^{2113})+2\alpha^2A^{2323}-2\gamma^2A^{3113}+\alpha\gamma(2A^{2313}-2A^{3123}+A^{3312}-A^{3321})]\frac{\omega_3}{m}$$
$$+\gamma[-\beta\gamma(A^{2113}+A^{3112})+\alpha\gamma(A^{3321}+A^{3312})+2\beta^2A^{2112}-2\alpha^2A^{3322}+\alpha\beta(2A^{2312}-2A^{3122}+A^{2321}-A^{2123})]\frac{\omega_2}{m}\Big\}$$
$$=\left[\frac{\alpha(\beta-\gamma)}{\beta\gamma}\frac{\omega_1}{m}-\frac{\gamma(\alpha-\beta)}{\alpha\beta}\frac{\omega_2}{m}+\frac{\beta(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}\right][B(\alpha)+B(\beta)+B(\gamma)-B(\alpha+\beta+\gamma)]$$
$$+\frac{2}{\alpha}\left[(\alpha+\gamma)\frac{\omega_2}{m}-(\alpha+\beta)\frac{\omega_3}{m}\right]S(\alpha,\beta,\gamma)-\frac{2}{\beta}\left[(\alpha+\beta)\frac{\omega_1}{m}+(\beta+\gamma)\frac{\omega_2}{m}\right]S(\beta,\alpha,\gamma)+\frac{2}{\gamma}\left[(\alpha+\gamma)\frac{\omega_1}{m}+(\beta+\gamma)\frac{\omega_3}{m}\right]S(\gamma,\alpha,\beta).\quad\text{(A13)}$$

$$v=-\frac{m^4}{16i\pi^2}[\alpha\beta(A^{2123}+A^{2321})+\alpha\gamma(A^{3312}+A^{3321})+\beta\gamma(A^{2113}+A^{3112})]$$
$$=2-\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)[B(\alpha)+B(\beta)+B(\gamma)-B(\alpha+\beta+\gamma)]$$
$$-\frac{2}{\alpha}\left(1-\frac{4\alpha}{\beta\gamma}\right)S(\alpha,\beta,\gamma)-\frac{2}{\beta}\left(1-\frac{4\beta}{\alpha\gamma}\right)S(\beta,\alpha,\gamma)-\frac{2}{\gamma}\left(1-\frac{4\gamma}{\alpha\beta}\right)S(\gamma,\alpha,\beta). \quad \text{(A14)}$$

$$y_1=-\frac{m^4}{16i\pi^2}\beta[\alpha(A^{2123}+A^{3122}+A^{2312})+\gamma(A^{2113}+A^{2311}+A^{3121})]$$
$$=-\alpha\left(\frac{2}{\beta}-\frac{1}{\beta+\gamma}\right)[A(\alpha)-A(\alpha+\beta+\gamma)]-\gamma\left(\frac{2}{\beta}-\frac{1}{\alpha+\beta}\right)[A(\gamma)-A(\alpha+\beta+\gamma)]-\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)B(\beta)$$
$$+\left(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha+\gamma}{2\beta}-\frac{1}{\gamma}-\frac{3}{\beta}+\frac{1}{\beta+\gamma}\right)B(\alpha)+\left(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha+\gamma}{2\beta}-\frac{1}{\alpha}-\frac{3}{\beta}+\frac{1}{\alpha+\beta}\right)B(\gamma)$$
$$-\left(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha+\gamma}{2\beta}-\frac{\alpha+\gamma}{\alpha\gamma}-\frac{3}{\beta}+\frac{1}{\alpha+\beta}+\frac{1}{\beta+\gamma}\right)B(\alpha+\beta+\gamma)-2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{\alpha}{\beta\gamma}-\frac{4}{\beta\gamma}\right)S(\alpha,\beta,\gamma)$$
$$-2\left(\frac{1}{\gamma}+\frac{1}{\beta}+\frac{\gamma}{\alpha\beta}-\frac{4}{\alpha\beta}\right)S(\gamma,\alpha,\beta)+2\left(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha+\gamma}{2\beta}-\frac{\alpha+\gamma}{\alpha\gamma}-\frac{5}{\beta}+\frac{4}{\alpha\gamma}\right)S(\beta,\alpha,\gamma); \quad \text{(A15)}$$

$$y_2=-\frac{m^4}{16i\pi^2}\alpha[\beta(A^{2321}+A^{2312}+A^{3122})+\gamma(A^{3321}+A^{3123}+A^{2313})]$$
$$=-\beta\left(\frac{2}{\alpha}-\frac{1}{\alpha+\gamma}\right)[A(\beta)-A(\alpha+\beta+\gamma)]-\gamma\left(\frac{2}{\alpha}-\frac{1}{\alpha+\beta}\right)[A(\gamma)-A(\alpha+\beta+\gamma)]-\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)B(\alpha)$$
$$+\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta+\gamma}{2\alpha}-\frac{1}{\gamma}-\frac{3}{\alpha}+\frac{1}{\alpha+\gamma}\right)B(\beta)+\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta+\gamma}{2\alpha}-\frac{1}{\beta}-\frac{3}{\alpha}+\frac{1}{\alpha+\beta}\right)B(\gamma)$$
$$-\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta+\gamma}{2\alpha}-\frac{\beta+\gamma}{\beta\gamma}-\frac{3}{\alpha}+\frac{1}{\alpha+\beta}+\frac{1}{\alpha+\gamma}\right)B(\alpha+\beta+\gamma)-2\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{\beta}{\alpha\gamma}-\frac{4}{\alpha\gamma}\right)S(\beta,\alpha,\gamma)$$
$$-2\left(\frac{1}{\gamma}+\frac{1}{\alpha}+\frac{\gamma}{\alpha\beta}-\frac{4}{\alpha\beta}\right)S(\gamma,\alpha,\beta)+2\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta+\gamma}{2\alpha}-\frac{\beta+\gamma}{\beta\gamma}-\frac{5}{\alpha}+\frac{4}{\beta\gamma}\right)S(\alpha,\beta,\gamma); \quad \text{(A16)}$$

$$y_3=-\frac{m^4}{16i\pi^2}\gamma[\alpha(A^{3312}+A^{2313}+A^{3123})+\beta(A^{3112}+A^{3121}+A^{2311})]$$
$$=-\alpha\left(\frac{2}{\gamma}-\frac{1}{\beta+\gamma}\right)[A(\alpha)-A(\alpha+\beta+\gamma)]-\beta\left(\frac{2}{\gamma}-\frac{1}{\alpha+\gamma}\right)[A(\beta)-A(\alpha+\beta+\gamma)]-\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)B(\gamma)$$
$$+\left(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha+\beta}{2\gamma}-\frac{1}{\beta}-\frac{3}{\gamma}+\frac{1}{\beta+\gamma}\right)B(\alpha)+\left(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha+\beta}{2\gamma}-\frac{1}{\alpha}-\frac{3}{\gamma}+\frac{1}{\alpha+\gamma}\right)B(\beta)$$
$$-\left(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha+\beta}{2\gamma}-\frac{\alpha+\beta}{\alpha\beta}-\frac{3}{\gamma}+\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\gamma}\right)B(\alpha+\beta+\gamma)-2\left(\frac{1}{\alpha}+\frac{1}{\gamma}+\frac{\alpha}{\beta\gamma}-\frac{4}{\beta\gamma}\right)S(\alpha,\beta,\gamma)$$
$$-2\left(\frac{1}{\beta}+\frac{1}{\gamma}+\frac{\beta}{\alpha\gamma}-\frac{4}{\alpha\gamma}\right)S(\beta,\alpha,\gamma)+2\left(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha+\beta}{2\gamma}-\frac{\alpha+\beta}{\alpha\beta}-\frac{5}{\gamma}+\frac{4}{\alpha\beta}\right)S(\gamma,\alpha,\beta); \quad \text{(A17)}$$

$$x_1=\frac{m^4}{8i\pi^2}\Big\{\alpha\beta\big[\beta A^{2121}-\tfrac12\alpha(A^{2123}+A^{3122}+A^{2312})-\tfrac12\gamma(A^{2113}+A^{2311}+A^{3121})\big]\frac{\omega_1}{m}$$

$$+\beta\gamma\big[\beta A^{2112}-\tfrac12\alpha(A^{2123}+A^{3122}+A^{2312})-\tfrac12\gamma(A^{2113}+A^{2311}+A^{3121})\big]\frac{\omega_2}{m}$$

$$-\tfrac12\beta^2\big[\alpha(A^{2123}+A^{3122}+A^{2312})-\gamma(A^{2113}+A^{2311}+A^{3121})\big]\frac{\omega_3}{m}\Big\}$$

$$=\alpha\Big[\Big(\frac{2\alpha}{\beta}-\frac{\alpha}{\beta+\gamma}+\frac{2\gamma}{\beta+\gamma}-\frac{2\alpha\beta}{(\beta+\gamma)^2}\Big)\frac{\omega_1}{m}+\gamma\Big(\frac{2}{\beta}+\frac{1}{\beta+\gamma}\Big)\frac{\omega_2}{m}-\frac{\beta}{\beta+\gamma}\frac{\omega_3}{m}\Big]\big[A(\alpha)-A(\alpha+\beta+\gamma)\big]$$

$$+\gamma\Big[\Big(\frac{2\gamma}{\beta}-\frac{\gamma}{\alpha+\beta}+\frac{2\alpha}{\alpha+\beta}-\frac{2\beta\gamma}{(\alpha+\beta)^2}\Big)\frac{\omega_2}{m}+\alpha\Big(\frac{2}{\beta}+\frac{1}{\alpha+\beta}\Big)\frac{\omega_1}{m}+\frac{\beta}{\alpha+\beta}\frac{\omega_3}{m}\Big]\big[A(\gamma)-A(\alpha+\beta+\gamma)\big]$$

$$-\Big[\alpha\Big(\frac{\alpha\gamma}{\beta^2}-\frac{\alpha-\gamma}{2\beta}-\frac1\alpha-\frac1\beta-\frac1\gamma+\frac{1}{\beta+\gamma}+\frac{2\beta}{(\beta+\gamma)^2}\Big)\frac{\omega_1}{m}+\gamma\Big(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha-\gamma}{2\beta}-\frac1\beta-\frac{1}{\beta+\gamma}\Big)\frac{\omega_2}{m}-\beta\Big(\frac{\alpha-\gamma}{2\beta}+\frac1\gamma-\frac{1}{\beta+\gamma}\Big)\frac{\omega_3}{m}\Big]B(\alpha)$$

$$-\Big[\gamma\Big(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha-\gamma}{2\beta}-\frac1\alpha-\frac1\beta-\frac1\gamma+\frac{1}{\alpha+\beta}+\frac{2\beta}{(\alpha+\beta)^2}\Big)\frac{\omega_2}{m}+\alpha\Big(\frac{\alpha\gamma}{\beta^2}-\frac{\alpha-\gamma}{2\beta}-\frac1\beta-\frac{1}{\alpha+\beta}\Big)\frac{\omega_1}{m}+\beta\Big(\frac{\gamma-\alpha}{2\beta}+\frac1\alpha-\frac{1}{\alpha+\beta}\Big)\frac{\omega_3}{m}\Big]B(\gamma)$$

$$+\Big[\frac{\alpha(\beta-\gamma)}{\beta\gamma}\frac{\omega_1}{m}-\frac{\gamma(\alpha-\beta)}{\alpha\beta}\frac{\omega_2}{m}+\frac{\beta(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}\Big]B(\beta)+\Big[\alpha\Big(\frac{\alpha\gamma}{\beta^2}-\frac{\alpha-\gamma}{2\beta}-\frac1\beta-\frac1\gamma-\frac{1}{\alpha+\beta}+\frac{1}{\beta+\gamma}+\frac{2\beta}{(\beta+\gamma)^2}\Big)\frac{\omega_1}{m}$$

$$+\gamma\Big(\frac{\alpha\gamma}{\beta^2}+\frac{\alpha-\gamma}{2\beta}-\frac1\beta-\frac1\alpha-\frac{1}{\beta+\gamma}+\frac{1}{\alpha+\beta}+\frac{2\beta}{(\alpha+\beta)^2}\Big)\frac{\omega_2}{m}-\beta(\alpha-\gamma)\Big(\frac{1}{2\beta}+\frac{\beta(\alpha+\beta+\gamma)}{\alpha\gamma(\alpha+\beta)(\beta+\gamma)}\Big)\frac{\omega_3}{m}\Big]B(\alpha+\beta+\gamma)$$

$$+2\alpha\gamma\Big(\frac{1}{\beta+\gamma}\frac{\omega_1}{m}+\frac{1}{\alpha+\beta}\frac{\omega_2}{m}\Big)-2\Big[\frac{\alpha(\alpha+\gamma)}{\beta\gamma}\frac{\omega_1}{m}+\frac{(\alpha+\gamma)(\alpha-\beta)}{\alpha\beta}\frac{\omega_2}{m}+\beta\Big(\frac1\alpha+\frac{\alpha}{\beta\gamma}\Big)\frac{\omega_3}{m}\Big]S(\alpha,\beta,\gamma)$$

$$-2\Big[\frac{\gamma(\alpha+\gamma)}{\alpha\beta}\frac{\omega_2}{m}+\frac{(\alpha+\gamma)(\gamma-\beta)}{\beta\gamma}\frac{\omega_1}{m}-\beta\Big(\frac1\gamma+\frac{\gamma}{\alpha\beta}\Big)\frac{\omega_3}{m}\Big]S(\gamma,\alpha,\beta)$$

$$-\Big[\alpha\Big(\frac{2\alpha\gamma}{\beta^2}-\frac{\alpha-\gamma}{\beta}-\frac4\alpha-\frac6\beta+\frac2\gamma\Big)\frac{\omega_1}{m}+\gamma\Big(\frac{2\alpha\gamma}{\beta^2}+\frac{\alpha-\gamma}{\beta}+\frac2\alpha-\frac6\beta-\frac4\gamma\Big)\frac{\omega_2}{m}-(\alpha-\gamma)\Big(1-\frac{2\beta}{\alpha\gamma}\Big)\frac{\omega_3}{m}\Big]S(\beta,\alpha,\gamma);\quad\text{(A18)}$$

$$x_2=\frac{m^4}{8i\pi^2}\Big\{\alpha\gamma\big[\gamma A^{3311}-\tfrac12\alpha(A^{3312}+A^{2313}+A^{3123})-\tfrac12\beta(A^{3112}+A^{3121}+A^{2311})\big]\frac{\omega_1}{m}$$

$$+\beta\gamma\big[\gamma A^{3113}-\tfrac12\alpha(A^{3312}+A^{2313}+A^{3123})-\tfrac12\beta(A^{3112}+A^{3121}+A^{2311})\big]\frac{\omega_3}{m}$$

$$-\tfrac12\gamma^2\big[\alpha(A^{3312}+A^{2313}+A^{3123})-\beta(A^{3112}+A^{3121}+A^{2311})\big]\frac{\omega_2}{m}\Big\}$$

$$=\alpha\Big[\Big(\frac{2\alpha}{\gamma}-\frac{\alpha}{\beta+\gamma}+\frac{2\beta}{\beta+\gamma}-\frac{2\alpha\gamma}{(\beta+\gamma)^2}\Big)\frac{\omega_1}{m}+\beta\Big(\frac2\gamma+\frac{1}{\beta+\gamma}\Big)\frac{\omega_3}{m}-\frac{\gamma}{\beta+\gamma}\frac{\omega_2}{m}\Big]\big[A(\alpha)-A(\alpha+\beta+\gamma)\big]$$

$$+\beta\Big[\Big(\frac{2\beta}{\gamma}-\frac{\beta}{\alpha+\gamma}+\frac{2\alpha}{\alpha+\gamma}-\frac{2\beta\gamma}{(\alpha+\gamma)^2}\Big)\frac{\omega_3}{m}+\alpha\Big(\frac2\gamma+\frac{1}{\alpha+\gamma}\Big)\frac{\omega_1}{m}+\frac{\gamma}{\alpha+\gamma}\frac{\omega_2}{m}\Big]\big[A(\beta)-A(\alpha+\beta+\gamma)\big]$$

$$-\Big[\alpha\Big(\frac{\alpha\beta}{\gamma^2}-\frac{\alpha-\beta}{2\gamma}-\frac1\alpha-\frac1\beta-\frac1\gamma+\frac{1}{\beta+\gamma}+\frac{2\gamma}{(\beta+\gamma)^2}\Big)\frac{\omega_1}{m}+\beta\Big(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha-\beta}{2\gamma}-\frac1\gamma-\frac{1}{\beta+\gamma}\Big)\frac{\omega_3}{m}-\gamma\Big(\frac{\alpha-\beta}{2\gamma}+\frac1\beta-\frac{1}{\beta+\gamma}\Big)\frac{\omega_2}{m}\Big]B(\alpha)$$

$$-\Big[\beta\Big(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha-\beta}{2\gamma}-\frac1\alpha-\frac1\beta-\frac1\gamma+\frac{1}{\alpha+\gamma}+\frac{2\gamma}{(\alpha+\gamma)^2}\Big)\frac{\omega_3}{m}+\alpha\Big(\frac{\alpha\beta}{\gamma^2}-\frac{\alpha-\beta}{2\gamma}-\frac1\gamma-\frac{1}{\alpha+\gamma}\Big)\frac{\omega_1}{m}+\gamma\Big(\frac{\beta-\alpha}{2\gamma}+\frac1\alpha-\frac{1}{\alpha+\gamma}\Big)\frac{\omega_2}{m}\Big]B(\beta)$$

$$
\begin{aligned}
&+\left[\frac{\alpha(\gamma-\beta)}{\beta\gamma}\frac{\omega_1}{m}-\frac{\beta(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}+\frac{\gamma(\alpha-\beta)}{\alpha\beta}\frac{\omega_2}{m}\right]B(\gamma)+\left[\alpha\left(\frac{\alpha\beta}{\gamma^2}-\frac{\alpha-\beta}{2\gamma}-\frac{1}{\beta}-\frac{1}{\gamma}-\frac{1}{\alpha+\gamma}+\frac{1}{\beta+\gamma}+\frac{2\gamma}{(\beta+\gamma)^2}\right)\frac{\omega_1}{m}\right.\\
&+\beta\left(\frac{\alpha\beta}{\gamma^2}+\frac{\alpha-\beta}{2\gamma}-\frac{1}{\gamma}-\frac{1}{\alpha}-\frac{1}{\beta+\gamma}+\frac{1}{\alpha+\gamma}+\frac{2\gamma}{(\alpha+\gamma)^2}\right)\frac{\omega_3}{m}-\left.\gamma(\alpha-\beta)\left(\frac{1}{2\gamma}+\frac{\gamma(\alpha+\beta+\gamma)}{\alpha\beta(\alpha+\gamma)(\beta+\gamma)}\right)\frac{\omega_2}{m}\right]B(\alpha+\beta+\gamma)\\
&+2\alpha\beta\left(\frac{1}{\beta+\gamma}\frac{\omega_1}{m}+\frac{1}{\alpha+\gamma}\frac{\omega_3}{m}\right)-2\left[\frac{\alpha(\alpha+\beta)}{\beta\gamma}\frac{\omega_1}{m}+\frac{(\alpha+\beta)(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}+\gamma\left(\frac{1}{\alpha}+\frac{\alpha}{\beta\gamma}\right)\frac{\omega_2}{m}\right]S(\alpha,\beta,\gamma)\\
&-2\left[\frac{\beta(\alpha+\beta)}{\alpha\gamma}\frac{\omega_3}{m}+\frac{(\alpha+\beta)(\beta-\gamma)}{\beta\gamma}\frac{\omega_1}{m}-\gamma\left(\frac{1}{\beta}+\frac{\beta}{\alpha\gamma}\right)\frac{\omega_2}{m}\right]S(\beta,\alpha,\gamma)\\
&\qquad-\left[\alpha\left(\frac{2\alpha\beta}{\gamma^2}-\frac{\alpha-\beta}{\gamma}-\frac{4}{\alpha}-\frac{6}{\gamma}+\frac{2}{\beta}\right)\frac{\omega_1}{m}+\beta\left(\frac{2\alpha\beta}{\gamma^2}+\frac{\alpha-\beta}{\gamma}+\frac{2}{\alpha}-\frac{6}{\gamma}-\frac{4}{\beta}\right)\frac{\omega_3}{m}-(\alpha-\beta)\left(1-\frac{2\gamma}{\alpha\beta}\right)\frac{\omega_2}{m}\right]S(\gamma,\alpha,\beta)\,;
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
x_3=\frac{m^4}{8i\pi^2}\Big\{&\alpha\gamma[\alpha A^{3322}-\tfrac{1}{2}\beta(A^{2321}+A^{2312}+A^{3122})-\tfrac{1}{2}\gamma(A^{3321}+A^{3123}+A^{2313})]\frac{\omega_2}{m}\\
&-\alpha\beta[\alpha A^{2323}-\tfrac{1}{2}\beta(A^{2321}+A^{2312}+A^{3122})-\tfrac{1}{2}\gamma(A^{3321}+A^{3123}+A^{2313})]\frac{\omega_3}{m}\\
&+\tfrac{1}{2}\alpha^2[\beta(A^{2321}+A^{2312}+A^{3122})-\gamma(A^{3321}+A^{3123}+A^{2313})]\frac{\omega_1}{m}\Big\}\\
=-\beta&\left[\left(\frac{2\beta}{\alpha}-\frac{\beta}{\alpha+\gamma}+\frac{2\gamma}{\alpha+\gamma}-\frac{2\alpha\beta}{(\alpha+\gamma)^2}\right)\frac{\omega_3}{m}-\gamma\left(\frac{2}{\alpha}+\frac{1}{\alpha+\gamma}\right)\frac{\omega_2}{m}-\frac{\alpha}{\alpha+\gamma}\frac{\omega_1}{m}\right][A(\beta)-A(\alpha+\beta+\gamma)]\\
+\gamma&\left[\left(\frac{2\gamma}{\alpha}-\frac{\gamma}{\alpha+\beta}+\frac{2\beta}{\alpha+\beta}-\frac{2\alpha\gamma}{(\alpha+\beta)^2}\right)\frac{\omega_2}{m}-\beta\left(\frac{2}{\alpha}+\frac{1}{\alpha+\beta}\right)\frac{\omega_3}{m}-\frac{\alpha}{\alpha+\beta}\frac{\omega_1}{m}\right][A(\gamma)-A(\alpha+\beta+\gamma)]\\
+&\left[\beta\left(\frac{\beta\gamma}{\alpha^2}-\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}+\frac{1}{\alpha+\gamma}+\frac{2\alpha}{(\alpha+\gamma)^2}\right)\frac{\omega_3}{m}-\gamma\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\alpha+\gamma}\right)\frac{\omega_2}{m}-\alpha\left(\frac{\beta-\gamma}{2\alpha}+\frac{1}{\gamma}-\frac{1}{\alpha+\gamma}\right)\frac{\omega_1}{m}\right]B(\beta)\\
-&\left[\gamma\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\gamma}+\frac{1}{\alpha+\beta}+\frac{2\alpha}{(\alpha+\beta)^2}\right)\frac{\omega_2}{m}-\beta\left(\frac{\beta\gamma}{\alpha^2}-\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\alpha+\beta}\right)\frac{\omega_3}{m}-\alpha\left(\frac{\gamma-\beta}{2\alpha}+\frac{1}{\beta}-\frac{1}{\alpha+\beta}\right)\frac{\omega_1}{m}\right]B(\gamma)\\
-&\left[\frac{\beta(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}+\frac{\gamma(\beta-\alpha)}{\alpha\beta}\frac{\omega_2}{m}+\frac{\alpha(\beta-\gamma)}{\beta\gamma}\frac{\omega_1}{m}\right]B(\alpha)-\left[\beta\left(\frac{\beta\gamma}{\alpha^2}-\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\gamma}-\frac{1}{\alpha+\beta}+\frac{1}{\alpha+\gamma}+\frac{2\alpha}{(\alpha+\gamma)^2}\right)\frac{\omega_3}{m}\right.\\
-\gamma&\left(\frac{\beta\gamma}{\alpha^2}+\frac{\beta-\gamma}{2\alpha}-\frac{1}{\alpha}-\frac{1}{\beta}-\frac{1}{\alpha+\gamma}+\frac{1}{\alpha+\beta}+\frac{2\alpha}{(\alpha+\beta)^2}\right)\frac{\omega_2}{m}-\left.\alpha(\beta-\gamma)\left(\frac{1}{2\alpha}+\frac{\alpha(\alpha+\beta+\gamma)}{\beta\gamma(\alpha+\beta)(\alpha+\gamma)}\right)\frac{\omega_1}{m}\right]B(\alpha+\beta+\gamma)\\
-2\beta\gamma&\left(\frac{1}{\alpha+\gamma}\frac{\omega_3}{m}-\frac{1}{\alpha+\beta}\frac{\omega_2}{m}\right)+2\left[\frac{\beta(\beta+\gamma)}{\alpha\gamma}\frac{\omega_3}{m}+\frac{(\beta+\gamma)(\alpha-\beta)}{\alpha\beta}\frac{\omega_2}{m}+\alpha\left(\frac{1}{\beta}+\frac{\beta}{\alpha\gamma}\right)\frac{\omega_1}{m}\right]S(\beta,\alpha,\gamma)\\
-2&\left[\frac{\gamma(\beta+\gamma)}{\alpha\beta}\frac{\omega_2}{m}+\frac{(\beta+\gamma)(\alpha-\gamma)}{\alpha\gamma}\frac{\omega_3}{m}+\alpha\left(\frac{1}{\gamma}+\frac{\gamma}{\alpha\beta}\right)\frac{\omega_1}{m}\right]S(\gamma,\alpha,\beta)\\
&\qquad+\left[\beta\left(\frac{2\beta\gamma}{\alpha^2}-\frac{\beta-\gamma}{\alpha}-\frac{4}{\beta}-\frac{6}{\alpha}+\frac{2}{\gamma}\right)\frac{\omega_3}{m}-\gamma\left(\frac{2\beta\gamma}{\alpha^2}+\frac{\beta-\gamma}{\alpha}+\frac{2}{\beta}-\frac{6}{\alpha}-\frac{4}{\gamma}\right)\frac{\omega_2}{m}-(\beta-\gamma)\left(1-\frac{2\alpha}{\beta\gamma}\right)\frac{\omega_1}{m}\right]S(\alpha,\beta,\gamma)\,;
\end{aligned}
\tag{A20}
$$

$$
t=-\left(\frac{\alpha\omega_1}{m}\right)^2-\left(\frac{\beta\omega_3}{m}\right)^2-\left(\frac{\gamma\omega_2}{m}\right)^2-2\frac{\alpha\beta\omega_1\omega_3}{m^2}-2\frac{\alpha\gamma\omega_1\omega_2}{m^2}+2\frac{\beta\gamma\omega_2\omega_3}{m^2}+\alpha\beta\gamma\,.
\tag{A21}
$$

t is a positive quantity. In fact, it can be easily shown that

$$t=\frac{4}{m^3}|\mathbf{k}^{(1)}\cdot\mathbf{k}^{(2)}\times\mathbf{k}^{(3)}|^2.$$

For small values of the arguments, the functions defined above have the following form:

$$x_1=-\frac{11}{360}\left(\alpha\beta(\alpha+\gamma)\frac{\omega_1}{m}+\beta\gamma(\alpha+\gamma)\frac{\omega_2}{m}+\beta^2(\alpha-\gamma)\frac{\omega_3}{m}\right), \tag{A22}$$

$$x_2=-\frac{11}{360}\left(\alpha\gamma(\alpha+\beta)\frac{\omega_1}{m}+\beta\gamma(\alpha+\beta)\frac{\omega_3}{m}+\gamma^2(\alpha-\beta)\frac{\omega_2}{m}\right), \tag{A23}$$

$$x_3=-\frac{11}{360}\left(\alpha\gamma(\beta+\gamma)\frac{\omega_2}{m}-\alpha\beta(\beta+\gamma)\frac{\omega_3}{m}-\alpha^2(\beta-\gamma)\frac{\omega_1}{m}\right), \tag{A24}$$

$$u=\frac{1}{60}\left(\alpha^2(\beta-\gamma)\frac{\omega_1}{m}-\gamma^2(\alpha-\beta)\frac{\omega_2}{m}+\beta^2(\alpha-\gamma)\frac{\omega_3}{m}\right), \tag{A25}$$

$$y_1=-(11/360)\beta(\alpha+\gamma), \tag{A26}$$

$$y_2=-(11/360)\alpha(\beta+\gamma), \tag{A27}$$

$$y_3=-(11/360)\gamma(\alpha+\beta), \tag{A28}$$

$$v=(1/60)(\alpha\beta+\alpha\gamma+\beta\gamma). \tag{A29}$$