Poincaré-Invariant Equations of Motion for Classical Particles*

D. G. CURRIE†‡
National Science Foundation Summer Institute for Theoretical Physics, University of Wisconsin, Madison, Wisconsin
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Classical particles, and interaction among them governed by second-order equations of motion for the positions of the particles, are considered. Equations of motion, defined for one instant in an arbitrary frame, are derived which are invariant under the Poincaré group. The equations of motion are considered invariant if, when the world-line solutions to the equations of motion are transformed, point by point, into a new frame, the new world lines obey the same second-order equation of motion. We illustrate the existence of a wide class of such invariant equations of motion. The further questions of causality and separability are mentioned.

I. INTRODUCTION

RECENTLY, there have been several approaches to a relativistic description of particles which interact directly. By a direct interaction we mean an interaction between particles which does not require the aid of an intervening dynamical field. The quantum mechanical formulation of such a direct interaction theory has been discussed recently by a number of authors.[1–5] One has been unable, however, to find an operator representing the position which had the proper transformation properties under Lorentz rotation. Without such an operator, the connection to experimental results must be made either by using the only quantity transforming properly, the momentum, or by an asymptotic connection to free particles states. The interest in a quantum theory of a finite number of particles is twofold; first, it may be useful as a practical description of interactions below production thresholds; second, the formal structure of a relativistic theory with interaction may be useful for the study of the problem of interaction in a quantum field theory.

To better understand some of the above questions, we will consider the classical theory which seems analogous to the above discussion, since the essential problem will appear in both the classical and quantum formulations. In the classical theory, we have canonical representations of the Poincaré group, instead of unitary representations. This approach to a relativistic classical theory of particles also has a venerable history.[6–10] Once again, it is difficult to find the position function which transforms properly under the Lorentz rotation. Bakamjian and Thomas noted this difficulty in connection with manifest causality,[11] i.e., that manifest causality, and a Hamiltonian formulation were not consistent with interaction. In any case, the requirement of manifest causality is a separate and independent postulate which will be considered separately. Even if manifest causality is not required, it has been shown that a position function which transforms properly does not exist when the representation for the Poincaré group is required to be canonical.[12–15] Thus one is led to suspect that the source of the difficulty for the quantum case does not lie in the uncertainty relations (as implied by various authors) since the same difficulty also appears in the classical theory.

To investigate the question in its simplest form, we shall deal with the classical-particle theory in which the position transforms correctly. From the beginning we start with the requirement that the position function has the proper transformation properties. We then proceed to build as much of the canonical representation formalism as possible. In general, we seek an extension of the direct interaction type of theory used for Galilean invariant systems. A possible form of a relativistic interaction which might be considered would be one in which the interaction propagates along the light cone.[16–18] This approach has a difficulty in common with classical electrodynamics, that is, there are runaway solutions. More precisely, if the theory is assumed

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† Supported, in part, by a National Science Foundation Postdoctoral Fellowship, and by the United States Air Force Office of Scientific Research.
‡ Permanent address: Northeastern University, Boston, Massachusetts.

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[9] E. C. G. Sudarshan, *1961 Brandeis University Summer Institute, Lectures in Theoretical Physics* (W. A. Benjamin and Company, Inc., New York, 1962), Vol. 1.
[10] E. C. G. Sudarshan, U. S. Atomic Energy Commission Report No. NYO 10250, 1963 (unpublished).
[11] *Manifest causality* in particle theory is the requirement that arbitrary variation of the position and velocity of any one of the particles may be performed without implicitly affecting the other particles, and that the other particles are not affected at points which are space-like to the original disturbance.
[12] D. G. Currie, J. Math, Phys. **4**, 1470 (1963).
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to be complete and consistent in itself (not a part of a larger theory which is quantized[19]) then has one to contend with the problem that there are either runaway solutions (in which the particle accelerates itself to the velocity of light) or pre-accelerations (in which the effects due to a signal generated at a finite time start in the infinite past).

We now return to more recent attempts which have been aimed at obtaining a representation for equations of motion for the position of the particle which are invariant under the Poincaré group. In this approach the question of the existence of a canonical representation, as well as the questions of separability of the interaction and causality, are left in the background. We shall mention two papers, those of Havas and Plebanski[20] and Van Dam and Wigner.[21]

In the Havas-Plebanski representation, the force on particle A at a given instant is determined by the position and velocity of particle B at a time which is simultaneous with the given instant in the rest frame of particle A.[22] In the investigation of Van Dam and Wigner, the relativistic generalization of a potential model is considered in which the acceleration, at a given time, of particle A is determined by the interval of the world line of particle B which is space-like to the particle A at the given time.

In neither of these investigations are the dynamical variables for a given instant (as determined in an arbitrary frame) directly related. It is therefore difficult to use these formulations as a basis for a Hamiltonian theory.

The present discussion is similar to, but not directly connected with, the model of Wigner and Van Dam since their acceleration functions are assumed to be nonanalytic in time, while the acceleration functions considered in this paper are analytic. However, their program can easily be extended to cover analytic potentials and then these two approaches could be two representations of the same interaction.

As a general comment on the two other approaches just mentioned, as well as our present approach, note that the procedure is to attack the question of the relativistic invariance of the equations of motion before the question of causality is considered. In the present discussion, we mean by *causality* the requirement that the velocity of signal propagation should be less than the speed of light. This is the "Einstein causality" of Haag[23] and the causal connection of Havas.[20] This form of causality is satisfied in a theory of particles and a passive field (where the particle produces a field, but the field does not act back on the particle). This is *not* causality in the sense of the Cauchy conditions, i.e., that the motion is determined from a set of second-order differential equations and a set of initial data.[24] Note, however, that the equations of motion considered in this paper obviously do satisfy the Cauchy causality.

In general the test of causality in a theory is to introduce a nonanalytic change in the motion of one of the particles and ask when a nonanalytic change occurs in the motion of the other particle. Despite the fact that one often considers relativistic invariance and causality at the same time, in the present discussion we consider (as do Havas and Plebanski, and Van Dam and Wigner) first the question of relativistic invariance, leaving for a later time the discussion of causality.

Another criterion conventionally applied to a theory of particle interaction is that the interaction is separable.[1,2] For our discussion, we begin by defining weak and strong separability. For all the following considerations, we assume for illustration that we have limited ourselves to interactions which are two-particle forces and which go to zero for large particle separations. For a many-particle system, the force due to many particles is then the vector sum of two-particle forces. We consider a group of particles which are divided into two clusters that are separated by the distance D. We now consider what happens as $D \to \infty$ while the diameter of the clusters remain bounded.

Weak separability is the requirement that as D goes to infinity the ratio of the change in acceleration due to a given change in the initial conditions in the distant cluster to the change in acceleration due to a given change in the initial conditions of the enveloping cluster goes to zero. We assume that neither change of initial conditions causes the diameter of the cluster to exceed some bound. Dealing with the dynamical variables defined at a single instant, as discussed in this paper, if we have only two-body forces and these forces add as vectors, then the requirement of weak separability may easily be satisfied.

Another stronger separability condition is more commonly encountered. This usually states that the total energy splits into a sum of the energies of each of the clusters as D goes to infinity. Thus a theory of particle interaction is said to have the property of *strong separability* if we can define a functional of the initial positions and velocities (or, for a Hamiltonian approach, the initial positions and momenta) which, as D goes to infinity, separates into the sum of two parts. Each of these parts is to consist of the functional applied to one of the two clusters.

Difficulties may arise if the functional is required to have specified transformation properties under Lorentz rotation, and perhaps other properties, like being the

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[20] P. Havas and J. Plebanski, Bull. Am. Phys. Soc. **5**, 433 (1961).
[21] H. Van Dam and E. P. Wigner, Phys. Rev. **138**, B1576 (1965).
[22] Havas and Plebanski also consider generalizations of this representation, but their generalizations are open to the same comments.
[23] R. Haag and B. Schroer, J. Math. Phys. **3**, 248 (1962).
[24] A discussion of some of these questions of causality may be found in the article by P. Havas in *Proceedings of the 1964 International Congress of Logic, Methodology, and Philosophy of Science*, edited by Y. Bar-Hillel (North-Holland Publishing Company, Amsterdam, 1965).

Hamiltonian. While the search for the functional is important for a simple discussion of the physical interpretation, we take the approach that the physically essential requirement is that of weak separability, and that after one has developed a relativistically invariant interaction, then one should attempt to display a functional which illustrates the strong separability of the system.

One final requirement, which is rather obviously necessary in classical particles, but somewhat harder to formulate for the quantum-particle theories, is that the particle velocity never exceeds the velocity of light. However, in the following development this point will appear naturally and be resolved in an obvious manner so its consideration will be delayed to a later section.

In the present discussion, we work at one instant, and require that the acceleration transform properly. (The acceleration at a given event on the world line transforms as a four-vector.) In Sec. II, we give a brief discussion of the meaning of invariance for the equation of motion. We shall illustrate the technique of specifying the equation of motion for the relative coordinate and velocity in a special frame, and then transforming to the general frame. In Sec. III, the change in position, velocity, acceleration, etc. in going from one frame to a Lorentz-rotated frame shall be determined. In Sec. IV, the equations of motion invariant under Lorentz rotation are obtained. The results are discussed in Sec. V.

II. INVARIANCE OF EQUATIONS OF MOTION

A second-order equation of motion for the position of a particle is a relation which determines the second time derivative of the position from a knowledge of the position and velocity of it and the other particles (the dynamical variables). Further, it will be assumed that this acceleration function of positions and velocities is differentiable with respect to each of its arguments. It can then be differentiated with respect to time to obtain, by substitution, all higher order time deviatives of the position in terms of the dynamical variables, and thus determine the motion. These sets of coefficients may or may not determine the entire motion. This depends upon the domain of analyticity of this expansion in time. We shall here restrict outselves to those equations of motion for which the expansion is entirely analytic, and thus we are able to determine the motion for all time. This restriction is far stronger than necessary, but the generalization shall not be considered in this publication.

We seek a second-order equation of motion such that the Lorentz transforms of solutions are also solutions to the same equation of motion, i.e., the same force function. We shall use a technique which defines the equation of motion in one frame, and induces the dependence on the total velocity so that it has the form of a second-order equation with the same force function in all other frames. We shall consider the invariances of space translation, and space rotation in two dimensions to give a background on the method, and to define our terminology. We shall deal with two particles.

The average, and half the relative coordinates are defined, respectively, by

$$\mathbf{X}\equiv(\mathbf{x}_1+\mathbf{x}_2)/2\,,\quad \mathbf{V}\equiv(\mathbf{v}_1+\mathbf{v}_2)/2\,,\quad \mathbf{A}\equiv(\mathbf{a}_1+\mathbf{a}_2)/2\,,$$
$$\mathbf{x}\equiv(\mathbf{x}_1-\mathbf{x}_2)/2\,,\quad \mathbf{v}\equiv(\mathbf{v}_1-\mathbf{v}_2)/2\,,\quad \mathbf{a}\equiv(\mathbf{a}_1-\mathbf{a}_2)/2\,. \tag{II.1a}$$

Thus the equations of motion in general have the form

$$\begin{aligned}\mathbf{a}\equiv\ddot{\mathbf{x}}&=\mathbf{f}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\,,\\ \mathbf{A}\equiv\ddot{\mathbf{X}}&=\mathbf{F}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\,.\end{aligned} \tag{II.1b}$$

Assuming the normal transformation of position under translation, we obtain a part of the representation of space translation in Ξ space. Thus the change under translation (δ) is

$$\delta\mathbf{x}=0\,,\quad \delta\mathbf{X}=\mathbf{b}\,,\quad \delta\mathbf{v}=0\,,\quad \delta\mathbf{V}=0\,,\quad \delta\mathbf{a}=0\,,\quad \delta\mathbf{A}=0\,. \tag{II.2}$$

Using (II.1) we can determine the change under translation in the acceleration as given by the equation of method

$$\begin{aligned}\mathbf{a}+\delta\mathbf{a}&=\mathbf{f}^\dagger(\mathbf{x}+\delta\mathbf{x},\,\mathbf{X}+\delta\mathbf{X},\,\mathbf{v}+\delta\mathbf{v},\,\mathbf{V}+\delta\mathbf{V})\\ &=\mathbf{f}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})+\mathbf{f}_x{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\delta\mathbf{x}+\mathbf{f}_{\mathbf{X}}{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\delta\mathbf{X}\\ &\quad+\mathbf{f}_v{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\delta\mathbf{v}+\mathbf{f}_{\mathbf{V}}{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\delta\mathbf{V}\,.\end{aligned} \tag{II.3}$$

We now impose invariance by requiring that the acceleration transform kinematically. That is, there are no outside sources to disturb the system, so $\delta\mathbf{a}=0$. After we note the values of $\delta\mathbf{x}$, $\delta\mathbf{X}$, $\delta\mathbf{v}$, and $\delta\mathbf{V}$ from from (II.2), the above equations become

$$0=\mathbf{f}_{\mathbf{X}}{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\,. \tag{II.4}$$

This implies that $\mathbf{f}^\dagger$ is independent of $\mathbf{X}$. In the same manner, for $\mathbf{F}^\dagger$ we get the equation

$$0=\mathbf{F}_{\mathbf{X}}{}^\dagger(\mathbf{x},\mathbf{X},\mathbf{v},\mathbf{V})\,, \tag{II.5}$$

so that we conclude that for the acceleration to be invariant under translation, $\mathbf{f}^\dagger$ and $\mathbf{F}^\dagger$ must be independent of $\mathbf{X}$.

Although we can easily guess the form for the rotationally invariant equations of motion in two dimensions, we shall obtain it by the expansion techniques in order to illustrate, for a simple example, the technique which shall be used for the construction of the Lorentz-invariant equations of motion. In order to concentrate on the relative coordinate, we presume that the force is independent of the velocities and the mean position.

Thus we have

$$\ddot{x}=f(x,y)=f_0(x)+f_1(x)y+f_2(x)y^2+\cdots=\sum_{n=0}^{\infty} f_n(x)y^n,$$
$$\ddot{y}=g(x,y)=g_0(x)+g_1(x)y+g_2(x)y^2+\cdots=\sum_{n=0}^{\infty} g_n(x)y^n, \tag{II.6}$$

where we have chosen to expand in y to effect a series solution of the equations. Again, the reason for this unsymmetrical approach is to develop the analogy to the Lorentz rotation. One might call this 'the method of considering the equations in a preferred frame' (the frame in which y vanished) and then rotating out to get the rotationally invariant second-order equations of motion. The presumption is that for space rotations the acceleration rotates like a two vector, or "kinematically." A rotation transforms the relative coordinates to

$$x\rightarrow x+\delta x=x-\theta y,\quad y\rightarrow y+\delta y=y+\theta x, \tag{II.7}$$

and the kinematic transformation of the velocity and acceleration are then

$$\dot{x}\rightarrow \dot{x}+\delta\dot{x}=\dot{x}-\theta\dot{y},\quad \dot{y}\rightarrow\dot{y}+\delta\dot{y}=\dot{y}+\theta\dot{x}, \tag{II.8a}$$

$$\ddot{x}\rightarrow \ddot{x}+\delta\ddot{x}=\ddot{x}-\theta\ddot{y},\quad \ddot{y}\rightarrow\ddot{y}+\delta\ddot{y}=\ddot{y}+\theta\ddot{x}, \tag{II.8b}$$

where we have assumed a small rotation, i.e., $\theta\ll\pi$. For the change in the acceleration for a small change in x and y we obtain

$$\ddot{x}+\delta\ddot{x}=f(x,y)+f_x(x,y)\delta x+f_y(x,y)\delta y,$$
$$\ddot{y}+\delta\ddot{y}=g(x,y)+g_x(x,y)\delta x+g_y(x,y)\delta y, \tag{II.9}$$

and if we now require that the accelerations rotate kinematically as in (II.8b), we obtain the differential equations wihch f and g must satisfy for the force to be rotationally invariant. The differential equations are

$$f(x,y)=xg_y-yg_x,\quad g(x,y)=yf_x-xf_y, \tag{II.10}$$

or we may get an equation for f (or equivalently, g) alone:

$$f=-y^2f_{xx}-x^2f_{yy}+2xyf_{xy}+yf_y+xf_x. \tag{II.11}$$

Guessing a solution of the form

$$f(x,y)=x\alpha(r^2)+y\gamma(r^2), \tag{II.12}$$

where $r^2\equiv x^2+y^2$, one may easily verify that (II.11) is satisfied for arbitrary functions α and γ. This implies that g has the form

$$g(x,y)=y\alpha-x\gamma \tag{II.13}$$

so

$$\ddot{\mathbf{r}}=\{\ddot{x},\ddot{y}\}=\mathbf{r}\alpha+\mathbf{r}_\perp\gamma, \tag{II.14}$$

where $\mathbf{r}_\perp\equiv\{y,-x\}$. (This pair of arbitrary functions may be resolved on any pair of vectors; the extra degree of arbitrariness is because the two-dimensional rotation group is Abelian.)

Now we attempt to solve (II.10) again by a series technique which will work when we have no guess available. Substituting (II.10), and setting coefficients of y^n to zero, we have

$$g_0=-xf_1,\quad g_n=f_{n-1}'-x(n+1)f_{n+1},\quad n>0 \tag{II.15a}$$

solving for highest f terms

$$f_1=-g_0/x,\quad f_{n+1}=(-g+f_{n-1})/x(n+1),\quad n>0, \tag{II.15b}$$

likewise for f

$$f_0=xg_1,\quad f_n=-g_{n-1}'+x(n+1)g_{n+1},\quad n>0 \tag{II.16a}$$

solving for highest g terms

$$g_1=f_0/x,\quad g_{n+1}=(f_n+g_{n-1}')/x,\quad n>0. \tag{II.16b}$$

Thus once we have fixed f_0 and g_0, all the higher order coefficients are fixed and we have determined the most general rotational invariant equation of motion. To further investigate this, we could put these in a form so that all coefficients are functions of the first-order functions, i.e., functionals of f_0 and g_0.

We may finally ask what type of equations of motion correspond to these expressions. The f_0 solutions is the familiar

$$\ddot{x}=f(x,y)=xF(x^2+y^2),$$
$$\ddot{y}=g(x,y)=yF(x^2+y^2), \tag{II.17}$$

while the other solution has the form

$$\ddot{x}=f(x,y)=-yG(x^2+y^2),$$
$$\ddot{y}=g(x,y)=xG(x^2+y^2). \tag{II.18}$$

Thus we find that we get the expected result by requiring the acceleration transform kinematically.

We are now prepared to do the same for the Lorentz rotation, but we will need a short detour to obtain the equations equivalent to (II.7) and (II.8).

III. REPRESENTATION OF LORENTZ ROTATION

In this section, we will obtain the Ξ space representation of Lorentz rotation (for a discussion of the representation of relativity mappings in Ξ space see Currie[25]). This may be done by two different methods. The most elementary method to obtain the representation would be to calculate directly from the Lorentz transformations of the events composing the world line. A sketch of this procedure appears in the Appendix. We shall here calculate directly the transformed position to terms which are first order in the Lorentz-rotation parameter, and use the commutation relations of the kinematic generators to obtain the transformation of the velocity, acceleration, and higher time derivatives.[12]

[25] D. G. Currie, Ann. Phys. (N. Y.) (to be published).

We thus proceed to calculate the change in the position function when measured in a moving frame. This we will do to first order in the relative velocity of the frames. Figure 1 illustrates one component of the quantity $q(t_1)$, which we will calculate. Our input data are the values of the position function $q(t)$ and all of its time derivatives at $t=0$. From these, we seek the value of the transformed position function at the origin of time in the new frame, i.e., the time $t'=0$. We want the value of the position in the new frame at the same value of time as used in the old frame. Writing the components of $\mathbf{q}$ as x, y, and z, the Lorentz transformation, with the relative velocity of the frames in the x direction, gives

$$\begin{aligned} x'(0)&=\gamma[x(t_1)+vt_1],\\ y'(0)&=y(t_1),\\ z'(0)&=z(t_1). \end{aligned} \tag{III.1}$$

Now we need only determine the value of t for which t' vanishes. But for this ,we have

$$t_1=\gamma[t_1'-vx'(t_1')]=-\gamma vx'(0). \tag{III.2}$$

Putting (III.2) into (III.1), we obtain

$$\begin{aligned} x'(0)&=\gamma[x(-\gamma vx'(0))-v\gamma vx'(0)],\\ y'(0)&=y(-\gamma vx'(0)),\\ z'(0)&=z(-\gamma vx'(0)). \end{aligned} \tag{III.3}$$

Now expanding the position function as a power series in t,

$$\mathbf{q}(t)=\mathbf{q}(0)+\dot{\mathbf{q}}(0)t+\tfrac{1}{2}\ddot{\mathbf{q}}(t)t^2+\cdots, \tag{III.4}$$

we insert this expansion into Eq. (III.3) and solve for $q'(0)$ in terms of $q(0)$, $\dot{q}(0)$, etc., yielding, after dropping terms in v^2,

$$\begin{aligned} x'(0)&=x(0)-vx(0)\dot{x}(0)+\cdots,\\ y'(0)&=y(0)-vx(0)\dot{y}(0)+\cdots,\\ z'(0)&=z(0)-vx(0)\dot{z}(0)+\cdots. \end{aligned} \tag{III.5}$$

Thus if we represent an infinitesimal Lorentz rotation by the "operator" $\mathcal{K}$ which acts in the space Ξ we may summarize (III.5) by

$$\mathcal{K}_iq_j(0)=q_i(0)\mathcal{H}q_j(0). \tag{III.6}$$

Now to determine how the velocity transforms, we consider the infinitesimal Lorentz rotator acting on the velocity, so

$$\begin{aligned} \mathcal{K}_i\dot{q}_j(0)&=\mathcal{K}_i\mathcal{H}q_j(0)=\mathcal{H}\mathcal{K}_iq_j(0)+\mathcal{P}_j(0)q_j(0)\\ &=\mathcal{H}(q_i(0)\mathcal{H}q_j(0))-\delta_{ij}\\ &=q_i(0)\ddot{q}_j(0)q_j(0)+\dot{q}_i(0)\dot{q}_j(0)-\delta_{ij}, \end{aligned} \tag{III.7}$$

where $\mathcal{H}$ is the generator of an infinitesimal time translation and we have used the communication relation,

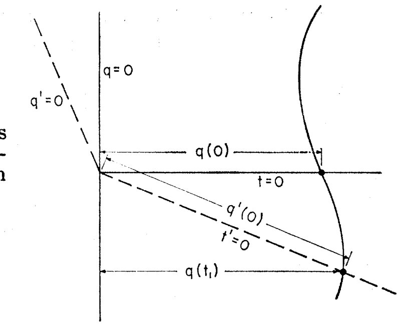

FIG. 1. Definitions of the position function as measured in different frames.

$\mathcal{K}_i\mathcal{H}=\mathcal{H}\mathcal{K}_i+\mathcal{P}_i$. In the same manner, we obtain

$$\begin{aligned} \mathcal{K}_iq^{(2)}{}_j&=q^{(0)}{}_iq^{(3)}{}_j+2q^{(1)}{}_iq^{(2)}{}_j+q^{(2)}{}_iq^{(1)}{}_j,\\ \mathcal{K}_iq^{(3)}{}_j&=q^{(0)}{}_iq^{(4)}{}_j+3q^{(1)}{}_iq^{(3)}{}_j\\ &\quad+3q^{(2)}{}_iq^{(2)}{}_j+q^{(3)}{}_iq^{(1)}{}_j, \end{aligned} \tag{III.8}$$

etc., where the number n in parentheses refers to the nth derivative of q with respect to time. We now might guess that the effect of an infinitesimal Lorentz rotation on the nth derivative of the position function is given by

$$\mathcal{K}_iq^{(n)}{}_j=\sum_{s=0}^{s=n}\binom{n}{s}q^{(s)}{}_iq^{(n+1-s)}-\delta_{n1}\delta_{ij} \tag{III.9}$$

and this expression may easily be verified by induction.

If we consider more than one particle, Eq. (III.6) takes the form

$$\mathcal{K}_iq^{(n)}{}_j{}^r=\sum_{s=0}^{s=n}\binom{n}{s}q^{(s)}{}_i{}^rq^{(n+1-s)}{}_j{}^r-\delta_{n1}\delta_{ij} \tag{III.10}$$

for the rth particle. For two particles, using the collective coordinates defined in (II.1), and a certain amount of algebra, we have

$$\begin{aligned} \mathcal{K}_iQ^{(n)}{}_j&=\sum_{s=0}^{s=n}\binom{n}{s}Q^{(s)}{}_iQ^{(n+1-s)}{}_j+q^{(s)}{}_iq^{(n+1-s)}{}_j,\\ &\qquad-\delta_{n1}\delta_{ij}, \end{aligned} \tag{III.11a}$$

$$\begin{aligned} \mathcal{K}_iq^{(n)}{}_j&=\sum_{s=0}^{s=n}\binom{n}{s}Q^{(s)}{}_iq^{(n+1-s)}{}_j\\ &\qquad+q^{(s)}Q_i{}^{(n+1-s)}{}_j. \end{aligned} \tag{III.11b}$$

In particular, for one dimension, we have[26]

$$\begin{aligned} \mathcal{K}X&=X\dot{X}+x\dot{x}=XV+xv,\\ \mathcal{K}V&=X\ddot{X}+\dot{X}\dot{X}+x\ddot{x}+\dot{x}\dot{x}-1=XA+V^2+xa+v^2-1,\\ \mathcal{K}A&=X\dot{A}+3VA+x\dot{a}+3va, \end{aligned} \tag{III.12a}$$

$$\begin{aligned} \mathcal{K}x&=X\dot{x}+x\dot{X}=Xv+xV,\\ \mathcal{K}v&=X\ddot{x}+2\dot{X}\dot{x}+x\ddot{X}=Xa+xA+2vV,\\ \mathcal{K}a&=X\dot{a}+x\dot{X}+3Va+3vA. \end{aligned} \tag{III.12b}$$

[26] For typographical reasons, the third derivative of x and X with respect to time has been written $\dot{a}$ and $\dot{A}$, respectively, and the fourth derivative of x as $\ddot{a}$.

IV. LORENTZ-INVARIANT EQUATION OF MOTION

Having obtained the transformation of the position, velocity, and acceleration under Lorentz rotation in the previous section, we can now apply the methods of Sec. II to obtain Lorentz-invariant equations of motion.

As before we require that the acceleration transform kinematically. We require that the function giving the acceleration be such as to guarantee this. The acceleration may be written

$$\ddot{x}_1 = f_1^\dagger(x_1,x_2,\dot{x}_1,\dot{x}_2)\,, \qquad \text{(IV.1)}$$

$$\ddot{x}_2 = f_2^\dagger(x_1,x_2,\dot{x}_1,\dot{x}_2)\,, \qquad \text{(IV.2)}$$

or, using the x and X defined in (II.1a), we may write this as

$$a=\ddot{x}=f^\dagger(x,X,v,V)\,,\quad A=\ddot{X}=F^\dagger(x,X,v,V)\,, \qquad \text{(IV.3)}$$

where we have used $v\equiv\dot{x}$, $V\equiv\dot{X}$, $a\equiv\ddot{x}$, $A\equiv\ddot{X}$.

In this and later sections, we considerd one space dimension. In the manner of Sec. II, invariance under space translation implies that f and F should be independent of X. Thus we have

$$a=f(x,v,V)\,,\quad A=F(x,v,V)\,. \qquad \text{(IV.4)}$$

For an infinitesimal change in the acceleration we have

$$\begin{aligned}a+\delta a&=f(x+\delta x,\ v+\delta v,\ V+\delta V)\\&=f(x,v,V)+f_x(x,v,V)\delta x+f_v(x,v,V)\delta v\\&\quad+f_v(x,v,V)\delta V\,,\end{aligned} \qquad \text{(IV.5)}$$

where, for a Lorentz transformation, δx, δv, δV are given by (III.12). This then yields the relation

$$\begin{aligned}\delta a=&\,\alpha(xV+Xv)f_x(x,v,V)+\alpha(xF+Xf+2vV)f_v(x,v,V)\\&+\alpha(XF+xf+VV+vv-1)f_V(x,v,V)\,,\end{aligned} \qquad \text{(IV.6a)}$$

where $a=f$ and $A=F$ have been used several times. Of course a similar equation holds for A, that is,

$$\begin{aligned}\delta A=&\,\alpha(xV+Xv)F_x(x,v,V)+\alpha(xF+Xf+2vV)F_v(x,v,V)\\&+\alpha(XF+xF+VV+vv-1)F_V(x,v,V)\,.\end{aligned} \qquad \text{(IV.6b)}$$

The requirement of Lorentz invariance is the requirement that the acceleration transform kinematically, i.e., as given in (III.12). From these, the kinematic transformation of the acceleration is given by

$$\delta a=\alpha(X\dot{a}+x\dot{X}+3Va+3vA)\,, \qquad \text{(IV.7a)}$$

$$\delta A=\alpha(X\dot{A}+x\dot{a}+3va+3VA)\,. \qquad \text{(IV.7b)}$$

Now expressing the time derivative of the acceleration in terms of the variables x, v, and V, we have

$$\begin{aligned}\dot{a}=\frac{d}{dt}f(x,v,V)=&\,f_x(x,v,V)v+f_v(x,v,V)a\\&+f_V(x,v,V)A\end{aligned} \qquad \text{(IV.8)}$$

and likewise for A. Putting these into (IV.7), we get

$$\begin{aligned}\delta a/\alpha=&\,Xvf_x+Xff_v+XFf_V+xvF_x\\&+xfF_v+xFF_V+3Vf+3vF\,,\end{aligned} \qquad \text{(IV.9a)}$$

$$\begin{aligned}\delta A/\alpha=&\,XvF_x+XfF_v+XFF_V+xvf_x\\&+xff_v+xFf_V+3vf+3VF\,,\end{aligned} \qquad \text{(IV.9b)}$$

which describes the kinematic transformation of the acceleration. The requirement of invariance consists of setting the left-hand side of (IV.9) equal to the left-hand side of (IV.6). We then get the following nonlinear partial differential equations for f and F:

$$\begin{aligned}&xVf_x+(xF+2vV)f_v+(xf+VV+vv-1)f_V\\&\quad=xvF_x+xfF_v+xFF_V+3Vf+3vF\,,\end{aligned} \qquad \text{(IV.10a)}$$

$$\begin{aligned}&xVF_x+(xF+2vV)F_v+(xf+VV+vv-1)F_V\\&\quad=xvf_x+xff_v+xFf_V+3vf+3VF\,.\end{aligned} \qquad \text{(IV.10b)}$$

These equations are equivalent to (II.10) for the rotational case. To obtain the Lorentz-invariant equations of motion, we need only solve them. In lieu of a direct solution, we shall try to obtain a power-series solution of the type suggested in Sec. II. Our parameter will be V, the velocity of the center of mass. Thus we write

$$\begin{aligned}f(x,v,V)&=f_0(x,v)+f_1(x,v)V+f_2(x,v)V^2+\cdots\,,\\F(x,v,V)&=F_0(x,v)+F_1(x,v)V+F_2(x,v)V^2+\cdots\,.\end{aligned} \qquad \text{(IV.11)}$$

We now mention a few restrictions we shall put upon the acceleration or force functions. These restrictions are not necessary and can be relaxed, but are put on to simplify these calculations. We require that the particles be identical, and that the force be invariant under reflection. The latter may be considered the remnant, in one dimension, of the rotational invariance in three dimensions, rather than a parity invariance. Under particle interchange

$$\begin{aligned}x&\to -x & v&\to -v, & a&\to -a,\\X&\to X, & V&\to V, & A&\to A,\end{aligned} \qquad \text{(IV.12)}$$

so

$$F(-x,\ -v,\ V)=F(x,v,V)\quad\text{and}\quad f(-x,\ -v,\ V)=-f(x,v,V)\,. \qquad \text{(IV.13)}$$

For the reflection invariance, we have

$$\begin{aligned}x&\to -x\,, & v&\to -v\,, & a&\to -a\,,\\X&\to -X\,, & V&\to -V\,, & A&\to -A\,,\end{aligned} \qquad \text{(IV.14)}$$

so

$$F(-x,\ -v,\ -V)=-F(x,v,V)\quad\text{and}\quad f(-x,\ -v,\ -V)=-f(x,v,V)\,. \qquad \text{(IV.15)}$$

Or, combining these two invariances, we can write

$$F(x,\ v,\ -V)=-F(x,v,V)\quad\text{and}\quad f(x,\ v,\ -V)=+f(x,v,V)\,. \qquad \text{(IV.16)}$$

These discrete symmetries tell us that f_1, F_0, F_2 vanish, so (IV.11) takes the form

$$\begin{aligned} f(x,v,V) &= f_0+f_2V^2+f_4V^4+\cdots, \\ F(x,v,V) &= F_1V+F_3V^3+F_5V^5+\cdots. \end{aligned} \tag{IV.17}$$

We proceed with the series solution by substituting (IV.17) into (IV.10a,b). The coefficient of V^0 in (IV.10a) is identically equal to zero. One-half of the equations we get by setting coefficients to zero will be identities. This is a result of the discrete symmetries put on the force functions. The coefficient of V^0 in (IV.10b) is

$$(xf_0+v^2-1)F_1 = xvf_{0x}+xf_0f_{0v}+3vf_0. \tag{IV.18}$$

Taking f_0 to be an arbitrary (infinitely differentiable) function of x and v, this equation determines F_1 in terms of f_0, i.e.,

$$F_1(x,v) = (xvf_{0x}+xf_0f_{0v}+3vf_0)/(xf_0+v^2-1). \tag{IV.19}$$

In the same manner, the coefficient of V in (IV.10a) gives us f_2 in terms of F_0, i.e.,

$$f_2 = \tfrac{1}{2}[xF_1F_1+3f_0+3vF_1+xvF_{1x}+xf_0F_{1v}-xf_{0x} -(xF_1+2v)f_{0v}]/[xf_0+v^2-1], \tag{IV.20}$$

where by (IV.19) F_1 is a function of f_0. This procedure may be continued to obtain the higher coefficients. Thus except for the possible vanishing of the denominator, the solution may always be determined, although it may be computationally rather complicated. Because of the weak separability, f_0 and all its derivatives go to zero as $x\rightarrow\infty$.

Now concerning the denominator, this is the previously mentioned point concerning the velocity of the particles. For example, if the force were independent of v, then it could always drive a particle to a velocity greater than that of light. For this force the denominator term may vanish, thus illustrating how this trouble shows itself in the series expression. On the other hand, if we have a force of the form $f_0=(1-v^2)g(x)$,[27] then the denominator term never vanishes, that is, the force is weakened for a particle moving at a velocity near c, so the force can never drive a particle to a velocity greater than that of light. There are other forms for the force function which will also guarantee that the denominator term remains finite, but we do not consider them further.

The apparent simplicity of the rotational form (II.14) comes from being able to sum the series to determine the general form of the solution. While such a closed form is a very useful calculational aid, the equation may be calculated from the series expression, except for possible convergence questions.

[27] D. G. Currie and E. Saletan, J. Math. Phys. (to be published).

V. CONCLUSION

Thus we have illustrated an expression for the Poincaré-invariant equations of motion. The general context is the same as that which had been considered earlier by Havas and Plebanski and simultaneously with this work by Van Dam and Wigner. That is, we seek to display equations of motion which are invariant under the transformation of the Poincaré group. We leave for later consideration the questions concerning causality.

The motivation for the present formulation is that it relates all the dynamical quantities at a single instant in any frame. It is thus a form which will permit the investigation of a Hamiltonian description of the motion That a Hamiltonian be admissible requires that the equations of motion satisfy a certain set of integrability conditions. The consideration of the equations of motion which satisfy these conditions and admit Hamiltonians, and the construction of these Hamiltonians, will be discussed in a later publication.

It is not difficult to expand this formalism to three dimensions, or to describe more than two particles. For the n-body problem, we need only require weak separability, i.e., separability for the accelerations, rather than for a functional representing the energy. Relaxing the requirement that the particles are identical may also be done; however, it will involve a calculation which is algebraically more involved than the above discussion.

In Galilean-invariant systems, we found that while the invariance limited the form of the equations of motion, there was still a wide class of equations of motion which were possible. From this wide class, a particular interaction could be chosen to describe a particular system. When one changes from the Galilean relativity group to the Poincaré relativity group, one might initially expect the same situation.

On the other hand, there has been built up a philosophical attitude that things may be very different from this initial expectation, i.e., that the Poincaré group imposes much more stringent conditions. Technical difficulty in finding the proper representation for the display of the equations of motion has permitted this philosophy to gain strength. However, as we have seen, there is as wide a class of equations of motion for the relativistic equations as for the Galilean-invariant equations of motion. The choice of the proper equations of motion must come from nature, in the form of additional requirements, rather than *a priori* from the elements of the relativity group. Of course, the outstanding candidate for this additional requirement is at present manifest causality, but these questions will be relegated to a later publication.

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APPENDIX

In this section, we will obtain the Ξ space representation of the Lorentz rotation by a direct (in principle) although algebraically complicated application of the Lorentz transformation (for a discussion of the representation of relativity mappings, see Currie[25]). Using Fig. 2, we assume that we are given the values of the position function and all its time derivatives at the event m_0. From these, we shall determine the values of the position function and all its time derivatives at the point m expressed in the Lorentz rotated frame. The technique is to express the position function, which describes the world line as a power series in the old frame, then change to the variables of the new frame.

The time and position of an event as expressed in the original frame is labeled by t and x. In the new frame, they are labeled by y and s. The relative velocity of the frames is $\alpha \equiv U/c$. U is the relative frame velocity and $\eta \equiv \tanh^{-1}\alpha$.

Having defined some of our notation, we now use the definition of Lorentz rotation

$$y_n = x_n \cosh\eta + ct_n \sinh\eta\,, \qquad \text{(A1)}$$

$$s_n = t_n \cosh\eta + x_n \sinh\eta/c\,, \qquad \text{(A2)}$$

where n is a general event on the world line. Now expressing the position of the general event n in terms of a power series using the derivatives evaluated at the event m_0, and further, dropping the subscript m_0, we have[26]

$$x_n = x + \dot{x}t_n + \ddot{x}t_n^2/2 + \dddot{x}t_n^3/6 + \ddddot{x}t_n^4/24 + \cdots. \qquad \text{(A3)}$$

Our object now is to determine the nth derivative of $y(s)$ with respect to s evaluated at the event n. We will later specialize on the event m which is simultaneous with the origin of time in the new frame. Thus we first wish to express the position of the event n in the new frame, i.e., y_n in terms of s_n and the derivatives of x with respect to t, evaluated at m_0. Thus we start with (A1) and use (A3) and

$$y_n = [x + \dot{x}t_n + \ddot{x}t_n^2/2 + \dddot{x}t_n^3/6 + \ddddot{x}t_n^4/24 + \cdots]\cosh\eta + ct_n \sinh\eta\,. \qquad \text{(A4)}$$

Now using the inverse of (A2)

$$t_n = s_n \cosh\eta - y_n \sinh\eta/c\,, \qquad \text{(A5)}$$

we obtain

$$\begin{aligned} y_n = \cosh\eta\{ & x + \dot{x}[s_n \cosh\eta - y_n \sinh\eta/c] \\ & + \tfrac{1}{2}\ddot{x}[s_n \cosh\eta - y_n \sinh\eta/c]^2 \\ & + \dddot{x}[s_n \cosh\eta - y_n \sinh\eta/c]^3/6 + \cdots \\ & - c[s_n \cosh\eta - y_n \sinh\eta/c]\tanh\eta\}\,. \end{aligned} \qquad \text{(A6)}$$

Expanding this in power series in $\eta = \tanh^{-1}(\alpha)$, and keeping only the first-order terms in η, we have

$$y_n = x + \dot{x}s_n + \tfrac{1}{2}\ddot{x}(s_n)^2 + \cdots - \eta[\dot{x}y_n + \ddot{x}y_n s_n + \tfrac{1}{2}\dddot{x}y_n s_n^2 + \ddddot{x}\cdots - c^2 s_n]/c + \cdots. \qquad \text{(A7)}$$

This expression has y_n on both sides, so we substitute this expression into itself repeatedly, to obtain

$$y_n = x - \eta x\dot{x}/c + s_n[\dot{x} - \eta(x\ddot{x}/c + \dot{x}\dot{x}/c - c)] + \tfrac{1}{2}s_n[\cdots]\,, \qquad \text{(A8)}$$

where we have arranged it in powers of s_n and kept only the first-order terms in n.

Now taking the derivative of y_n with respect to s_n, and denoting it with a prime, we have (after neglecting higher terms in η)

$$\begin{aligned} y_n' &= \dot{x} - \eta(x\ddot{x} + \dot{x}\dot{x} - c^2)/c + s_n[\cdots]\,, \\ y_n'' &= \ddot{x} - \eta(x\dddot{x} + 3\dot{x}\ddot{x})/c + s_n[\cdots]\,, \\ y_n''' &= \dddot{x} - \eta(x\ddddot{x} + 4\dot{x}\dddot{x} + 3\ddot{x}\ddot{x})/c + s_n[\cdots]\,. \end{aligned} \qquad \text{(A9)}$$

Thus we have the values of the derivatives of y in the new form for an arbitrary event, labeled n. We are interested in the event m, which has the property that s_m vanishes. We can use (A8) and (A9) in an obvious manner to obtain this, by setting s_n to zero. We thus obtain[28]

$$\begin{aligned} y_n &= x - \eta x\dot{x}/c\,, \\ y_m' &= \dot{x} - \eta(x\ddot{x} + \dot{x}\dot{x} - c^2)/c\,, \\ y_m'' &= \ddot{x} - \eta(x\dddot{x} + 3\dot{x}\ddot{x})/c\,, \\ y_m''' &= \dddot{x} - \eta(x\ddddot{x} + 4\dot{x}\dddot{x} + 3\ddot{x}\ddot{x})/c\,. \end{aligned} \qquad \text{(A10)}$$

Thus we have obtained the representation of the Lorentz rotation on Ξ directly, without the use of the infinitesimal generator formalism employed in the body of the paper.

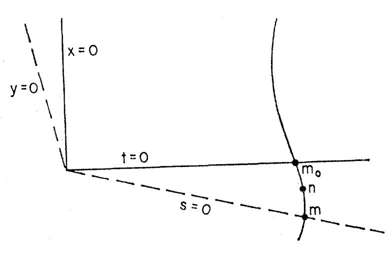

FIG. 2. Location of events used in derivation of Ξ space representation of Lorentz rotation.

[28] As $c \to \infty$, $\eta \to U/c$ and $\dot{y}_m \to \dot{x} + U$ while $y_m^{(r)} \to x^{(r)}$ for $r \neq 1$, which is the proper Galilean limit.