Light Fluctuations Due to an Intergalactic Flux of Gravitational Waves*

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A formula is derived that gives the time dependence of the intensity of a light beam that has passed through an arbitrarily varying gravitational field. The formula is valid to second order in the gravitational field strength and is valid only in the geometrical-optical limit for the light beam. An application of the formula to the case where there is only a single source of gravitational field, such as a binary star system, shows that the effect is exceedingly small. Then the case where space is filled uniformly with gravitational radiation is considered. It is found that for a small light source at a distance of a few billion light years the largest effect that can be expected amounts to an intensity variation of a thousandth of a percent. At larger distances it is conceivable that there would be observable intensity fluctuations; however, the formulas derived in this paper are not valid at such great distances.

I. INTRODUCTION

A FEW years ago it was suggested by Bertotti[1] that gravitational waves should interact with light waves in a way that could make the effect observable on an astronomical scale.

The reason for believing in such an effect is the following: It is known that under some circumstances when light waves travel through a gravitational field the field acts like a refracting medium. In this analogy the gravitational potentials take the place of the refractive index; indeed this analogy was used by Einstein to calculate the bending of light as it passes near a star.[2] It then seems reasonable that if the potentials vary in time, a light wave passing through the field will be bent to different degrees at different times and also that a beam of light will change in intensity as a function of time. The analog in ordinary refracting media is the twinkling of stars as viewed from below the atmosphere and is just due to fluctuations in the density and temperature of parts of the atmosphere.[3]

The purpose of the present paper is to investigate the twinkling of stars which may be caused by time-varying gravitational fields in interstellar and intergalactic space. Only the limiting case of geometrical optics (of the light) will be treated in this paper.

In Secs. II and III a formula is derived which gives the intensity of the observed light in terms of the gravitational fields through which the light passes. Most of the rest of the paper is devoted to applying the formula to various cases. Section IV deals with the case of a single source of gravity and it is shown the effect is probably unobservable. Section V deals with the case of a universe filled uniformly with gravitational radiation. Various quantities of interest are calculated and it is shown that the light fluctuations are probably unobservable. Even under the rather extreme conditions assumed for the calculation the intensity probably does not vary over a thousandth of a percent. A more exact calculation could conceivably show, however, that for extremely distant sources [$\simeq 10^{10}$ light years (l.y.)] there would be an observable effect. Sections VI and VII deal with two points that had been omitted from previous discussions and Secs. VIII and IX give the conclusions and acknowledgments.

II. GEOMETRICAL OPTICS

Maxwell's equations for the potentials in the presence of a gravitational field can be written[4,5]:

$$A_\mu{}^{;\nu}{}_{;\nu}+R_\mu{}^\nu A_\nu=-4\pi J_\mu\,,\quad A^\mu{}_{;\mu}=0\,. \tag{1}$$

A_μ is the Maxwell 4-potential, $R_\mu{}^\nu$ is the Ricci tensor and J_μ is the 4-current density. The Maxwell fields $F_{\mu\nu}$ are

$$F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}=A_{\nu,\mu}-A_{\mu,\nu}\,. \tag{2}$$

We will look for a solution of (1) of the form:

$$A_\mu=B_\mu\exp(iS)\,. \tag{3}$$

In the limit of geometrical optics the (scalar) phase, S, is a rapidly varying function of x^μ whereas the amplitude, B_μ, is a slowly varying function of x^μ. In the absence of a gravitational field S is just $k_\mu x^\mu$, where k_μ is the (constant) frequency 4-vector of the light wave. The frequencies contained in B_μ are those characteristic of the gravitational field and are assumed to be very much smaller than the frequency of the light wave. We set $J_\mu=0$ and insert (3) into (1).

$$B_\mu{}^{;\nu}{}_{;\nu}+R_\mu{}^\nu B_\nu-S^{,\nu}S_{,\nu}B_\mu=0\,, \tag{4}$$

$$2S^{,\nu}B_{\mu;\nu}+S^{,\nu}{}_{;\nu}B_\mu=0\,. \tag{5}$$

With the approximations inherent in geometrical optics

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[1] J. Weber (private communication).

[2] A. Einstein, A. A. Lorentz, H. Minkowski, and H. Weyl, *The Principle of Relativity* (Dover Publications, Inc., New York, 1923), p. 99.

[3] L. A. Chernov, *Wave Propagation in a Random Medium* (McGraw-Hill Book Company, Inc., New York, 1960).

[4] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961), p. 59.

[5] Summation over repeated indices is used throughout the paper. Greek indices run from 1–4; Latin indices run from 1–3. The signature of the metric is −2. Usually c and G will be set equal to one.

mentioned above we see that the solution of (4) in the first approximation is

$$S^{,\nu}S_{,\nu}=0. \tag{6}$$

This is just the usual equation that governs geometrical optics. The theory of geometrical optics (Hamilton's theory) tells us that we may introduce curves that are normal to the surfaces of constant phase, S.[6] These curves are the paths of light rays and $S_{,\mu}$ is their tangent vector field. On each ray this tangent can be written in a different form. Define

$$S^{,\mu}\equiv k^\mu=\lambda\frac{d}{du}x^\mu(u)\equiv\lambda U^\mu. \tag{7}$$

The functions $x^\mu(u)$ are the parametric equations of the ray and u is a "special parameter" on the ray.[7] λ is a constant on each ray but may vary from ray to ray; it can be chosen to make $k^\mu=S^{,\mu}$ on each ray [Eq. (7)]. k^μ is the frequency 4-vector of the wave and it is a null vector [Eq. (6)].

$$k_\mu k^\mu=0;\quad U_\mu U^\mu=0. \tag{8}$$

The rays are geodesics as can be seen by differentiating (8) and using (7). The parameters u and λ will be defined in more detail below. We can get an equation for $B_\mu B^\mu\equiv B^2$ by contracting (5) with B^μ.

$$\frac{d}{du}B^2+\lambda^{-1}k_{\nu;\nu}B^2=0. \tag{9}$$

Before proceeding with the solution of (9) we will define what is meant by the observed energy flux in the light beam. The energy flux, F_0, measured by the observer is given by the Poynting vector measured in his rest frame[8]:

$$F_0=n_0{}^a(T_{a4})_0. \tag{10}$$

The unit 3-vector $n_0{}^a$ points in the direction of the ray. Both $n_0{}^a$ and $(T_{a4})_0$ refer to the observer's (instantaneous Lorentz) *rest* frame. In this frame $n_0{}^a$ can be thought of as proportional to the 4-vector k^μ with the component parallel to the observer's world line subtracted off. In general then

$$n_0{}^\mu=N^{-1}[k^\mu-(k_\nu v_0{}^\nu)v_0{}^\mu],\quad n_{0\mu}n_0{}^\mu=-1. \tag{11}$$

Here $v_0{}^\mu=dx^\mu(s)/ds$ is the 4-velocity of the observer, $x^\mu(s)$ is his trajectory and s is his proper time. The normalization factor N is given by

$$N=k_\nu v_0{}^\nu\equiv\omega_0, \tag{12}$$

where ω_0 is the frequency of the light beam as measured by the observer [as can be seen by evaluating (12) in the observer's rest frame]. The stress tensor $T_{\mu\nu}$ can be calculated in the geometrical optical limit to be

$$T_{\mu\nu}=\frac{1}{4\pi}[F_\mu{}^\rho F_{\rho\nu}-\tfrac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma}]\simeq-\frac{1}{4\pi}k_\mu k_\nu B^2. \tag{13}$$

Equation (10) can now be written in the form[9]:

$$F_0=n_0{}^\mu(T_{\mu\nu})_0v_0{}^\nu\simeq(1/4\pi)\omega_0{}^2B^2. \tag{14}$$

We see that we only need to find B^2 from (9) in order to calculate F_0.

III. SOLUTION FOR B^2

In order to solve (9) for B^2 we need to evaluate $\lambda^{-1}k^\mu{}_{;\mu}$. It is a scalar and so we can define a convenient set of coordinates in which to evaluate it. Let one of these new coordinates $x^{1'}$ be the parameter u. Then

$$k^{\mu'}=\lambda\frac{d}{du}x^{\mu'}=\lambda\delta_1{}^\mu, \tag{15}$$

and

$$\lambda^{-1}k^\mu{}_{;\mu}=\frac{\partial}{\partial u}\ln[\lambda\sqrt{(-g')}], \tag{16}$$

where g' is the determinant of the metric $g_{\mu\nu}'$. In terms of the transformation coefficients between the primed and unprimed systems,

$$\sqrt{(-g')}=\eta_{\mu\nu\rho\sigma}\frac{\partial x^\mu}{\partial x^{1'}}\frac{\partial x^\nu}{\partial x^{2'}}\frac{\partial x^\rho}{\partial x^{3'}}\frac{\partial x^\sigma}{\partial x^{4'}}. \tag{17}$$

$\eta_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor of Levi-Civita.[10] The rest of the primed coordinates are conveniently defined as follows.[11] Choose an event P' on the world line C' of a point source of light and construct all of the null geodesics which are emitted at this event [Fig. 1(a)]. The coordinate u can be though of as a radial coordinate measured along these geodesics. One of these rays will intersect the world line C of the observer at the event P when the proper time, s, of the observer is s_1. We define the time coordinate $x^{4'}$ for each ray that intersects C to be s, the proper time of arrival as measured by the observer. For completeness define the value of $x^{4'}$ for all the other rays that leave P' to be s_1. The value of u for any event on C' will be defined to be zero; on C it will be defined to be a constant σ. The special parameter u can always be defined

[6] J. L. Synge, *Relativity: The General Theory* (North-Holland Publishing Company, Amsterdam, 1960), p. 25. Hereafter this book will be referred to as "Synge." The notation used in this paper is slightly different from that in "Synge." The main difference is that Synge's metric signature is +2 and the roles of Greek and Latin indices are interchanged.

[7] Synge, Ref. 6, p. 7.

[8] The subscript 0 will denote quantities pertaining to the observer and the subscript s will denote those pertaining to the light source.

[9] T. Fulton and F. Rohrlich, Ann. Phys. **9**, 499 (1960), Eq. (3.1).

[10] Synge, Ref. 6, p. 18.

[11] Synge, Ref. 6, p. 83. These coordinates are similar to Synge's "optical coordinates."

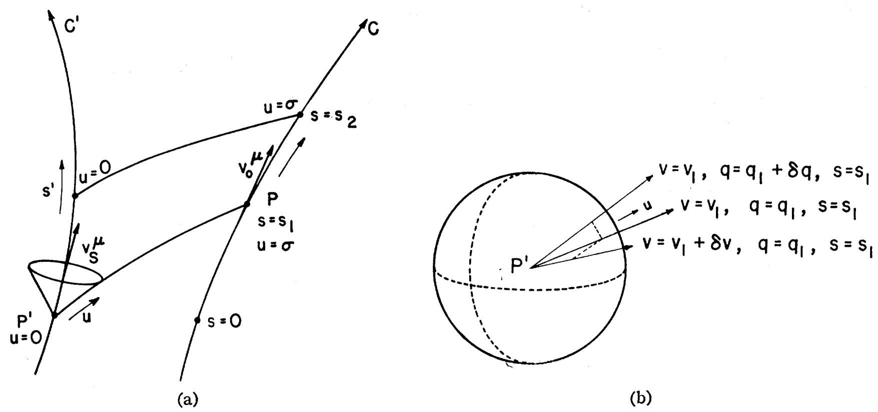

FIG. 1. (a) World lines of a light source C' and observer C, and two null geodesics connecting them. The various symbols are explained in the text. (b) Coordinate labels of three light rays leaving the source.

to make this true.[12] There are two more coordinates to be defined. Construct an infinitesimal spacial sphere around the event P' [Fig. 1(b)]. On the surface of this sphere we can set up a two-dimensional coordinate system. For our purposes it will be sufficient to define these coordinates only in the neighborhood of the ray $P'P$ of Fig. 1(a). Let the coordinates be p and q and let their values be p_1 and q_1 for the ray $P'P$. The coordinates of two neighboring rays are shown in Fig. 1(b). With these definitions (17) becomes

$$\sqrt{(-g')}=\eta_{\mu\nu\rho\sigma}\frac{\partial x^\mu}{\partial u}\frac{\partial x^\nu}{\partial p}\frac{\partial x^\rho}{\partial q}\frac{\partial x^\sigma}{\partial s}. \tag{18}$$

Now $\partial x^\mu/\partial u=U^\mu$ which is the tangent to the curves along which u varies. Similarly,

$$\frac{\partial x^\nu}{\partial p}\equiv a^\nu,\quad \frac{\partial x^\rho}{\partial q}\equiv b^\rho,\quad \frac{\partial x^\sigma}{\partial s}\equiv t^\sigma$$

are tangents to their respective coordinate lines. With this notation we define a quantity A.

$$A\equiv\lambda\sqrt{(-g')}=\eta_{\mu\nu\rho\sigma}k^\mu a^\nu b^\rho t^\sigma. \tag{19}$$

Inserting (19) and (16) into (9) gives

$$B^2=h/A, \tag{20}$$

where h is independent of u.

The reason for this rather detailed (though geometrically simple) set of definitions will now become evident because we have a way to calculate k, a, b, t in (19). In Fig. 1(b) the vector distance of (or deviation of) the ray labeled by $p_1+\delta p$ from the original ray $P'P$ is just given by $a^\mu\delta p$. This latter quantity is the deviation between two geodesics that are an infinitesimal distance apart. There is a well-known and fundamental equation that tells how $a^\mu\delta p$ varies as we move along the rays, namely, the "equation of geodesic deviation."[13]

[12] Synge, Ref. 6, p. 7.
[13] Synge, Ref. 6, p. 19.

It is

$$(\delta^2/\delta u^2)a^\mu-K^\mu{}_\nu a^\nu=0, \tag{21}$$

where

$$K^\mu{}_\nu=R^\mu{}_{\rho\nu\sigma}U^\rho U^\sigma, \tag{22}$$

and $R^\mu{}_{\rho\nu\sigma}$ is the Riemann-Christoffel curvature tensor.

$$R^\mu{}_{\rho\nu\sigma}=\Gamma^\mu{}_{\rho\nu,\sigma}-\Gamma^\mu{}_{\rho\sigma,\nu}+\Gamma^\mu{}_{\alpha\sigma}\Gamma^\alpha{}_{\rho\nu}-\Gamma^\mu{}_{\alpha\nu}\Gamma^\alpha{}_{\rho\sigma}, \tag{23}$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel symbol of the second kind. Note that the "angle" between a^ρ and U^ρ is independent of u.

$$\frac{d}{du}(a_\rho U^\rho)=U^\rho\frac{\delta}{\delta u}a_\rho=U^\rho\frac{\delta}{\delta p}U_\rho=\frac{1}{2}\frac{d}{dp}(U^\rho U_\rho)=0. \tag{24}$$

All of the vectors that appear in (19) satisfy an equation of geodesic deviation so that the problem reduces to solving (21).

Equation (21) is rather complicated in that it involves covariant derivatives. It can be greatly simplified by introducing an orthonormal set of basis vectors (tetrad), $\lambda_{(\alpha)}{}^\mu$.[14] The subscript in parentheses is not a tensor index but merely denotes the αth basis vector ($\alpha=1-4$); the summation convention is used on these tetrad indices. The main properties of the tetrad are

$$\begin{aligned}&\lambda_{(\alpha)}{}^\mu\lambda_{(\beta)\mu}=\eta_{\alpha\beta}, \quad \lambda^{(\alpha)\mu}\equiv\eta^{\alpha\beta}\lambda_{(\beta)}{}^\mu,\\ &\lambda_{(\alpha)}{}^\mu\lambda^{(\alpha)\nu}=g^{\mu\nu}, \quad \eta^{\alpha\beta}=\mathrm{diag}(-1,-1,-1,1).\end{aligned} \tag{25}$$

Such a tetrad can be defined at some event on the world line of the *light source* C'. It is constructed at other events on C' by parallel transport along C', and it is constructed at any other event in space-time by parallel transport along the future pointing null geodesic that connects C' to the event. That is,

$$\frac{\delta}{\delta s'}\lambda_{(\alpha)}{}^\mu\big|_{\text{on } C'}=0,\quad \frac{\delta}{\delta u}\lambda_{(\alpha)}{}^\mu=0, \tag{26}$$

where s' is the proper time of the source. It is easy to show that the relations in (25) are preserved everywhere.

[14] Synge, Ref. 6, p. 8.

The invariant components of a tensor are defined as

$$T^{(\alpha\beta\cdots}{}_{\gamma\delta\cdots)}\equiv\lambda^{(\alpha)}{}_{\mu}\lambda^{(\beta)}{}_{\nu}\ldots\lambda_{(\gamma)}{}^{\rho}\lambda_{(\delta)}{}^{\sigma}\ldots T^{\mu\nu\cdots}{}_{\rho\sigma\cdots}. \quad (27)$$

They do not depend on the coordinate system chosen; however, they do depend on the particular tetrad chosen. The particular tetrad that we will use will be defined more precisely below.

All of our equations can be written in terms of invariant components. In particular

$$A=\epsilon_{(\alpha\beta\gamma\delta)}k^{(\alpha)}a^{(\beta)}b^{(\gamma)}t^{(\delta)}, \quad (28)$$

$$\frac{d^2}{du^2}l^{(\alpha)}-K^{(\alpha}{}_{\beta)}l^{(\beta)}=0,\quad \frac{d}{du}(U_{(\alpha)}l^{(\alpha)})=0, \quad (29)$$

$$K^{(\alpha}{}_{\beta)}=R^{(\alpha}{}_{\gamma\beta\delta)}U^{(\gamma)}U^{(\delta)}. \quad (30)$$

In (28) $\epsilon_{(\alpha\beta\gamma\delta)}$ is the ordinary flat-space permutation symbol and in (29) $l^{(\alpha)}$ represents any of the scalars $k^{(\alpha)}$, $a^{(\alpha)}$, $b^{(\alpha)}$, $t^{(\alpha)}$. The advantage of this procedure can be seen in (29) where only ordinary derivatives appear. Equation (29) can be converted to an integral equation.

$$l^{(\alpha)}(u)=l^{(\alpha)}(0)+u\frac{d}{du}l^{(\alpha)}(0)+\int_0^u du'\int_0^{u'}du'' \times K^{(\alpha}{}_{\beta)}(u'')l^{(\beta)}(u''). \quad (31)$$

This equation can be solved by iteration but first we will define the initial conditions and the tetrad more precisely.[15]

Equation (29) for the wave vector k^α is trivial because k^α is tangent to a geodesic.

$$(d/du)k^\alpha=0. \quad (32)$$

The invariant components of the wave vector are constant along any particular ray. Define $\lambda_{(4)}{}^{\mu}$ on C' to be equal to $v_s{}^{\mu}$ the 4-velocity of the source. Then

$$k_{(4)}=\lambda_{(4)}{}^{\mu}k_\mu=v_s{}^{\mu}k_\mu=\omega_s, \quad (33)$$

where ω_s is the frequency of the light beam as measured by the source.

We are interested in a point source of light; therefore choose

$$a^\alpha(0)=b^\alpha(0)=0;\quad \frac{d}{du}a^\alpha(0)=\theta^\alpha,\quad \frac{d}{du}b^\alpha(0)=\varphi^\alpha. \quad (34)$$

θ and φ are related to the (small) angular divergence of the light beam.

The vector t^α is best specified by its values on C' and C rather than its initial value and slope. This is because we want the two rays which are $t^{(\alpha)}ds$ apart to both start from C' and both end on C [see Fig. 1(a)] and this is assured by specifying the initial and final values of t^α. A formula similar to (31) can be derived for this case[16] but it will not be needed. All we will need are the initial and final values.

$$t^\alpha(0)=t_s{}^\alpha=\lambda^\alpha{}_\mu\frac{d}{ds}x^\mu\bigg|_{\text{on } C'}=v_s{}^\alpha\frac{ds'}{ds}\bigg|_{\text{on } C'}=v_s{}^\alpha\frac{\omega_0}{\omega_s};$$
$$t^\alpha(\sigma)=v_0{}^\alpha. \quad (35)$$

From (14) and (20) we see that we only need A evaluated at the observer so that we have left to find a^α and b^α on C. The solution to (31) correct to second order in the gravitational field is

$$\begin{aligned} a^\beta &= u\theta^\beta+\int_0^u du'\int_0^{u'}du''K^\beta{}_\gamma a^\gamma \\ &\simeq u\theta^\beta+\theta^\gamma\int_0^u du'\int_0^{u'}du''u''K^\beta{}_\gamma \\ &\quad +\theta^\delta\int_0^u du'\int_0^{u'}du''K^\beta{}_\gamma\int_0^{u''}dv\int_0^v dv'v'K^\gamma{}_\delta \\ &\equiv u\theta^\beta+M^\beta{}_\gamma(u)\theta^\gamma+P^\beta{}_\gamma(u)\theta^\gamma. \end{aligned} \quad (36)$$

$$b^\beta\simeq u\varphi^\beta+M^\beta{}_\gamma(u)\varphi^\gamma+P^\beta{}_\gamma(u)\varphi^\gamma. \quad (37)$$

The functions $M^\beta{}_\gamma(u)$ which are first order in $K^\beta{}_\gamma$ and $P^\beta{}_\gamma(u)$ which are second order in $K^\beta{}_\gamma$ can be evaluated explicitly once the gravitational fields are known.

In order to simplify the remaining calculations, the reference tetrad will be defined explicitly. We already have defined $\lambda_{(4)}{}^\mu$ to be parallel to the source's time axis (33). Choose $\lambda_{(3)}{}^\mu$ to point in the spacial direction of the wave vector. $\lambda_{(1)}{}^\mu$, $\lambda_{(2)}{}^\mu$, and $\lambda_{(3)}{}^\mu$ are just the $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ unit vectors in the locally Minkowskian frame of the source. Then

$$k^\alpha=\omega_s(\delta_3{}^\alpha+\delta_4{}^\alpha). \quad (38)$$

Choose

$$\theta^\alpha=\theta\delta_1{}^\alpha,\quad \varphi^\alpha=\varphi\delta_2{}^\alpha. \quad (39)$$

The light rays define a small rectangular pyramid whose apex is at the source and which points along the 3-axis. Finally we note that $v_0{}^\alpha$ can be interpreted as the velocity of the observer relative to the source and in general it has all components present. Inserting (36)–(39) into (28) and evaluating it at the source we obtain

$$\begin{aligned} A_0\equiv A(\sigma) &= \omega_0\sigma^2\theta\varphi+\omega_0\sigma\theta\varphi(M^1{}_1+M^2{}_2)_0 \\ &\quad +\omega_0\theta\varphi[\sigma(P^1{}_1+P^2{}_2)+(M^1{}_1M^2{}_2-M^2{}_1M^1{}_2)]_0. \end{aligned} \quad (40)$$

We have used the fact that [from (22)]

$$K^\alpha{}_\beta U^\beta\equiv0 \quad (41)$$

and that

$$\omega_0/\omega_s=v_0{}^{(4)}-v_0{}^{(3)}. \quad (42)$$

[15] For the rest of the paper the parentheses around tetrad indices will be dropped; in cases where confusion may arise they will be reinstated. As a precaution against confusion, whenever possible the first letters of the Greek alphabet will be tetrad indices $(\alpha,\beta\cdots)$ and the later letters will be tensor indices $(\mu,\nu\cdots)$. For convenience quantities like k^α will be called 4-vectors whereas they are really four scalars.

[16] Synge, Ref. 6, p. 23.

The last equation is just the usual formula for the Doppler shift written in terms of 4-velocities rather than 3-velocities.[17]

Formula (40) has a simple physical (geometrical) interpretation. In the absence of gravity only the first term survives and it is just equal to the cross-sectional area of the beam (times ω_0) at the observer. The rest of the terms can then be interpreted as the change in the cross section due to differential bending of light rays by the gravitational field (see Fig. 2).

If we make use of (41) and the definitions of $M^\alpha{}_\beta$ (36) we find

$$M^1{}_1+M^2{}_2=M^\alpha{}_\alpha=\int\int u''K^\alpha{}_\alpha du''. \tag{43}$$

But

$$K^\alpha{}_\alpha=R^\alpha{}_{\beta\alpha\gamma}U^\beta U^\gamma=R_{\beta\gamma}U^\beta U^\gamma. \tag{44}$$

This is zero in free space. We are not in free space because there is at least a nonzero electromagnetic stress-tensor around. But $T_{\alpha\beta}U^\alpha U^\beta=0$ for the Maxwell stress tensor of the light beam and this implies (44) is zero also. Of course, there may be other stresses around such as those caused by a matter distribution. These can give nonzero results for (44), however we will defer consideration of this case until Sec. VI [where we will find that for the case considered the contribution from (44) is small] and take the case where there is no first-order effect.

$$K^\alpha{}_\alpha=0. \tag{45}$$

A detailed look at the first-order terms in (36) and (37) when (45) holds shows that what happens is that when a^γ increases, b^γ decreases such that their product is constant. Geometrically what happens is that if the initial cross section of the light beam is square (for instance), it becomes deformed into a rectangle or diamond of the same area. This is a rather general property of free gravitational fields.[18]

Using (41) and (45), we find

$$P^1{}_1+P^2{}_2=P^\alpha{}_\alpha \tag{46}$$

and

$$M^1{}_1M^2{}_2-M^2{}_1M^1{}_2=-\tfrac{1}{2}M^\alpha{}_\beta M^\beta{}_\alpha. \tag{47}$$

A_0 can finally be written as

$$A_0=\omega_0\sigma^2\theta\varphi\left[1+\frac{\Delta A_0}{A_0}\right], \tag{48}$$

$$\frac{\Delta A_0}{A_0}=\frac{1}{\sigma}\int_0^\sigma du\int_0^u du' K^\alpha{}_\beta(u')\int_0^{u'} dv\int_0^v dv'v'K^\beta{}_\alpha(v') -\frac{1}{2\sigma}\left[\int_0^\sigma du\int_0^u du'u'K^\alpha{}_\beta(u')\right] \times\left[\int_0^\sigma du\int_0^u du'u'K^\beta{}_\alpha(u')\right]. \tag{49}$$

[17] J. D. Jackson, *Classical Electrodynamics* (John Wiley & Sons, Inc., New York, 1962), p. 363.
[18] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).

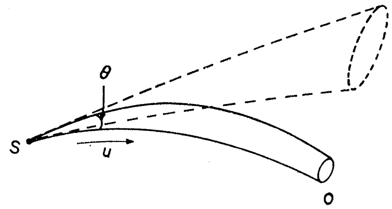

FIG. 2. A light beam in free space (dashed) and the beam when a gravitational field is present (solid). Different rays are bent to different degrees and so the cross-sectional area changes as well as the direction of the beam.

We note that (49) is independent of the choice made in (38) and (39) and it is also independent of our choice of tetrads.

In order to complete the derivation of the energy received by the observer we have to compute the function h in (20). This can be done by assuming that in the rest frame of the source the energy flux and frequency of the light beam are independent of time. The flux near the source can be obtained from (11) and (14) by replacing the subscript 0 by s. For a point source it is more meaningful to use the power radiated per unit solid angle P_Ω, rather than flux. Construct an infinitesimal sphere of proper radius r around the source. The light beam in (49) passes through a small area $r^2\theta\varphi$ and so the power ΔP passing through this area is

$$\Delta P\equiv P_\Omega\theta\varphi=n_s{}^\alpha T_{\alpha\beta}v_s{}^\beta r^2\theta\varphi=\frac{h\omega_s{}^2}{4\pi A(u)}r^2\theta\varphi. \tag{50}$$

$A(u)$ is obtained from (28), (33)–(35). In the limit that u (and r) go to zero, (50) becomes

$$P_\Omega\theta\varphi=\frac{h\omega_s{}^2}{4\pi\omega_0}\left(\frac{dr}{du}\right)_s^2, \tag{51}$$

which gives us h in terms of the constant P_Ω. Combining (14), (20), (48), (49), and (51), we obtain

$$F_0=\frac{P_\Omega}{\sigma^2}\left(\frac{\omega_0}{\omega_s}\right)^2\left(\frac{dr}{du}\right)_s^{-2}\left(1-\frac{\Delta A_0}{A_0}\right). \tag{52}$$

If we assume that the characteristic length of variation of the gravitational field is large compared to the telescope objective, then the total power P_0 received by the telescope (of area A_T) is

$$P_0=P_\Omega\frac{A_T}{\sigma^2}\left(\frac{\omega_0}{\omega_s}\right)^2\left(\frac{dr}{du}\right)_s^{-2}\left(1-\frac{\Delta A_0}{A_0}\right). \tag{53}$$

The factor $(dr/du)_s{}^2$ can be evaluated by using (25).

$$\left(\frac{dr}{du}\right)_s^2=-\lambda_{(a)\mu}\lambda^{(a)}{}_\nu\frac{dx^\mu}{du}\frac{dx^\nu}{du}\bigg|_s=-(g_{\mu\nu}-v_{s\mu}v_{s\nu})U^\mu U^\nu|_s =(v_s{}^\mu U_\mu)_s{}^2=(\lambda^{-1}\omega_s)^2=\left(\frac{\omega_s}{\omega_0}\right)^2(v_0{}^\mu U_\mu)^2. \tag{54}$$

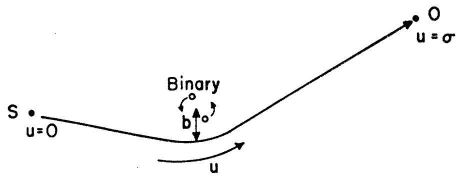

FIG. 3. The configuration used for a calculation of the light intensity variations induced by a binary star system that is near the path between the light source S and observer O.

The quantity $\sigma(v_0{}^\mu U_\mu)$ in flat space is just the observer-source distance as measured by the observer. It is natural to define this scalar to be the observer-source distance L in general.[19]

Our final result is

$$P_0=\left(P_\Omega\frac{A_T}{L^2}\right)\left(\frac{\omega_0}{\omega_s}\right)^4\left(1-\frac{\Delta A_0}{A_0}\right). \qquad (55)$$

The physical interpretation of the three factors in (55) is simple. The first factor is just the power that the observer would measure if there were no gravitational effects and if the source were at rest with respect to the observer. The second factor comes from three effects. One factor of (ω_0/ω_s) is just due to the Doppler shift of the emitted frequency. Another factor comes from a change in the rate of emission of photons by the source because of its motion with respect to the observer (time dilation plus retardation).[20] The last two factors of (ω_0/ω_s) are due to the aberration of the emitted light because of the source's relative motion. The last factor in (55) contains the gravitational effect and can be interpreted as due to a change in the cross-sectional area of the light beam caused by differential bending of the light rays induced by the gravitational fields through which they pass.

IV. SOME MODEL CALCULATIONS

As a first example we will consider a light beam that passes close by a binary star system (Fig. 3). A look at (49) would indicate that the radiation field from the binary may give a large effect because there would be appreciable contributions to the integrals from all along the light path. A closer study shows this to be untrue, however, because of the transverse character of gravitational radiation. At large distances from the binary system the direction of the gravitational radiation is almost parallel to the light beam and under this circumstance the quantity $K^\alpha{}_\beta$ approaches zero.[21] Further study of the integrals then shows that the near fields (Newtonian fields) give the largest contribution. Under this condition there is a much faster way to calculate the effect than by (49). Liebes[22] has calculated the intensification of starlight that passes through the Newtonian field around a star. His technique is valid when the gravitational field is static (or quasistatic); his result is not restricted to weak fields as is (49). Evaluation of (49) for the binary system gives a result in agreement with that of Liebes and since the effect is very small, only the final answer will be given

$$\frac{\Delta A_0}{A_0}\simeq-3\left(\frac{8Gmd}{c^2b^3}\right)^2L_1^2\left(1-\frac{L_1}{L}\right)^2\cos2(\omega t+\varphi). \qquad (56)$$

The stars, each of mass m and a distance d apart, rotate in the x-y plane and the light beam travels in the z direction and passes at a distance b from the binary. The frequency of the binary is ω and it has been assumed that ωb is much less than one. L is the source-observer distance and L_1 is the source-binary distance. Equation (56) is the largest time-varying part of the effect and arises from the interference between the static and dynamic fields around the binary. In (56) we see that the effect is small if the binary is near the source or observer. This is because if the binary is near the source the light rays are very close together and so the "force" tending to give relative bending is small [Eq. (29)]; if the binary is near the observer the "force" is relatively large but the rays do not have enough time to converge or diverge much by the time they reach the observer. If we take $m=M_\odot=2\times10^{33}$ g, a binary period of ten years, $b=1$ l.y., $L=2\times10^9$ l.y., and $L_1=10^9$ l.y. we get:

$$\frac{\Delta A_0}{A_0}\simeq-10^{-14}\cos2(\omega t+\varphi), \qquad (57)$$

an extremely small effect. A similar calculation using the lowest quadrupole vibrational mode of a perfect fluid sphere as the source of gravitation gives a result comparable to (57). The other factors in the received power (55) have been estimated and seem to have at most a logarithmic dependence on L. They give much smaller contributions than (57) does to the received power.

We saw above that the wave field does not contribute to the result when the source of gravity is near the light beam. We now consider a more distant gravity source, in fact, we will consider the effect of a gravitational plane wave pulse on the beam. In the linearized version of Einstein's theory the deviations $h_{\mu\nu}$ of the metric from the Minkowski metric $\eta_{\mu\nu}$ can be written[23]

$$h_{\mu\nu}=E(n_\mu n_\nu+n_\mu{}^*n_\nu{}^*)f(\alpha_\rho x^\rho), \qquad (58)$$

where E is the (real) amplitude of the wave, α_ρ is a null vector giving the direction of propagation and n_μ is a complex vector orthogonal to α_ρ.[24]

$$n_\mu{}^*n^\mu=-1,\quad n_\mu n^\mu=0,\quad n_\mu\alpha^\mu=0,\quad \alpha_\mu\alpha^\mu=0. \qquad (59)$$

The (real) function f will be chosen to be an arbitrarily

[19] Synge, Ref. 6, p. 86.
[20] G. C. McVittie, *Fact and Theory in Cosmology* (The Macmillan Company, New York, 1961), p. 114.
[21] L. Witten, *Gravitation: An Introduction to Current Research* (John Wiley & Sons, Inc., New York, 1962), p. 215–217.
[22] S. Liebes, Phys. Rev. **133**, B835 (1964).
[23] Reference 21, p. 218
[24] See Ref. 23.

shaped pulse which, for convenience, will be chosen to be zero at the source and observer. The Riemann tensor (23) is then given by

$$R_{\mu\nu\rho\sigma}=\tfrac{1}{2}E[(\alpha_\mu n_\nu-\alpha_\nu n_\mu)(\alpha_\rho n_\sigma-\alpha_\sigma n_\rho)+\text{c.c.}]f''(\alpha_\lambda x^\lambda), \tag{60}$$

where f'' is the second derivative of f with respect to its argument. Assume that the source and observer are in flat space and at rest with respect to each other; then the difference between invariant components and tensor components in (49) are of higher order in the Riemann tensor and can be neglected. When we insert (60) into (49) we have to express the argument of f'' as a function of u. Choose the path of the light ray along the z axis (to lowest order), then $x=y=0$, $z\rightarrow u$, and $t\rightarrow t+u$. This puts the source at the origin of coordinates and makes t the time that the ray leaves the source. After integrating by parts a few times we obtain for the leading term (in the distance L) in $\Delta A_0/A_0$:

$$\frac{\Delta A_0}{A_0}\simeq-\frac{E^2\xi^2}{2L}\int_0^L u(L-u)[f'(\alpha_4 t+\xi u)]^2du. \tag{61}$$

Here L, the source-observer distance, replaces σ to a sufficient approximation and $\xi=\alpha_\mu U^\mu=\alpha_4(1-\cos\theta)$ where θ is the angle between the direction of the light beam and the gravitational wave. Take f to be localized in a small region of length ΔL at a distance L_1 from the observer. Choose $\alpha_4=(\Delta L)^{-1}$; then f is nonzero only when its argument is between zero and one. Normalize f such that the integral of its square is unity; this is consistant with calling E in (58) the amplitude of the wave. Under these conditions and if f is reasonably smooth then the integral of the square of f' is also of order unity. Then

$$\frac{\Delta A_0}{A_0}\simeq-\tfrac{1}{2}E^2L_1\left(1-\frac{L_1}{L}\right)\frac{1}{\Delta L}(1-\cos\theta). \tag{62}$$

It should be noted that this result is independent of the details of f; indeed, even if f were a truncated sine wave with N wiggles the result would depend mainly on the total length of the wave and only to a small extend on its frequency. The result in (62) is quite general and does not depend critically on the plane wave character of f. Therefore we will regard f as a gravitational wave of dimension ΔL in all directions, i.e., a highly directed pulse of radiation. The intensification of the light beam [given by the negative of (62)] lasts for a time of order $\Delta L/c(1-\cos\theta)$; this is just the time that the light beam is inside the wave pulse.

As a model for calculating the dimensionless amplitude E, consider the extreme case of an explosion of a massive star or galaxy such that the explosion has a large dynamic quadrupole moment. Such an explosion could consist of two large masses being blown apart such that they move in opposite directions at high velocity. Rather than calculate E in detail we can make a reasonable guess at the result by analogy with results in electrodynamics[25]

$$E\simeq\frac{GM}{c^2r}\left(\frac{v}{c}\right)^2, \tag{63}$$

where M is the mass in the explosion that moves with velocity v and r is the distance from the explosion to the light ray. The factor GM/c^2r is characteristic of all gravity calculations involving finite source sizes and the factor $(v/c)^2$ is characteristic of quadrupole radiation. Using $M=10^8M_\odot$, $r=10^3$ l.y., $v/c=10^{-2}$, $L=2L_1=10^9$ l.y., $\Delta L=1$ l.y., $\theta=90°$, Eqs. (62) and (63) give

$$\Delta A_0/A_0\simeq-10^{-15}. \tag{64}$$

Even with this extreme example of an explosion of a galaxy the result is negligible. It would appear that it is difficult to get an observable effect from a single source.

V. RADIATION-FILLED UNIVERSE

A. Hubble's Constant

The next thing is to calculate the effect of many sources. As a reasonable and simplifying assumption we will assume that space is filled homogeneously and isotropically with gravitational radiation of arbitrary frequency spectrum and then try to see what spectrum (if any) can give an observable effect.

First of all we will make a calculation of Hubble's constant in order to get some information about the spectrum, and also to show the calculational technique that will be employed. We will assume that the curvature of space is caused by this distribution of gravitational radiation; that is, we will assume that gravitational radiation is the major constituent of the universe. This assumption may even be correct because the measured energy density in space is not large enough to account for the measured red shift, at least in those Friedmann universes where the cosmological constant is taken to be zero.[26] The results that we obtain should always be thought of as upper limits to the actual power spectrum.

For convenience in calculating we will express the radiation as a discrete sum of plane waves. These waves will be confined to a large cube of dimension L_0 so that the components of the wave vectors of the various waves will be integral multiples of $2\pi/L_0$. Eventually we will pass to the limit of $L_0\rightarrow\infty$. In order to make the problem tractable we will use an iterative method to solve the vacuum field equations $G_{\mu\nu}=0$. This is done by rearranging the terms in the equations such that all the linear terms are on the left and the nonlinear terms are on the right. The general form of the resulting equations is that of a wave equation with source terms. Since

[25] Reference 17, p. 526.
[26] Reference 20, p. 145.

the source terms are at least quadratic in the potentials, they will be all equal to zero in the first iteration. The resulting homogeneous wave equations have the usual plane wave solutions.

$$h_{\mu\nu}=\sum_k E_{\mu\nu}(\mathbf{k})e^{ik_\sigma x^\sigma}+\text{c.c.}, \tag{65}$$

where

$$E_{\mu\nu}=E(\mathbf{k})(n_\mu n_\nu+n_\mu^* n_\nu^*), \quad E_{\mu\nu}k^\nu=0. \tag{66}$$

The definitions of the various quantities are the same as in (58) with α_μ replaced by ik_μ. It should be noted that the sum on k in (65) is over its spacial components; k_4 is determined by the wave equation.

$$k_4=|\mathbf{k}|\geqslant 0. \tag{67}$$

The second iteration is performed by inserting (65) into the right-hand side (RHS) of the rearranged field equations mentioned above, retaining only terms to second order in $h_{\mu\nu}$, and solving the resulting inhomogeneous wave equations. Instead of doing this we will get the answer correct to second order by treating the RHS as an ordinary stress-energy tensor and inserting it into the exact field equations.

$$G_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}\ {}_{\mu\nu}R=-8\pi T_{\mu\nu}\equiv\text{RHS}. \tag{68}$$

$$\begin{aligned}\text{RHS}=&-\tfrac{1}{2}[h^{\alpha\beta}(h_{\mu\beta,\nu}+h_{\nu\beta,\mu}-h_{\mu\nu,\beta})]_{,\alpha}\\&+\tfrac{1}{4}(h^{\alpha\beta}h_{\alpha\beta})_{,\mu\nu}-\Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}+\Gamma^\alpha{}_{\mu\nu}\Gamma^\beta{}_{\alpha\beta}.\end{aligned} \tag{69}$$

Consider the third term

$$\begin{aligned}\Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}=\sum_k\sum_l\ &[B_{\mu\alpha}{}^\beta(\mathbf{k})B_{\nu\beta}{}^\alpha(\mathbf{l})e^{i(k_\sigma+l_\sigma)x^\sigma}\\&+B_{\mu\alpha}{}^\beta(\mathbf{k})B^*{}_{\nu\beta}{}^\alpha(\mathbf{l})e^{i(k_\sigma-l_\sigma)x^\sigma}+\text{c.c.}],\end{aligned} \tag{70}$$

where

$$B_{\nu\beta}{}^\alpha(\mathbf{k})=\tfrac{1}{2}i(E_\nu{}^\alpha k_\beta+E_\beta{}^\alpha k_\nu-E_{\nu\beta}k^\alpha). \tag{71}$$

The quantity that is of interest in the usual cosmological equations is not the actual stress tensor but rather the smoothed stress tensor. In keeping with these approximations we will ensemble average (69) and (70). To do this we assume that each wave is only correlated with itself.

$$\begin{aligned}\langle B_{\mu\alpha}{}^\beta(\mathbf{k})B_{\nu\beta}{}^\alpha(\mathbf{l})\rangle&=\langle B_{\mu\alpha}{}^\beta(\mathbf{k})B_{\nu\beta}{}^\alpha(\mathbf{k})\rangle\delta^3{}_{kl}\\&=\langle|B_{\mu\alpha}{}^\beta(\mathbf{k})B_{\nu\beta}{}^\alpha(\mathbf{k})|e^{2i\varphi(\mathbf{k})}\rangle\delta^3{}_{kl}=0,\end{aligned} \tag{72}$$

where $\varphi(\mathbf{k})$ is the phase of the wave and is uniformly distributed between 0 and 2π.

$$\begin{aligned}\langle B_{\mu\alpha}{}^\beta(\mathbf{k})B^*{}_{\nu\beta}{}^\alpha(\mathbf{l})\rangle&=\langle|B_{\mu\alpha}{}^\beta(\mathbf{k})B^*{}_{\nu\beta}{}^\alpha(\mathbf{k})|\rangle\delta^3{}_{kl}\\&=\tfrac{1}{4}\langle|E^*{}_\beta{}^\alpha E_\alpha{}^\beta|\rangle k_\mu k_\nu\delta^3{}_{kl}\\&=\tfrac{1}{2}\langle|E(\mathbf{k})|^2\rangle k_\mu k_\nu\delta^3{}_{kl},\end{aligned} \tag{73}$$

so that

$$\langle\Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\nu\beta}\rangle=\sum_k\langle|E(\mathbf{k})|^2\rangle k_\mu k_\nu. \tag{74}$$

It turns out that the rest of the terms in (69) average to zero. From (58):

$$T_{\mu\nu}=-\frac{1}{8\pi}\langle\text{RHS}\rangle=\frac{1}{8\pi}\sum_k\langle|E(\mathbf{k})|^2\rangle k_\mu k_\nu. \tag{75}$$

Since we assumed that the radiation is isotropic, E only depends on the magnitude of $\mathbf{k}$. We now convert from a sum to an integral and integrate over the directions of $\mathbf{k}$.

$$T_{\mu\nu}=\tfrac{1}{2}\lim_{L_0\to\infty}\left(\frac{L_0}{2\pi}\right)^3\int_0^\infty\langle|E(k)|^2\rangle k^4dk\times\text{diag}(-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},1). \tag{76}$$

This is just the form for the stress tensor for a radiation-filled universe.[27] We denote the density ρ_0 and the pressure p_0 by

$$\rho_0=3p_0=\tfrac{1}{2}\lim_{L_0\to\infty}\left(\frac{L_0}{2\pi}\right)^3\int_0^\infty\langle|E(k)|^2\rangle k^4dk. \tag{77}$$

Let us define the power spectrum $P(k)$ of the Riemann tensor by[28]

$$\langle K^{\alpha\beta}K_{\alpha\beta}\rangle\equiv\int_0^\infty P(k)dk\quad(\text{units of cm}^{-4}), \tag{78}$$

where $K_{\alpha\beta}$ is given by (22). (It would be more sensible to define the power spectrum through an average value of some scalar quadratic form of the Riemann tensor itself. It turns out that all of these are identically zero for plane waves and so we have to settle for $\langle K^{\alpha\beta}K_{\alpha\beta}\rangle$.) A calculation of $\langle K^{\alpha\beta}K_{\alpha\beta}\rangle$ in the same manner as above gives

$$\langle K^{\alpha\beta}K_{\alpha\beta}\rangle=\frac{64\pi}{5}\lim_{L_0\to\infty}\left(\frac{L_0}{2\pi}\right)^3\int_0^\infty\langle|E(k)|^2\rangle k^6dk. \tag{79}$$

Then

$$\rho_0=3p_0=\frac{5}{128\pi}\int_0^\infty P(k)\frac{dk}{k^2}. \tag{80}$$

The relation between Hubble's constant and the density depends on how far the universe is along in its expansion phase. If we assume that it still has a long time to go before it reaches its maximum size, we find by manipulating Tolman's equations[29] that

$$H^2\simeq\frac{8\pi}{3}\rho_0=\frac{5}{48}\int_0^\infty P(k)\frac{dk}{k^2}, \tag{81}$$

where H is Hubble's constant ($H\cong 100$ km/sec Mpc $\cong 10^{-10}$ yr^{-1}) (Mpc = megaparsec). Putting in the c's

[27] R. C. Tolman, *Relativity, Thermodynamics and Cosmology* (Oxford at the Clarendon Press, Oxford, 1934), p. 219.

[28] For a formal definition of power spectrum see: A. van der Ziel, *Noise* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1954), Chap. 12.

[29] Reference 27, p. 413.

and converting from wave number to (angular) frequency, we get

$$\int_0^\infty P(\omega)\frac{d\omega}{\omega^2}\simeq\frac{48}{5}\frac{H^2}{c^4}\simeq 10^{-76}\ \mathrm{sec}^2\ \mathrm{cm}^{-4}. \tag{82}$$

The power spectrum must drop off faster than ω at low frequencies and must not increase as fast as ω at high frequencies in order that the integral in (82) be finite. As an example of a simple power spectrum that has these properties let

$$P(\omega)=P_0\frac{\omega_0^2\omega^2}{\omega_0^4+\omega^4}. \tag{83}$$

This reaches a maximum at about ω_0 with a peak value of about P_0. From (82)

$$P_0/\omega_0\simeq 10^{-76}. \tag{84}$$

An experimental upper limit for $P(\omega)$ in the vicinity of one cycle per hour has been found to be 10^{-75} cm^{-4} sec [30] so that if we arbitrarily set $\omega_0=10^{-3}$ sec^{-1} we get

$$P_0\simeq 10^{-79}\ \mathrm{cm}^{-4}\ \mathrm{sec}. \tag{85}$$

This would perhaps indicate that in the near future experiments will be performed that will actually be able to set cosmologically meaningful limits on $P(\omega)$.

B. Doppler Shift

We will now consider fluctuations, due to the above flux, in the first-order Doppler shift and in the first order jitter of the apparent position of a light source.

The Doppler shift can be obtained from (54).

$$\frac{\omega_0}{\omega_s}=\frac{v_0{}^\mu U_\mu}{v_s{}^\nu U_\nu}. \tag{86}$$

A calculation of this ratio to first order, using Synge's notation[31] yields:

$$\frac{\omega_0}{\omega_s}=1-\int_0^\sigma(\sigma-u)K_{(44)}(u,0)du -\int_0^s\frac{ds'}{U_{(4)}}\int_0^\sigma K_{(44)}(u,s')du. \tag{87}$$

Our notation differs from Synge's in that we use the opposite signature for the metric and we use the opposite direction for increasing u, otherwise the symbols mean the same thing[6]; in particular, the parameter s is the observer's proper time. Proceeding in the same manner as above we can calculate the leading term in the power spectrum of the first-order Doppler shift.

$$\left\langle\left(\frac{\omega_0}{\omega_s}\right)^2\right\rangle-\left\langle\frac{\omega_0}{\omega_s}\right\rangle^2\simeq 5/3c^4\int_0^\infty P(\omega)\frac{d\omega}{\omega^4}\equiv\int_0^\infty P_{DS}(\omega)d\omega. \tag{88}$$

First of all we note that in order for this to be finite, $P(\omega)$ must drop off faster than ω^3 for small ω. Using

$$P(\omega)=P_0\frac{\omega_0\,\omega^4}{\omega_0^6+\omega^6}, \tag{89}$$

we find that (84) still holds. [Indeed, (84) holds approximately for any broad peaked spectrum.] The quantity that can be obtained from observations is only a portion of the Doppler-shift spectrum, namely that ranging from a period of at most a hundred years (of observation) down to a period equal roughly to a photographic exposure time. Let ω_1 $(=2\pi/100\ \mathrm{yr})$ be the lower limit on the integral in (88). A study of (89) indicates that the optimum value of ω_0 is roughly ω_1 and this then gives for the largest possible mean-square Doppler shift

$$\left\langle\left(\frac{\omega_0}{\omega_s}\right)^2\right\rangle-\left\langle\frac{\omega_0}{\omega_s}\right\rangle^2\simeq c^4\frac{P_0}{\omega_1^3}\simeq 10^{-16}. \tag{90}$$

This corresponds to a line shift of about one part in 10^8. It is difficult to see how to make this much larger and still be consistent with (82).

A calculation of the second-order Doppler shift using Synge's technique has not been attempted; however, our calculation of Hubble's constant (81) corresponds to the average second-order shift. The second-order fluctuations would be extremely tedious to calculate and no attempt has been made on it.

C. Jitter in Position

We proceed with a calculation of the fluctuations in the apparent position of the source. We have already defined in (11) a unit spacelike vector that points in the direction of the light ray. Let

$$r^\mu=(U_\nu v_0{}^\nu)^{-1}[U^\mu-(U_\rho v_0{}^\rho)v_0{}^\mu];\quad r_\mu r^\mu=-1. \tag{91}$$

We can define a unit vector s^μ in the average direction of the ray as

$$s^\mu=N^{-1}\langle U^\mu-(U_\rho v_0{}^\rho)v_0{}^\mu\rangle;\quad s_\mu s^\mu=-1, \tag{92}$$

where N is a normalizing factor (which in general is *not* equal to $\langle U_\rho v_0{}^\rho\rangle$). It is convenient to define a rank two tensor $l_{\mu\nu}$.

$$l_{\mu\nu}\equiv r_\mu s_\nu-r_\nu s_\mu. \tag{93}$$

We see that in the rest frame of the observer $l_{\mu\nu}$ is nothing more than the cross product of the two unit vectors and is therefore equal in magnitude to the sine of the angle between them θ. The invariant $\frac{1}{2}l_{\mu\nu}l^{\mu\nu}$ is

[30] R. L. Forward, D. Zipoy, J. Weber, S. Smith, and H. Benioff, Nature **189**, 473 (1961).

[31] Synge, Ref. 6, pp. 91–95.

just the square of the angle between the two vectors (for small angles).

$$\langle\theta^2\rangle \simeq \tfrac{1}{2}\langle l_{\mu\nu}l^{\mu\nu}\rangle. \tag{94}$$

If we actually used (94) to calculate $\langle\theta^2\rangle$, we would have to calculate $l_{\mu\nu}$ to second order. On the other hand, for a ray going in 3-direction we know that we only have to calculate r_1, r_2, s_1, s_2 to first order to get $\langle\theta^2\rangle$ to second order. Using (94) corresponds to calculating $\cos\theta$ first and then using the trigonometric identity to get $\sin\theta$. The other way corresponds to calculating $\sin\theta$ directly. It is most easily done using Synge's notation; the result is

$$\langle\theta^2\rangle = 5/3c^4\int_0^\infty P(\omega)\frac{d\omega}{\omega^4}. \tag{95}$$

The formula (95) is identical to (88) (this seems to be a fluke), and according to (90) it is numerically small. Equation (90) corresponds to measuring angular positions to 10^{-5} sec of arc.[31a]

D. Twinkling

We now come to the calculation of the intensity fluctuations in the beam of light. Equation (55) gives the quantities that should be calculated; however, only the area factor will be evaluated and the other two factors, L and ω_0/ω_s, will be assumed to be small. The only cases where this can cause trouble is if the first two factors would cancel or dominate the contribution from the area fluctuations. The average second-order Doppler shift is given by Hubble's law and is proportional to the first power of the source-observer distance L. We will see that the average value of $\Delta A_0/A_0$ is proportioned to L^2, a different functional dependence from the Doppler shift. It seems unlikely then that the fluctuations in these quantities will have the same functional form and thus cancel each other. It does seem likely, however, that the area fluctuations will be proportional to a higher power of L than the Doppler fluctuations just because this is true of their average values. Since we will only be interested in large distances, the area fluctuations should dominate. A somewhat similar argument holds for the fluctuations in L itself. In what follows we will assume that the dominant contribution to the intensity fluctuations comes from the area fluctuations.

There is another effect that has been left out of (55). This is just the intensity variation caused by variation in the incident direction of the light beam. We have seen that this is the same size as the Doppler shift in first order and so we will assume the same is approximately true in second order and neglect it.

We first compute the average value of $\Delta A_0/A_0$ given in (49) using (22), (23), and (65). After ensemble averaging we obtain

$$\left\langle\frac{\Delta A_0}{A_0}\right\rangle = -\sum_k\left[\langle D^{\alpha\beta}(\mathbf{k})D^*{}_{\alpha\beta}(\mathbf{k})\rangle h(k', -k', L) + \text{c.c.}\right], \tag{96}$$

where the coefficients $D^{\alpha\beta}(\mathbf{k})$ are the Fourier coefficients of $K^{\alpha\beta}(x)$, k' is just $k_\mu U^\mu$ and $L=\sigma$ is the source observer distance. The factor h comes from evaluating the trigonometric integrals that arise in (49).

$$h(k', l', u) = \frac{1}{\sigma}\int_0^u d\alpha\int_0^\alpha d\beta e^{il'\beta}\int_0^\beta d\gamma\int_0^\gamma d\epsilon\epsilon e^{ik'\epsilon} - \frac{1}{2\sigma}\left[\int_0^u d\alpha\int_0^\alpha d\beta\beta e^{il'\beta}\right]\left[\int_0^u d\alpha\int_0^\alpha d\beta\beta e^{ik'\beta}\right]. \tag{97}$$

Evaluation of the averaged Fourier coefficients gives

$$\langle D^{\alpha\beta}(\mathbf{k})D^*{}_{\alpha\beta}(\mathbf{k})\rangle = \tfrac{1}{2}\langle|E(k)|^2\rangle(k_\mu U^\mu)^4, \tag{98}$$

so that

$$\left\langle\frac{\Delta A_0}{A_0}\right\rangle = -\sum_k\langle|E(k)|^2\rangle(k_\mu U^\mu)^4\,\mathrm{Re}h(k', -k', L). \tag{99}$$

A calculation gives

$$\mathrm{Re}h(k', -k', L) = (k')^{-4}\left[(\tfrac{1}{6}\rho^2-2)+\frac{4}{\rho^2}(1-\cos\rho)\right], \quad \rho = k'L. \tag{100}$$

After substituting (100) into (99), going to continuous $\mathbf{k}$, integrating over the directions of $\mathbf{k}$ and substituting for $\langle|E(k)|^2\rangle$ we get the result.

$$\left\langle\frac{\Delta A_0}{A_0}\right\rangle = -\frac{5L^2}{72}\int_0^\infty dk\frac{P(k)}{k^2}\left[1-\frac{9}{\alpha^2}-\frac{9\sin^2\alpha}{\alpha^4}+\frac{9}{\alpha^3}\int_0^\alpha\frac{\sin 2x}{x}dx\right], \quad \alpha = kL. \tag{101}$$

For $\alpha \gg 1$ the bracket is unity whereas for $\alpha \ll 1$ it is $(2/25)\alpha^2$. If we assume that $P(k)$ peaks at wavelengths that are short compared to the source-observer distance L, the bracket is unity and we obtain [using (81)]:

$$\left\langle\frac{\Delta A_0}{A_0}\right\rangle = -\tfrac{2}{3}H^2L^2 \to -\frac{2}{3}\left(\frac{HL}{c}\right)^2 \simeq (10^{-28}L)^2 \lesssim 1. \tag{102}$$

Thus, we find that the average fractional change in intensity can be of order unity for distant sources ($L\sim 10^{10}$ l.y.). Of course, this result is to be expected

[31a] *Note added in proof:* This random-walk effect does not increase with L as would be expected. The effect would be large when the light ray travels in the same direction as some of the gravitational waves and thus could interact for an appreciable time. However the transverseness of the gravitational field causes the interaction to vanish for this configuration. If the gravitational wave propagated at less than light velocity, the effect would increase with L in the usual manner.

because of the analogy between our calculation and Tolman's radiation-filled universe.

The calculation of the power spectrum of area fluctuations (and therefore the received power) is most unambiguously carried out by noting that in general the power spectrum of a random variable $x(t)$ is twice the Fourier transform of its corrolation function, $\langle x(t)x(t+s)\rangle-\langle x(t)\rangle^2$, a function that only depends on the magnitude of s.[32] Proceeding as before we obtain the power spectrum for the intensity fluctuations, $P_I(k)$.

$$P_I(k)=2\sum_m\sum_l\langle D^{\alpha\beta}(\mathbf{m})D^{*\gamma\delta}(\mathbf{m})\rangle\langle D_{\alpha\beta}(\mathbf{l})D^*_{\gamma\delta}(\mathbf{l})\rangle$$
$$\times[\tfrac{1}{2}|f(m',l',L)|^2(\delta(m+l+k)+\delta(m+l-k))$$
$$+|f(m',-l',L)|^2\delta(m-l-k)],\quad(103)$$

where: $f(m',l',L)=h(m',l',L)+h(l',m',L)$, and D, h, m', l' have been defined near (97). A lengthy calculation gives

$$\langle D^{\alpha\beta}(\mathbf{m})D^{*\gamma\delta}(\mathbf{m})\rangle\langle D_{\alpha\beta}(\mathbf{l})D^*_{\gamma\delta}(\mathbf{l})\rangle$$
$$=\tfrac{1}{8}\langle|E(m)|^2\rangle\langle|E(l)|^2\rangle(m')^4(l')^4(1+\cos 4(\varphi_l-\varphi_k)).\quad(104)$$

The angles φ_l and φ_k are the azimuthal angles of $\mathbf{k}$ and $\mathbf{l}$ with respect to $\hat{U}$ as the polar axis.

A calculation of $f(m',l',L)$ gives

$$f(m',l',L)=\frac{4L^4}{\alpha\beta}e^{\frac{1}{2}i(\alpha+\beta)}\left[\frac{\cos\frac{1}{2}(\alpha+\beta)}{(\alpha+\beta)^2}+\frac{4}{\alpha^2\beta^2}\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\right.$$
$$\left.-2\frac{\alpha^2+3\alpha\beta+\beta^2}{\alpha\beta(\alpha+\beta)^3}\sin\tfrac{1}{2}(\alpha+\beta)\right],\quad(105)$$

where $\alpha=m'L$, $\beta=l'L$.

If we insert (104) and (105) into (103), go to continuous $\mathbf{l}$, $\mathbf{m}$ and study the integral we find that when $\alpha+\beta$ [in (105)] is near zero the integral is large whereas for all other values of α and β it is comparatively small. We note that we are mainly interested in the case where L is very large and so we use an asymptotic method for evaluating the integral. Doing this we find that the last term in (103) dominates (because m and l are positive) and we eventually obtain

$$P_I(k)\simeq\left(\frac{5}{24}\right)^2\left(\frac{\pi}{5}\right)^{1/2}L^3\int_0^\infty\frac{P(k+l)P(l)}{(k+l)^5}dl.\quad(106)$$

In (106) we have also made the assumption that the peak in $P(l)$ occurs at a wavelength that is short compared to L. For the opposite case the integrand has an additional (small) factor of the order of $(lL)^5$.

In order to evaluate this we have to know $P(l)$. Assume, as before, that it is peaked at ω_0 and has a maximum value of P_0. As a first case assume that ω_0 is much smaller than the frequencies of interest in $P_I(\omega)$, but assume that $\omega_0L/c\gg1$. Then

$$P_I(\omega)\simeq\left(\frac{5}{24}\right)^2\left(\frac{\pi}{5}\right)^{1/2}L^3c^5\frac{P(\omega)}{\omega^5}\int_0^\infty P(\omega')d\omega'$$
$$\simeq10^{-2}L^3c^5\frac{P(\omega)}{\omega^5}\omega_0P_0.\quad(107)$$

If we say $P(\omega)\cong P_0(\omega_0/\omega)^2$ for $\omega\gg\omega_0$:

$$P_I(\omega)\simeq10^{-2}\frac{L^3c^5}{\omega_0^2}\left(\frac{P_0}{\omega_0}\right)^2\left(\frac{\omega_0}{\omega}\right)^7\simeq10^{-101}\frac{L^3}{\omega_0^2}\left(\frac{\omega_0}{\omega}\right)^7.$$

For $L\simeq10^9 l.y.=10^{27}$ cm,

$$P_I(\omega)\simeq\frac{10^{-20}}{\omega_0^2}\left(\frac{\omega_0}{\omega}\right)^7\ \mathrm{sec}.$$

If we consider measuring the intensity fluctuations around a frequency ω in a bandwidth of $\Delta\omega$ we get for the mean-square intensity fluctuation:

$$\left\langle\frac{\Delta I^2}{I^2}\right\rangle\simeq P_I(\omega)\Delta\omega\simeq\frac{10^{-20}}{\omega_0^2}\left(\frac{\omega_0}{\omega}\right)^7\Delta\omega.$$

Let the periods corresponding to ω_0, ω, $\Delta\omega$ be 100 yr, 1 yr, and 1 yr respectively. Then

$$\langle\Delta I^2/I^2\rangle\simeq10^{-23},$$

which corresponds to an rms intensity fluctuation of one part in 10^{12}.

As a second case assume that ω_0 is much larger than the frequencies of interest. Then (106) becomes

$$P_I(\omega)\simeq\left(\frac{5}{24}\right)^2\left(\frac{\pi}{5}\right)^{1/2}L^3c^5\int_0^\infty\frac{P^2(\omega')}{(\omega')^5}d\omega'$$
$$\simeq10^{-101}\frac{L^3}{\omega_0^2},\quad(108)$$

which is independent of ω. For $L=10^9$ l.y. and ω_0 and $\Delta\omega$ corresponding to 1 yr and 10 yr, respectively,

$$\langle\Delta I^2/I^2\rangle\simeq10^{-14}.\quad(109)$$

So we see that even at distances equal to about one-tenth of the Hubble radius of the universe the effect is very small. Before trying to use our formula for even larger values of L we should try to make an estimate of its region of validity.

E. Region of Validity

By iterating the equation of geodesic deviation such as was done in (36) and by taking the absolute values of the result we could show that an upper limit to the nth iteration is given by $(|K|L^2)^n/n!d$. $|K|$ is the maximum value of any component of $|K^\alpha{}_\rho|$ and d is the diameter

[32] Reference 28, p. 316.

of the telescope objective. Although this result can be used to show that the series converges, it does not give a very useful result for an upper limit on the size of succeeding terms because the value of $|K|$ can be very large.

The above result did not make use of the fact that K is a fluctuating quantity and therefore that there would be cancellation taking place when it is integrated. We can obtain a qualitative but useful relation between succeeding terms in the iteration if we take the ratio of the average square of succeeding terms. When this is done we find that this ratio is of the order of $\langle K^2\rangle L\lambda\!\!\!^-_0$. $\langle K^2\rangle$ is the average square of any component of $K^\alpha{}_\rho$ and we have assumed a power spectrum which is peaked at $\omega_0=c/\lambda\!\!\!^-_0$. We can put this ratio into a more significant form by making use of the relation between $\langle K^2\rangle$ and Hubble's constant; the ratio then turns out to be $HL/c=L/L_0$, where L_0 is the Hubble radius of the universe ($\cong 10^{10}$ l.y.). So we see that our expansion is in the usual form of cosmological formulas, namely, an expansion in powers of the fractional distance to the "edge of the universe." We can be reasonably certain that if we use values of L up to a few times 10^9 l.y. in our formulas that the result is correct to an order of magnitude or so. Even then the intensity fluctuations are only about a thousandth of a percent.

This last result is close enough to being measurable to make it worthwhile to go back and investigate our assumptions to see whether they are valid and, if not, whether they can be modified in order to enhance the effect. We have already chosen as big a Riemann tensor as is possible and we have pushed L about as far as it can go. Indeed, in order to push L further we would at least have to find an assymtotic solution to the geodesic deviation equation that was valid for large L. Moreover, we would have to take into account the usual cosmological solutions because we are in a region where the large scale curvature of space is important. No attempt has been made to carry out this more difficult investigation but in lieu of this it would be useful to know whether our calculation constitutes a lower or an upper bound on the exact solution. Unfortunately there are (rather unconvincing) arguments for either conclusion.

We can go back further to our assumption of geometrical optics of the light rays. In order that diffraction be unimportant, we have to require that the diffracting region covers many Fresnel zones. This means that

$$l^2 \gg \lambda_\epsilon L\,, \tag{110}$$

where l is the size of the diffracting region and λ_ϵ is the electromagnetic wavelength. The size of the diffracting region is of the order of the distance over which the gravitational fields are correlated[3] and this in turn can be shown to be about equal to the wavelength of the gravitational waves being considered. Using $L=10^{10}$ l.y. we find that the calculations are valid for optical waves when the fluctuation periods are longer than minutes and for ratio waves for fluctuation periods longer than days.

F. Finite Sources

The formula for the intensity fluctuations assumes that the source of light is a point; it has to be modified if the source is large. An approximate expression for this modification can be obtained without a detailed calculation.

Divide up the source into many infinitesimal areas ΔA_i, and call the intensity at the observer from the ith area $F_i\Delta A_i$. The flux F_i is made up of a stationary part $\langle F_i\rangle$ and a fluctuating part ΔF_i, which has zero average value. Then the total average intensity $\langle I\rangle$ at the observer is

$$\langle I\rangle=\langle\sum_i F_i\Delta A_i\rangle=\sum\langle F_i\rangle\Delta A_i=\langle F\rangle A_s\,, \tag{111}$$

where A_s is the area of the source and it has been assumed that the source is uniformly bright; $\langle F_i\rangle=\langle F\rangle$. The mean square fluctuation in the total intensity is

$$\begin{aligned}\langle\Delta I^2\rangle&\equiv\langle I^2\rangle-\langle I\rangle^2=\langle\sum\sum F_iF_j\Delta A_i\Delta A_j\rangle-\langle F\rangle^2A_s{}^2\\&=\sum\sum\langle\Delta F_i\Delta F_j\rangle\Delta A_i\Delta A_j\equiv\sum\langle\Delta F_i{}^2\rangle l^2\Delta A_i\\&=\langle\Delta F^2\rangle l^2A_s\,.\end{aligned} \tag{112}$$

The correlation length l can be thought of as the average distance over which the intensity fluctuations are correlated. The fractional change in intensity is then given by

$$\frac{\langle\Delta I^2\rangle}{\langle I\rangle^2}=\frac{\langle\Delta F^2\rangle}{\langle F\rangle^2}\frac{l^2}{A_s}\simeq\frac{\langle\Delta F^2\rangle}{\langle F\rangle^2}\left(\frac{l}{D}\right)^2, \tag{113}$$

where D is the diameter of the source.

For an isotropic flux of gravitational radiation of a single frequency, it can readily be shown that l is approximately equal to the reciprocal of the wave number. Therefore, we see that if the source size is small compared to the wavelength of interest, there is no additional effect whereas if the source is large our result drops down as the inverse area of the source.

VI. FIRST-ORDER EFFECT

We now consider the case where (45) does not hold and so there is a first-order effect on the beam area. As a simple model assume that the space is filled uniformly and isotropically with a perfect fluid which is at rest (for convenience). Then if we proceed in the same manner as above we arrive at an expression for the average charge in area.

$$\left\langle\frac{\Delta A_0}{A_0}\right\rangle\simeq-\frac{4\pi}{3}L^2\langle\rho_0\rangle\simeq\left(\frac{HL}{c}\right)^2\simeq(10^{-28}L)^2\lesssim 1\,, \tag{114}$$

where $\langle\rho_0\rangle$ is the average density of matter and we have used the first part of (81) to relate $\langle\rho_0\rangle$ to H. (It is approximately valid for many of the Friedmann universes.) Equation (114) is essentially the same as (102), as it must be. To get fluctuations we will assume that there is a homogeneous and isotropic flux of sound waves traveling through this fluid. We can define the power spectrum of the density fluctuations by

$$\frac{\langle\Delta\rho^2\rangle}{\langle\rho_0\rangle^2}\equiv\int_0^\infty P_a(\omega)d\omega. \tag{115}$$

We can then proceed as above and calculate the area fluctuation. The result for the power spectrum of the intensity fluctuations $P_I'(\omega)$ is

$$P_I'(\omega)\simeq\frac{32\pi^2}{9}(5\pi)^{1/2}L^3\langle\rho_0\rangle^2\frac{P_a(\omega)}{\omega}. \tag{116}$$

If we assume that $P_a(\omega)$ is peaked around ω_0 and if we take the extreme case where the density fluctuations are about 100% then (116) becomes numerically equal to (108) (approximately) and we find that the effect is small. Two other effects are important for this case. First, the correlation length is of the order of an acoustic wavelength which is about four or five orders smaller than a gravitational wavelength of the same frequency. Consequently, if the source is much larger than a large star, the right side of Eq. (116) is decreased because of finite source size, (113). Second, the short correlation length affects the geometrical optical limit according to (110) and therefore (116) is only valid out to a few light years; beyond this distance diffraction predominates. An extensive treatment of light fluctuations due to an atmosphere has been carried out by Chernov.[3] He has treated both the geometrical limit and the general case and finds that the geometrical case increases as L^3 [as in (106) and (116)] whereas the diffraction limited case increases as L. By analogy we would expect (116) to be reduced considerably in the diffraction limited case. There can be exceptions to all this if the fluid consists of electromagnetic radiation or possibly neutrinos. Then the effect is comparable to the case when gravitational radiation dominates (108) because the correlation length is once again given by the speed of light.

VII. ORDINARY FLUCTUATIONS

A source of light fluctuations that has not been considered in much detail is that due to fluctuations or inhomogeneities in the ordinary index of refraction of the interestellar or intergalactic gas. As mentioned above, the theory has been developed by Chernov, among others, but apparently has not been applied to fluctuations in distant gas clouds (Chernov applied it to the earth's atmosphere to account for the ordinary twinkling of stars). A straight application of Chernov's formulas to the matter distribution considered in Sec. VI above indicates that for this case the effect is small. The main thing that makes the effect small is the short correlation length combined with finite source size. This does not mean that there are no interesting effects to be found in the application of Chernov's formulas to astronomical objects but rather it merely means that for the case considered here there is no appreciable effect. Haddock and Sciama[33] have recently proposed a technique for measuring the ionized hydrogen content of intergalactic space. Perhaps an application of Chernov's formulas would give additional information about the ion or gas density.

VIII. CONCLUSIONS AND COMMENTS

We have seen that the intensity fluctuations of light from distant sources is unmeasurable under the approximations considered in this paper. To get the maximum possible effect we had to assume that gravitational radiation is at least an important constituent of the universe if not the most important. A recent paper by Dicke *et al.*[34] indicates that this is not an impossible situation. They conjecture that during the early stages of expansion in an oscillating universe the primeval "fireball" could reach temperatures in excess of 10^{10} °K and that electromagnetic radiation, neutrinos and even gravitational radiation could be in thermal equilibrium. Subsequently these radiations would be degraded in energy due to the general adiabatic expansion of the universe and would now have a frequency spectrum that was peaked in the radio-frequency region or higher. This is just the sort of spectrum we need to get the maximum effect from our calculations [Eq. (108)]. Our formula is valid out to a few billion light-years and gives intensity fluctuations of up to about a thousandth of a percent. It was pointed out that if a formula which was valid very near the "edge" of the observable universe were obtained, it may show that the effect was observable. As yet no such calculation has been attempted. An example of an effect which increases rapidly as the Hubble radius is approached is the cosmological red shift. For a distance of 10^9 l.y. the red shift is about 10% whereas at the Hubble radius (10^{10} l.y.) the red shift is (by definition) infinite. We could hope for a similar situation here.

When intensity fluctuations were first seen in the quasistellar source 3*C*273,[35] it was hoped, of course, that the above effect could account for them. The distance to 3*C*273 is only about two billion light years, however, and so we see that the fluctuations must be caused by something else.

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Instantaneous and Asymptotic Conservation Laws for Classical Relativistic Mechanics of Interacting Point Particles

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The present article consists of two parts. First, we assume that the conservation laws for energy and linear momentum are valid and that these quantities are the sums of the energies and linear momenta of the individual particles, i.e., that there is no interaction energy and no interaction momentum. We then repeat a familiar argument and show that there can then be no interaction between the particles, that is, their world lines are straight. In the second part of the paper the interaction quantities for energy, linear and angular momenta, and the center-of-mass law are derived for the equations of motion proposed in an earlier paper. We then study these interaction quantities in the asymptotic region of collision processes, in order to arrive at asymptotic conservation laws. We find, in agreement with the earlier paper, that the interaction energy and the linear interaction momenta vanish asymptotically. This, however, is not true in general for the interaction angular momenta and center-of-mass motion. Asymptotic interaction angular momentum is present in all theories, such as classical electrodynamics, which lead to inverse-square-law forces.

INTRODUCTION AND SUMMARY

We wish to discuss some aspects of the relativistic dynamics of a system of n interacting classical point particles. The history of each particle, $i=1, 2, \cdots, n$, will be described by an orbit in Minkowski space. This orbit will be given, parametrically, in terms of the proper time τ_i, i.e., the orbit i is given by a vector-valued function $x_{i\alpha}(\tau_i)$, where α refers to the time and the spatial coordinates. Proper time is the same as arc length so that

$$\sum_\alpha l_\alpha[\dot{x}_{i\alpha}(\tau_i)]^2=1\,, \tag{1}$$

where the dot refers to differentiation with respect to the argument τ_i and where $-l_0=l_1=l_2=l_3=-1$, and we assume that $\dot{x}_{i0}(\tau_i)>0$, i.e., that the proper time runs in the same direction as the actual time.

The components of the linear momentum and the energy of the individual particle i at time t_1 will be defined as follows: Let τ_{i1} be the solution of the equation

$$x_{i0}(\tau_{i1})=t_1\,, \tag{2}$$

then the expression for the energy and the components of the linear momentum at t_1 will be

$$P_{i\alpha}(t_1)=m_i\dot{x}_{i\alpha}(\tau_{i1})\,. \tag{3}$$

Henceforth, we shall use the term linear momentum for both energy and linear momentum. Similarly we shall use the word angular momentum for center-of-mass momentum and angular momentum, i.e., all six components of the usual antisymmetric tensor. The components of the angular momentum of particle i at time t_1 are

$$M_{i\alpha\beta}(t_1)=x_{i\alpha}(\tau_{i1})P_{i\beta}(t_1)-P_{i\alpha}(t_1)x_{i\beta}(\tau_{i1})\,. \tag{4}$$

In Sec. 1 we repeat, in detail, a familiar derivation of a "no interaction theorem"[1] by showing that if (a) the

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[1] See, for instance, P. G. Bergmann, *The Special Theory of Relativity*, Handbuch der Physik IV (Springer, Berlin, 1962), p. 147. The theorem discussed here has a purely kinematical basis. Several "no interaction" theorems with a dynamical origin have appeared in the recent literature: D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963); D. G. Currie, J. Math. Phys. **4**, 1470 (1963); J. T. Cannon and T. F. Jordan, *ibid.* **5**, 299 (1964); H. Ekstein, Consistence of Relativistic Particle Theories, Université d'Aix-Marseille, (1964, unpublished), H. Leutwyler, Nuovo Cimento **37**, 556 (1965).