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Instantaneous and Asymptotic Conservation Laws for Classical Relativistic Mechanics of Interacting Point Particles

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The present article consists of two parts. First, we assume that the conservation laws for energy and linear momentum are valid and that these quantities are the sums of the energies and linear momenta of the individual particles, i.e., that there is no interaction energy and no interaction momentum. We then repeat a familiar argument and show that there can then be no interaction between the particles, that is, their world lines are straight. In the second part of the paper the interaction quantities for energy, linear and angular momenta, and the center-of-mass law are derived for the equations of motion proposed in an earlier paper. We then study these interaction quantities in the asymptotic region of collision processes, in order to arrive at asymptotic conservation laws. We find, in agreement with the earlier paper, that the interaction energy and the linear interaction momenta vanish asymptotically. This, however, is not true in general for the interaction angular momenta and center-of-mass motion. Asymptotic interaction angular momentum is present in all theories, such as classical electrodynamics, which lead to inverse-square-law forces.

INTRODUCTION AND SUMMARY

We wish to discuss some aspects of the relativistic dynamics of a system of n interacting classical point particles. The history of each particle, $i=1, 2, \cdots, n$, will be described by an orbit in Minkowski space. This orbit will be given, parametrically, in terms of the proper time τ_i, i.e., the orbit i is given by a vector-valued function $x_{i\alpha}(\tau_i)$, where α refers to the time and the spatial coordinates. Proper time is the same as arc length so that

$$\sum_\alpha l_\alpha[\dot{x}_{i\alpha}(\tau_i)]^2=1\,, \tag{1}$$

where the dot refers to differentiation with respect to the argument τ_i and where $-l_0=l_1=l_2=l_3=-1$, and we assume that $\dot{x}_{i0}(\tau_i)>0$, i.e., that the proper time runs in the same direction as the actual time.

The components of the linear momentum and the energy of the individual particle i at time t_1 will be defined as follows: Let τ_{i1} be the solution of the equation

$$x_{i0}(\tau_{i1})=t_1\,, \tag{2}$$

then the expression for the energy and the components of the linear momentum at t_1 will be

$$P_{i\alpha}(t_1)=m_i\dot{x}_{i\alpha}(\tau_{i1})\,. \tag{3}$$

Henceforth, we shall use the term linear momentum for both energy and linear momentum. Similarly we shall use the word angular momentum for center-of-mass momentum and angular momentum, i.e., all six components of the usual antisymmetric tensor. The components of the angular momentum of particle i at time t_1 are

$$M_{i\alpha\beta}(t_1)=x_{i\alpha}(\tau_{i1})P_{i\beta}(t_1)-P_{i\alpha}(t_1)x_{i\beta}(\tau_{i1})\,. \tag{4}$$

In Sec. 1 we repeat, in detail, a familiar derivation of a "no interaction theorem"[1] by showing that if (a) the

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[1] See, for instance, P. G. Bergmann, *The Special Theory of Relativity*, Handbuch der Physik IV (Springer, Berlin, 1962), p. 147. The theorem discussed here has a purely kinematical basis. Several "no interaction" theorems with a dynamical origin have appeared in the recent literature: D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963); D. G. Currie, J. Math. Phys. **4**, 1470 (1963); J. T. Cannon and T. F. Jordan, *ibid.* **5**, 299 (1964); H. Ekstein, Consistence of Relativistic Particle Theories, Université d'Aix-Marseille, (1964, unpublished), H. Leutwyler, Nuovo Cimento **37**, 556 (1965).

components of the total linear momentum are given by

$$P_\alpha(t_1)=\sum_i P_{i\alpha}(t_1)\,, \tag{5}$$

and are conserved, i.e., independent of t, in every coordinate system of the special theory of relativity, (b) the particles do not coincide, i.e., there are no pairs i, k, and proper times τ_i and τ_k so that $x_{i\alpha}(\tau_i)=x_{k\alpha}(\tau_k)$ for $\alpha=0, 1, 2, 3$, and if (c) asymptotically, that is either for all $\tau_i\rightarrow\infty$, or all $\tau_i\rightarrow-\infty$, the orbits are straight i.e., the $x_{i\alpha}$ become constant, then for $n=2$, 3, or 4 the orbits are straight throughout. Even if the preceding conclusion were valid only for $n=2$,i.e., for all pairs i, k, this would ordinarily be interpreted as the absence of interaction between pairs of particles except at coincidence. A coincidence of two world lines is, however, infinitely unlikely as soon as the dimensionality of space-time is 3 or more. It follows that the total linear momentum can be given by (5) only in the asymptotic regions. In the region of interaction one must add to the right side of (5) interaction linear momenta.[2]

In Sec. 2 we give the expressions for the interaction linear momenta and the interaction angular momenta of the dynamics which is described by the equations of motion which we proposed in an earlier paper.[3] These expressions are then studied, in Sec. 3, in the asymptotic regions of collision processes. The interaction angular momenta approach constants in these regions. In Sec. 4 we illustrate, by a kinematical argument, that such asymptotic interaction angular momenta are present in all relativistic theories, such as classical electrodynamics and gravity theory, which involve inverse-square-law forces.

1. PROOF OF A "NO INTERACTION THEOREM"

Let us make the assumptions (a), (b), and (c), mentioned in the preceding section. According to assumption (a) the total linear momentum $P_\alpha(t)$ is given by (5) and does not depend on t, and this is true for every coordinate system of the special theory of relativity. The asymptotic condition (c) implies that P_α, given by (5), transforms as a vector under proper inhomogeneous Lorentz transformations. In order to see this, consider two coordinate systems such that the coordinates x_α' in the second coordinate system are related to those in the first frame x_α by

$$x_\alpha'=\sum_\beta L_{\alpha\beta}x_\beta\,. \tag{6}$$

[2] That one must introduce interaction energy and linear momentum for classical relativistic mechanics has been pointed out also by L. Brillouin, Compt. Rend. **259**, 2361 (1964). For a simplified derivation of the conservation laws from invariance postulates, see W. Macke, Forsch. Fortschr. **39**, 193 (1965).

[3] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965). The status of the classical theory of interacting point particles, prior to this article was summarized by P. Havas, in *Statistical Mechanics of Equilibrium and Non-equilibrium*, edited by J. Meixner (North Holland Publishing Company, Amsterdam, 1965). Also see D. G. Currie, Phys. Rev. **142**, 817 (1966).

Let us assume that $\dot{x}_{i\alpha}(\tau_i)$ become constant for $\tau_i\rightarrow\infty$. It is then possible to choose t_1 in the first frame such that $\dot{x}_{i\alpha}(\tau_i)$ become independent of τ_i for $\tau_i\geq\tau_{i1}$, where the τ_{i1} are given by (2). The time t_1' in the second coordinate system can, furthermore, be chosen so large that $\tau_{i1}'>\tau_{i1}$ for all i, where the τ_{i1}' are given by (2) in the second coordinate system, that is, by

$$t_1'=x_{i0}'(\tau_{i1}')=\sum_\alpha L_{0\alpha}x_{i\alpha}(\tau_{i1}')\,. \tag{7}$$

One can achieve this by choosing t_1' larger than the largest of the $x_{i0}'(\tau_{i1})$. Let us now calculate the components of the linear momentum in the second coordinate system. We obtain

$$P_\alpha'=\sum_i m_i\sum_\beta L_{\alpha\beta}\dot{x}_{i\beta}(\tau_{i1}')\,. \tag{8}$$

Since the $\dot{x}_{i\alpha}$ are under the conditions independent of the τ_i, the τ_{i1}' can be replaced in (6) by τ_{i1} and we have

$$P_\alpha'=\sum_i m_i\sum_\beta L_{\alpha\beta}\dot{x}_{i\beta}(\tau_{i1})=\sum_\beta L_{\alpha\beta}P_\beta\,. \tag{9}$$

Hence, the components of the linear-momentum transform as the components of a vector. It is in order to assure this that the asymptotic condition (c) was introduced.

Let us view the linear momentum in the interacting (nonasymptotic) region as given by the two frames. The first frame gives for the total linear momentum $\sum_i m_i\dot{x}_{i\alpha}(\tau_{i1})$, where the points $x_{i\alpha}(\tau_{i1})$ are simultaneous at t_1 in that frame. The second frame gives $\sum_i m_i\dot{x}_{i\alpha}(\tau_{i1})$, where the points $x_{i\alpha}(\tau_{i1})$ are simultaneous, at time t_1', in the second coordinate frame. The total linear momenta are conserved in their respective frames, i.e., independent of t_1 and t_1'. Furthermore they are asymptotically related by (9), which means that they are always so related. Hence,

$$\sum_i m_i\dot{x}_{i\alpha}'(\tau_{i1}')=P_\alpha'=\sum_\beta L_{\alpha\beta}P_\beta =\sum_\beta\sum_i L_{\alpha\beta}m_i\dot{x}_{i\beta}(\tau_{i1}) \tag{10}$$

is valid for all values of t_1 and t_1'. Since, however, $\sum_\beta L_{\alpha\beta}\dot{x}_{i\beta}(\tau_{i1})=\dot{x}_{i\alpha}'(\tau_{i1})$, we obtain the result that the sum of the α components of the individual momenta is the same not only on any plane of constant time in the coordinate system considered but also on any plane which represents constant time in any other coordinate system. Replacing the primed variables by unprimed ones, and conversely, we have

$$\sum_i m_i\dot{x}_{i\alpha}(\tau_{i1})=\sum_{i1}m_i\dot{x}_{i\alpha}(\tau_{i1}')\,, \tag{11}$$

where the points $x_{i\alpha}(\tau_{i1}')$ are simultaneous from the point of view of any coordinate system. This is illustrated in Fig. 1. Equation (11) immediately implies the no interaction theorem, as we shall see next.

Consider first a closed system of two point particles. Let $x_{1\alpha}(\tau_1)$ be the coordinates of an arbitrary point on the orbit of the first particle, and let $x_{2\alpha}(\tau_2)$ and $x_{2\alpha}(\tau_2')$ be the two points which are simultaneous with $x_{1\alpha}(\tau_1)$ in two different coordinate frames. Equation (11) gives $m_1\dot{x}_{1\alpha}(\tau_1)+m_2\dot{x}_{2\alpha}(\tau_2)=m_1\dot{x}_{1\alpha}(\tau_1)+m_2\dot{x}_2\dot{x}_{2\alpha}(\tau_2')$, so that

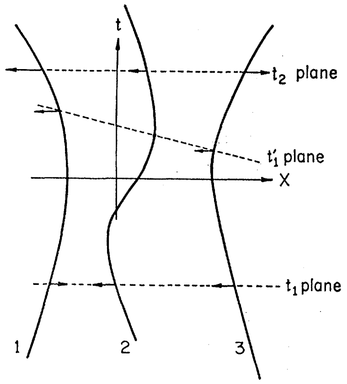

FIG. 1. Interpretation of Eq. (11). The world lines of particles 1, 2, and 3 are assumed to be coplanar and to lie entirely in the x-t plane. The x components of the momenta of three particles are shown at times t_1 and t_2, Their sum must be equal because of the conservation law for linear momentum. It follows, however, from (11) that the sum of these components is the same also if taken on the t_1' plane which represents the points which are simultaneous from the point of view of any other coordinate system.

$\dot{x}_{2\alpha}(\tau_2)=\dot{x}_{2\alpha}(\tau_2')$. We conclude that the linear momentum in the second orbit does not change in the section of the orbit which is outside the light cones based on the point $x_{1\alpha}(\tau_1)$. Using several points $x_{1\alpha}(\tau_1)$ as bases, it follows that $\dot{x}_{2\alpha}(\tau_2)$ does not depend on τ_2. A similar argument is possible also for $n=3$, i.e., three orbits, unless they are coplanar. Let us choose a point on each of the first two orbits, say, the points with the parameters τ_1, τ_2, so that these are in a spacelike relation to each other. There is, in general, still a finite section on the third orbit the points of which are in some coordinate system simultaneous with the selected points on the first two orbits. The momentum vector of the third particle must then be constant along that finite section. In general, the entire third orbit can be decomposed into such sections so that the momentum vector $m_3\dot{x}_{3\alpha}(\tau_3)$ must be constant throughout. This amounts to the absence of three-particle forces in the usual language. The same argument is possible even for $n=4$.

2. INSTANTANEOUS CONSERVATION LAWS FOR A DYNAMICS OF INTERACTING POINT PARTICLES

We shall now give the expressions for the interaction linear momenta and angular momenta for those equations of motion which we proposed in an earlier paper.[3] We then study these expressions in the asymptotic regions of collision processes. For forces between pairs of particles which are, in the nonrelativistic sense, proportional to the inverse of the square of the distance between the particles, the interaction angular momenta approach constants in the asymptotic regions. This leads to a modification in the formulation of the asymptotic conservation laws for angular momenta.

The equations of motion considered before are

$$m_i\ddot{x}_{i\alpha}(\tau_i)=\sum_{k\neq i}\int_{-\infty}^{\infty}d\tau_k\hat{F}_{ik\alpha}(\tau_i,\tau_k)\,,\qquad(12a)$$

where

$$\hat{F}_{ik\alpha}(\tau_i,\tau_k)=-\hat{F}_{ki\alpha}(\tau_k,\tau_i)=[x_{i\alpha}(\tau_i)-x_{k\alpha}(\tau_k)]\times[2\varphi_{ik}(\rho_{ik})\zeta_{ik}+\varphi_{ik}'(\rho_{ik})\sigma_{ik}\sigma_{ki}/\rho_{ik}].\qquad(12b)$$

In this, $\varphi_{ik}=\varphi_{ki}$ is an essentially arbitrary function representing the interaction between particles i and k which, however, is to be set equal to zero for imaginary ρ_{ik} so that the integration in (12a) covers effectively only a finite range; $\varphi_{ik}'(\rho_{ik})=d\varphi_{ik}(\rho_{ik})/d\rho_{ik}$ and where

$$\rho_{ik}=\{\textstyle\sum_\alpha-l_\alpha[x_{i\alpha}(\tau_i)-x_{k\alpha}(\tau_k)]^2\}^{1/2},\qquad(13a)$$

$$\zeta_{ik}=\textstyle\sum_\alpha l_\alpha\dot{x}_{i\alpha}(\tau_i)\dot{x}_{k\alpha}(\tau_k)\,,\qquad(13b)$$

$$\sigma_{ik}=\textstyle\sum_\alpha l_\alpha\dot{x}_{i\alpha}(\tau_i)[x_{i\alpha}(\tau_i)-x_{k\alpha}(\tau_k)]\,,\qquad(13c)$$

$$\sigma_{ki}=\textstyle\sum_\alpha l_\alpha\dot{x}_{k\alpha}(\tau_k)[x_{k\alpha}(\tau_k)-x_{i\alpha}(\tau_i)].\qquad(13d)$$

All the quantities are functions of τ_i and τ_k. We shall assume that the functions $\varphi_{ik}(\rho)$ go to zero rapidly if ρ becomes larger than the "range" R_{ik} of the interaction. The orbits, incidentally, can be calculated by an iterative procedure from the positions and velocities of the particles at one instant of time in an arbitrary Lorentz frame. It can be shown that this iterative procedure converges to a unique set of orbits for distant collisions.

Since $\varphi_{ik}(\rho)=\varphi_{ki}(\rho)$, Eqs. (12) imply that the contribution, from a small interval δ_k around τ_k on k, to the change in linear momentum of i during a small interval δ_i around τ_i on i,is equal and opposite to the contribution from δ_i to the change of linear momentum of k during δ_k. That means that linear momentum is exchanged without loss between the orbits i and k. Since, furthermore, the exchanged linear momentum has the direction of the line connecting $x_{i\alpha}(\tau_i)$ and $x_{k\alpha}(\tau_k)$, angular momentum is also transferred without loss between the orbits i and k.

The situation is analogous to that for central forces between pairs of particles in Newtonian mechanics. The only difference is that in Newtonian theory the transfer of linear momentum and angular momentum is instantaneous, whereas here the times $x_{i0}(\tau_i)$ and $x_{k0}(\tau_k)$ will not be the same in general. Hence, there is linear momentum (and angular momentum) which has left i and not yet arrived at k. These quantities "in transfer" constitute the interaction momentum and angular momentum. Since no linear momentum is lost during the interaction, the sum of the linear momenta of the particles, which is given by the right side of (5), and the linear momentum in transit is conserved and this sum will be called total linear momentum. This is illustrated, for a single exchange of linear momentum,[4]

[4] An interaction of this type has been proposed by P. Havas and J. Plebanski, Bull. Am. Phys. Soc. 5, 433 (1960).

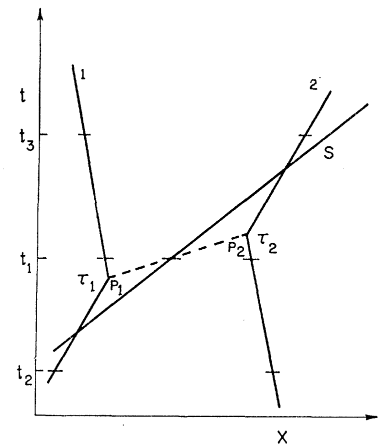

FIG. 2. World lines of two interacting particles with equal masses. Linear momentum Δ leaves particle 1 at P_1 and arrives at particle 2 and P_2. The sum of the linear momenta must be the same at t_2, t_1, and t_3. Hence, an interaction momentum must be present at t_1; its x component is positive. Similarly, an interaction momentum must be present in the coordinate system in which the points of the line S are simultaneous; its x component is negative.

in Fig. 2. The sum at t_1 of the linear momenta of i and k and of the interaction momentum is equal to a similar sum at t_2 or t_3. This sum is also equal to a similar sum for the total linear momentum across any plane S the normal to which is a timelike vector.

If the total interaction is a sum of interactions between pairs of particles, as implied by (12), the same holds for the components V_α of the interaction momentum

$$V_\alpha(t_1)=\sum_{i\neq k} V_{ik\alpha}(t_1). \qquad (14a)$$

The $V_{ik\alpha}$ are, for the equations of motion (12),

$$V_{ik\alpha}(t_1)=\int_{\tau_{i1}}^{\infty} d\tau_i \int_{-\infty}^{\tau_{k1}} d\tau_k \hat{F}_{ik\alpha}(\tau_i,\tau_k). \qquad (14b)$$

This is the momentum transferred by particle k to particle i which has left k before t_1 but will arrive at i after that time. Actually, the integrations in (14b) have only a finite range because φ_{ik} and φ_{ik}' have to be set equal to zero for imaginary ρ_{ik}. Hence, the upper limit of the integration with respect to τ_i can be replaced by the τ_i of the point at which the advanced light cone from k, with its apex at τ_{k1}, intersects the world line of i. This τ_i will be denoted by ik_{adv}. Similarly, the lower limit of the intergation with respect to τ_k can be replaced by the τ_k of the point where the retarded light cone from i, with its apex at τ_{i1}, intersects the world line of k. This τ_k will be denoted by ki_{ret}. All this is illustrated in Fig. 3.

The interaction angular momentum $W_{\alpha\beta}$ is also a sum of terms due to pairs

$$W_{\alpha\beta}(t_1)=\sum_{i\neq k} W_{ik\alpha\beta}(t_1), \qquad (15a)$$

where

$$W_{ik\alpha\beta}(t_1)=\int_{\tau_{i1}}^{\infty} d\tau_i \int_{-\infty}^{\tau_{k1}} d\tau_k [x_{i\alpha}(\tau_i)\hat{F}_{ik\beta}(\tau_i,\tau_k) - x_{i\beta}(\tau_i)\hat{F}_{ik\alpha}(\tau_i,\tau_k)]. \qquad (15b)$$

The remarks about the finite range of the integration, made after (14b), apply also in this case.

With these interaction momenta, the conservation laws can be easily verified by direct calculation. Thus, for

$$P_\alpha(t_1)=\sum_i m_i\dot{x}_{i\alpha}(\tau_{i1})+\sum_{i\neq k} V_{ik\alpha}(t_1), \qquad (16a)$$

we have

$$\frac{dP_\alpha(t_1)}{dt_1}=\sum_i \frac{m_i\ddot{x}_{i\alpha}(\tau_{i1})}{\dot{x}_{i0}(\tau_{i1})}+\sum_{i\neq k}\frac{1}{\dot{x}_{i0}(\tau_{i1})}\frac{\partial V_{ik\alpha}}{\partial\tau_{i1}} +\frac{1}{\dot{x}_{k0}(\tau_{k1})}\frac{\partial V_{ik\alpha}}{\partial\tau_{k1}}$$
$$=\sum_{i\neq k}\frac{1}{\dot{x}_{i0}(\tau_{i1})}\left[\int_{-\infty}^{\infty} d\tau_k \hat{F}_{ik\alpha}(\tau_{i1},\tau_k) - \int_{-\infty}^{\tau_{k1}} d\tau_k \hat{F}_{ik\alpha}(\tau_{i1},\tau_k)\right]+\sum_{i\neq k}\frac{1}{\dot{x}_{k0}(\tau_{k1})} \times\int_{\tau_{i1}}^{\infty} d\tau_i \hat{F}_{ik\alpha}(\tau_i,\tau_{k1}). \qquad (16b)$$

Interchanging i and k in the last term gives, because of the antisymmetry of $F_{ik\alpha}$ in i and k, just the term necessary to cancel the preceding term. The conservation law for angular momentum can be verified in the same way.

It is disappointing, perhaps, that the conservation laws must be formulated in terms of interaction quantities which contain integrals over the actual orbits rather than quantities which refer to a single time only. However, this had to be expected in view of the force law (12a) which contains a similar integral.

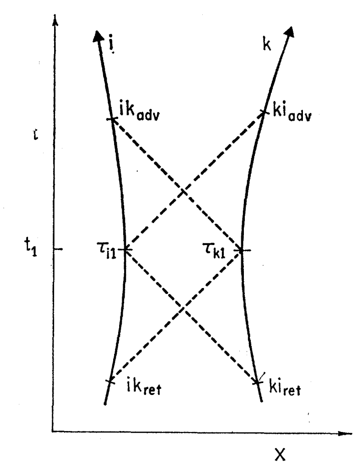

FIG. 3. The intersection of the orbit k with the advanced light cone based on $x_{i\alpha}(\tau_{i1})$ is called ki_{adv}. In a similar way ki_{ret} is the point where orbit k intersects the retarded light cone based on $x_{i\alpha}(\tau_{i1})$.

3. ASYMPTOTIC CONSERVATION LAWS

We assume that the world lines become widely separated for large values of $|t_1|$ and wish to evaluate (14b) and (15b) under this assumption. This implies the evaluation of double integrals, over τ_i and τ_k. However, the integral over τ_k under the assumption of wide separation has been evaluated before; it is the first term of (20), Ref. 3. This gives

$$\int_{-\infty}^{\tau_{k1}} d\tau_k \hat{F}_{ik\alpha}(\tau_i,\tau_k)=\{\zeta_{ik}(\tau_i,ki_{\text{ret}})[x_{i\alpha}(\tau_i)-x_{k\alpha}(ki_{\text{ret}})]-\sigma_{ik}(\tau_i,ki_{\text{ret}})\dot{x}_{k\alpha}(ki_{\text{ret}})\}I, \quad (17\text{a})$$

and I is given by (20b) [$\sigma_{ki}(\tau_i,ki_{\text{ret}})$ is negative]

$$I=-\tfrac{1}{2}[\sigma_{ki}(\tau_i,ki_{\text{ret}})]^{-3}\varphi_{ik}{}^3, \quad (17\text{b})$$

$$\varphi_{ik}{}^3=\int_0^\infty \varphi_{ik}(\rho)\rho^3 d\rho, \quad (17\text{c})$$

where ki_{ret} is the value of τ_k which corresponds to the point on the world line of k where the retarded light cone from τ_i intersects it. Since ζ_{ik}, σ_{ik}, σ_{ki} all depend on τ_i and τ_k, the value of these variables is given explicitly in the Eqs. (17). It is noteworthy that the value of the integral on the left does not depend on τ_{k1}. The reason for this is that the integrand is very small except in the neighborhood of the value $\tau_k=ki_{\text{ret}}$ so that the region around τ_{k1} contributes significantly only if τ_i is near the advanced light cone from τ_{k1}. For this τ_i, (17) is indeed inaccurate but this is only a small part of the total integral over τ_i in (14b) or (15b).

It should be possible to calculate the expressions (14b) and (15b) for the interaction momenta, assuming only that the spatial part of the distance of every point on the world line of k, which is spatially located with respect to the point τ_{i1} on the world line of i, is large as compared with the range of φ_{ik}, and that this holds for all pairs i, k. This assumption is already implicit in (17a). Such a calculation would give an expression for the interaction momenta which is valid throughout distant collisions, as also in a gas with low density, in addition to being valid in the asymptotic region of all collisions. Both calculation and result become much simpler, however, if we interpret the asymptotic regions as the regions in which $x_{i\alpha}(\tau_i)$ can be approximated by $\tau_i\dot{x}_{i\alpha}$, with a constant $\dot{x}_{i\alpha}$, in the sense that $x_{i\alpha}(\tau_i)-\tau_i\dot{x}_{i\alpha}$ is small as compared with $\tau_i\dot{x}_{i\alpha}$. Using this interpretation of asymptotic, we shall obtain expressions for the interaction momenta which are valid only a very long time before, or a very long time after, an arbitrary collision but are not valid *during* any collision, however distant.

Let us derive, first, a more simple expression for the integral in (17a). For this purpose, we first calculate $x_{k\alpha}(ki_{\text{ret}})$. This is given by the equation

$$0=\sum_\alpha l_\alpha[x_{i\alpha}-x_{k\alpha}(ki_{\text{ret}})]^2=\sum_\alpha l_\alpha(\tau_i\dot{x}_{i\alpha}-ki_{\text{ret}}\dot{x}_{k\alpha})^2 =\tau_i{}^2+(ki_{\text{ret}})^2-2\zeta_{ik}\tau_i ki_{\text{ret}}. \quad (18\text{a})$$

Since all velocities are assumed to be independent of τ_i, τ_k, this holds also for ζ_{ik}. From (18a)

$$x_{k\alpha}(ki_{\text{ret}})=\dot{x}_{k\alpha}\tau_i[\zeta_{ik}-(\zeta_{ik}{}^2-1)^{1/2}]. \quad (18\text{b})$$

From this

$$\sigma_{ik}(\tau_i,ki_{\text{ret}})=\sum_\alpha l_\alpha\dot{x}_{i\alpha}[x_{i\alpha}(\tau_i)-x_{k\alpha}(ki_{\text{ret}})] =\tau_i[1-\zeta_{ik}{}^2+\zeta_{ik}(\zeta_{ik}{}^2-1)^{1/2}], \quad (18\text{c})$$

$$\sigma_{ki}(ki_{\text{ret}},\tau_i)=\sum_\alpha l_\alpha\dot{x}_{k\alpha}[x_{k\alpha}(ki_{\text{ret}})-x_{i\alpha}(\tau_i)] =-\tau_i(\zeta_{ik}{}^2-1)^{1/2}. \quad (18\text{d})$$

Hence (17a) and (17b) give

$$\int_{-\infty}^{\tau_{k1}} d\tau_k\hat{F}_{ik\alpha}(\tau_i,\tau_k)=\frac{\zeta_{ik}\dot{x}_{i\alpha}-\dot{x}_{k\alpha}}{2\tau_i{}^2(\zeta_{ik}{}^2-1)^{3/2}}\varphi_{ik}{}^3, \quad (19)$$

For (14b), this has to be integrated with respect to τ_i from τ_{i1} to the maximum value at which (17a) remains valid which is near ik_{adv}. However, even the integral from τ_{i1} to ∞ has a τ_{i1} in the denominator so that the linear interaction momenta go asymptotically to zero as $\tau_{i1}\to\infty$. This result was obtained already. [3] The calculation also shows that the small inaccuracy of Eqs. (18) does not affect the result.

The interaction angular momentum is given by (15b).

$$W_{ik\alpha\beta}(t_1)=-\int d\tau_i[\dot{x}_{i\alpha}\tau_i(\zeta_{ik}\dot{x}_{i\beta}-\dot{x}_{k\beta})-\dot{x}_{i\beta}\tau_i(\zeta_{ik}\dot{x}_{i\alpha}-\dot{x}_{k\alpha})]\varphi_{ik}{}^3 Z_{ik}/2\tau_i{}^2, \quad (20\text{a})$$

where

$$Z_{ik}=(\zeta_{ik}{}^2-1)^{-3/2}. \quad (20\text{b})$$

The lower limit of the integral in (20a) is τ_{i1}. It upper limit is given by ik_{adv} which is the limit of the validity of (17a). For $\tau_i>ik_{\text{adv}}$, the integrand must be replaced by 0, the correct value of the left side of (17a) if τ_i is in the light cone of all τ_k. Hence,

$$W_{ik\alpha\beta}(t_1)=\tfrac{1}{2}Z_{ik}\varphi_{ik}{}^3(\dot{x}_{i\beta}\dot{x}_{k\alpha}-\dot{x}_{i\alpha}\dot{x}_{k\beta})\ln(ik_{\text{adv}}/\tau_{i1}), \quad (21\text{a})$$

where ik_{adv} is given by an equation similar to (18a)

$$(ik_{\text{adv}})^2+\tau_{k1}{}^2-2\zeta_{ik}\tau_{k1}(ik_{\text{adv}})=0. \quad (21\text{b})$$

This permits the calculation of ik_{adv}. Finally, τ_{i1} and τ_{k1} have to be expressed in terms of t_1; since only the logarithms of these quantities appear in (21), $t_1/\dot{x}_{i0}$ and $t_1/\dot{x}_{k0}$ are sufficiently accurate expressions for them. We obtain, finally,

$$W_{ik\alpha\beta}(t_1)=\tfrac{1}{2}Z_{ik}\varphi_{ik}{}^3(\dot{x}_{i\beta}\dot{x}_{k\alpha}-\dot{x}_{i\alpha}\dot{x}_{k\beta}) \times\{\ln(\dot{x}_{i0}/\dot{x}_{k0})+\ln[\zeta_{ik}+(\zeta_{ik}{}^2-1)^{1/2}]\}. \quad (22)$$

The last ln drops out if one forms the symmetric expression $W_{ik\alpha\beta}(t_1)+W_{ki\alpha\beta}(t_1)$ which appears in the total interaction angular momentum (15a). Hence, the final expression in the asymptotic region, as defined above, is

$$W_{\alpha\beta}=\sum_{i<k} Z_{ik}\varphi_{ik}{}^3(\dot{x}_{i\beta}\dot{x}_{k\alpha}-\dot{x}_{i\alpha}\dot{x}_{k\beta})\ln(\dot{x}_{i0}/\dot{x}_{k0}). \tag{23}$$

Z_{ik} is given by (20b); $\varphi_{ik}{}^3$ by (17c).[5]

In the center-of-mass system for a collision of two particles, the interaction term (23) vanishes for the spatial components and all the components vanish if the two masses are equal. However, no component of $W_{\alpha\beta}$ will vanish in general in a collision of three or more particles and the aforementioned cancellations occur also only in the center-of-mass coordinate system.

4. KINEMATICAL ORIGIN OF THE INTERACTION ANGULAR MOMENTA

The existence of the aysmptotic interaction momenta has, to our knowledge, not been pointed out before.[6] It is worth while to mention, therefore, that their existence can be demonstrated also kinematically and that they are due to the relatively long-range nature of the inverse-square law of force.

The most simple case which shows this is a collision of two charged particles of equal mass, moving along the x axis, separating from each other. One has, asymptotically, in the coordinate system in which the center of mass is at rest

$$x_1=-x_2=a+vt-(e^2/4v^2)\ln t/t_0, \tag{24}$$

v being the terminal velocity. Transforming this to a moving coordinate system

$$x'=Cx+St \quad t'=Sx+Ct$$

$C^2-S^2=1$, one obtains, assuming that the logarithmic term in (24) is small as compared with vt and neglecting a

$$x_1'=\frac{Cv+S}{C+Sv}t'-\frac{e^2}{4v^2(C+Sv)}\ln\frac{t'/t_0}{C+Sv} \tag{25a}$$

and

$$x_2'=\frac{-Cv+S}{C-Sv}t'+\frac{e^2}{4v^2(C-Sv)}\ln\frac{t'/t_0}{C-Sv}. \tag{25b}$$

In order to calculate $M'=x_1'dt'/d\tau_1+x_2'dt'/d\tau_2-t'(dx_1'/d\tau_1+dx_2'/d\tau_2)$, one may note that the derivatives of x_1' and x_2' do not contain a logarithmic term; $dt'/d\tau_1=(C+Sv)(1-v^2)^{-1/2}$, $dt'/d\tau_2=(C-Sv)(1-v^2)^{-1/2}$ Hence one obtains

$$M'=\frac{e^2}{4v^2(1-v^2)^{1/2}}\ln\frac{C+Sv}{C-Sv}, \tag{26}$$

Since the whole M tensor vanishes in the original coordinate system, this (26) must be compensated by the interaction angular momentum. Clearly, if there is a logarithmic term in the distance-time relationship, the occurrence of an interaction angular momentum is a purely kinematical effect in the sense of the no-interaction theorem of Sec. 1. In classical electrodynamics, the asymptotic interaction angular momentum is part of the asymptotic angular momentum of the field. This part is distinguished from radiation by the way it transforms under Lorentz transformations.

[5] For $\varphi_{ik}(\rho)=e_ie_k(d/d\rho^2)\delta(\rho^2)$, one obtains the equations of motion proposed by J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).

[6] The presence of interaction angular momentum in collision processes has been recognized by a number of authors. It seems, however, that one has, so far, assumed this angular momentum to vanish asymptotically, i.e., as the particles separate. See, for instance, H. Yilmaz, *Introduction to the Theory of Relativity and the Principles of Modern Physics* (Blaisdell Publishing Company, New York, 1965), pp. 56–57. Also, see J. W. Dettman and A. Schild, Phys. Rev. **95**, 1059 (1954).