Calculations of Deuteron Stripping in a Soluble Model*†

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(Received 11 June 1965; revised manuscript received 4 October 1965)

We present preliminary results of calculations based on a soluble three-body model recently proposed by Amado. Angular distributions for the deuteron stripping process are presented for several energies and the results are compared with the general behavior of experimental stripping patterns although no attempt is made to describe a particular nucleus. Various approximation schemes such as the Butler theory and the distorted-wave Born approximation are applied to our model amplitudes. The possibility of describing real nuclei with this model is also discussed.

I. INTRODUCTION

DEUTERON stripping and pickup are, by their nature, three-body problems. We shall show that these reactions can be studied in the context of a soluble three-body model recently proposed by Amado,[1] and used successfully to describe the three-nucleon system.[2,3] Our approach involves the use of particularly simple two-particle interactions (separable potentials) which reproduce the basic features of the low-energy two-body systems, such as, for example, bound state wave functions, scattering lengths, and effective ranges.[4] These potentials reduce the three-body problem to the solution of coupled sets of one dimensional Fredholm integral equations, and allow us to take into account exactly complicated three-body effects such as bound-state breakup and the coupling between elastic and inelastic channels.[5] The model stripping amplitudes so obtained exhibit many characteristic features of "real" deuteron stripping. This model enables us to speculate upon the nature of the basic stripping mechanism, and also offers the possibility of studying and understanding the various approximate theoretical procedures presently used to investigate stripping reactions.

In Sec. II we describe our model; in Sec. III we give results and in Sec. IV our discussion and conclusions.

II. THE MODEL

Prior to a discussion of results and conclusions, we briefly describe our model which deals with five spinless particles n, p, d, A, and B; n, p, and A possess no internal degrees of freedom. With no loss of generality the mass of A is chosen to be infinite and the masses of n, p, and d to be those of the neutron, proton, and deuteron, respectively. For simplicity we permit *only* the S-wave interactions

$$\begin{aligned} n+p &\rightleftarrows d\,, \\ n+A &\rightleftarrows B\,, \end{aligned} \tag{1}$$

which may be described by a second-quantized interaction Hamiltonian of the form

$$H_I=\lambda_{dnp}\sum_{\mathbf{d},\mathbf{n},\mathbf{p}} f_{dnp}\left[\frac{(\mathbf{n}-\mathbf{p})^2}{2}\right]\delta_{\mathbf{d}-\mathbf{n}-\mathbf{p}}\Phi_d{}^\dagger\psi_n\psi_p +\lambda_{BnA}\sum_{\mathbf{n}} f_{BnA}(n^2)\Phi_B{}^\dagger\psi_n\psi_A+\text{h.c.} \tag{2}$$

λ_{dnp} and λ_{BnA} are the renormalized coupling constants associated with each vertex, and the Φ's and ψ's are annihilation operators for the appropriate particles. Note that we allow no interaction between p and A, and thus restrict ourselves to the study of (d,p) reactions. It is necessary either to omit a direct p-A interaction *or* to treat the neutron and proton as identical particles[4] in order that the resulting coupled integral equations satisfied by our model amplitudes are sufficiently few in number to be solved straightforwardly on present day computers. The following reactions occur in this model.

$$\begin{aligned} &(\text{a})\quad d+A\to d+A\,,\\ &(\text{b})\quad d+A\to p+B\,,\\ &(\text{c})\quad d+A\to p+n+A\,,\\ &(\text{d})\quad p+B\to p+B\,,\\ &(\text{e})\quad p+B\to d+A\,,\\ &(\text{f})\quad p+B\to p+n+A\,. \end{aligned} \tag{3}$$

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* Supported in part by a grant from the National Science Foundation.
† The computations for this research were performed on an IBM 7094 at the M.I.T. Computation Center, Cambridge, Massachusetts.
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[1] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[2] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. Letters **13**, 574 (1964).
[3] R. Aaron, R. D. Amado, and Y. Y. Yam, Phys. Rev. **136**, B650 (1964).
[4] The three-particle Schrödinger equation with separable potentials between pairs was first solved by A. N. Mitra, Nucl. Phys. **32**, 529 (1962). This approach and ours give equivalent results when all wave function renormalization constants are zero. For calculations using this approach and further references, see B. S. Bhakar and A. N. Mitra, Phys. Rev. Letters **14**, 143 (1965), and A. N. Mitra, Phys. Rev. **139**, B1472 (1965). The latter publication discusses stripping from a different point of view than ours. Mitra treats the neutron and proton as identical particles.
[5] The alternative approach of using "realistic" potentials to describe the two-body scattering as completely as possible, and then taking three-body effects into account approximately has also been carried out. See, for example, J. M. Blatt and L. M. Delves, Phys. Rev. Letters **12**, 544 (1964); L. M. Delves, J. N. Lyness, and J. M. Blatt, *ibid.* **12**, 542 (1964).

For a more detailed description of this model we suggest that the reader consult Refs. 1–3.

The basic collision diagram, the Born approximation for deuteron stripping, is represented schematically in Fig. 1. The vertices in this diagram are described by form factors

$$f_{BnA}(q^2)=\lambda_{BnA}/(q^2+\beta_{BnA}{}^2)\,, \tag{4}$$

$$f_{dnp}(q^2)=\lambda_{dnp}/(q^2+\beta_{dnp}{}^2)\,, \tag{5}$$

where q is the momentum transfer to the exchanged neutron, and λ and β are constants. We set the deuteron and nucleus B wave function renormalization constants equal to zero; thus our present calculation is completely equivalent to solving the three-body Schrödinger equation with particles n and A, and n and p interacting via separable potentials $f_{BnA}(q^2)f_{BnA}(p^2)$ and $f_{dnp}(q^2)\times f_{dnp}(p^2)$, respectively.[4]

In this note we are speaking in the quasiparticle language of Amado[1] and Weinberg[6] rather than that of the Schrödinger equation. To see the connection between the two approaches, one should realize, for example, that when the nucleus B wave function renormalization constant is zero, the form factor $f_{BnA}(q^2)$ is just $(q^2+\alpha_B{}^2)\psi_B(q^2)$ where $\alpha_B{}^2$ is the binding energy of nucleus B, and $\psi_B(q^2)$ is the bound state Schrödinger wave function.

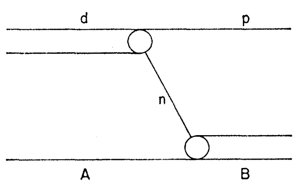

FIG. 1. Schematic representation of Born approximation to stripping amplitude.

Elastic d-A and p-B scattering and stripping and pickup amplitudes are then obtained by solving sets of coupled integral equations, following partial-wave analysis, as described in Refs. 1–3. In our calculation we have summed all diagrams involving the exchange of a neutron to form either a deuteron or nucleus B. In Fig. 2 we write the amplitudes for p-B elastic scattering and stripping as a sum of all possible diagrams. These sums do *not* represent a perturbation expansion. The formal sum of these diagrams is given by the set of integral equations shown symbolically in Fig. 3. Note that despite the fact that there is no direct p-A interaction in our Hamiltonian, p-B elastic scattering will certainly be affected by changing the B-n-A vertex function. This indirect p-A interaction arising from the B-n-A vertex is, however, not equivalent to the p-A distorting interaction encountered in the now standard distorted-wave Born approximation (DWBA) calculations of stripping, where the latter includes Coulomb and average nuclear interactions.

[6] S. Weinberg, Phys. Rev. **130**, 776 (1963).

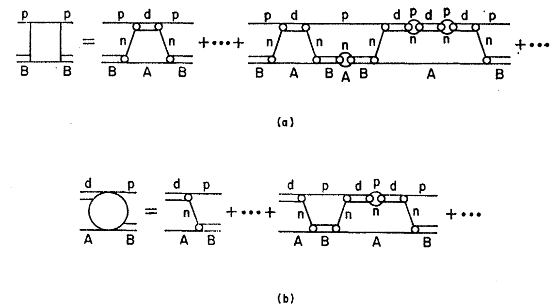

FIG. 2. (a) The sum of graphs for p-B scattering showing typical higher order terms. The box represents the full p-B amplitude. In general there may be an arbitrary number of both ladder rungs and bubbles on internal d or B lines; (b) the sum of graphs for stripping. The circle represents the full stripping amplitude.

III. RESULTS

We use units in which $\hbar=2m=1$ (where m is the nucleon mass) and the binding energy of the deuteron $\epsilon_d=0.5$. In the following calculation we have chosen the parameters in the n-p system to give the binding energy of the deteron and a triplet scattering length of 5.38 F. This choice is

$$\begin{aligned}\beta_{dnp}&=6.255\alpha_d\,,\\ \lambda_{dnp}{}^2&=32\pi\alpha_d\beta_{dnp}(\alpha_d+\beta_{dnp})^3\,,\\ \alpha_d&=(\epsilon_d/2)^{1/2}.\end{aligned} \tag{6}$$

In the n-A system the relations between the parameters are

$$\begin{aligned}\lambda_{BnA}{}^2&=8\pi\alpha_B\beta_{BnA}(\alpha_B+\beta_{BnA})^3\,,\\ \alpha_B&=(\epsilon_B)^{1/2}\,,\end{aligned} \tag{7}$$

where ϵ_B is the binding energy of nucleus B. The bound-state wave function of nucleus B, $\psi_B(r)$ is

$$\psi_B(r)=[2\alpha_B\beta_{BnA}(\alpha_B+\beta_{BnA})/(\alpha_B-\beta_{BnA})^2]^{1/2}\times(e^{-\alpha_B r}-e^{-\beta_{BnA} r})/r\,. \tag{8}$$

The rms radius of nucleus B is obtained from the

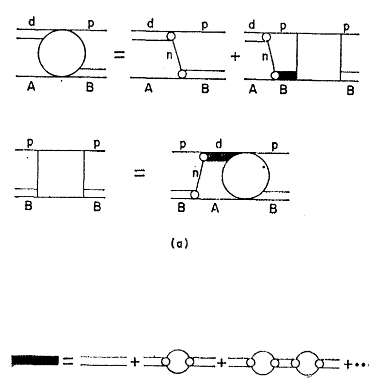

FIG. 3. (a) Diagrammatic representations of coupled integral equations for stripping and elastic p-B amplitudes. Pickup and elastic d-A amplitudes are coupled analogously; (b) renormalized propagator for either d or B.

relation

$$r_{\mathrm{rms}}^2=\int_0^\infty r^4\psi_B^2(r)dr$$

$$=\frac{\beta_{BnA}^3(\alpha_B+\beta_{BnA})^3+\alpha_B^3(\alpha_B+\beta_{BnA})^3-16\alpha_B^3\beta_{BnA}^3}{2\alpha_B^2\beta_{BnA}^2(\alpha_B+\beta_{BnA})^2(\alpha_B-\beta_{BnA})}. \tag{9}$$

We initially choose $\epsilon_B=4\epsilon_d=8.90$ MeV, and $\beta_{BnA}=4.0$ which leads to $r_{\mathrm{rms}}=1.69$ F.

We now solve the integral equations discussed in the previous section. We find a three-body discrete state with binding energy 14.5 MeV, but shall not discuss this state further since in this note we are interested in the stripping and elastic scattering amplitudes only. The deuteron stripping $(d+A\rightarrow p+B)$ differential cross sections finally obtained are shown in Fig. 4 for several energies. The resemblance of these cross sections to those of real stripping is remarkable, both as a function of energy and angle. Even the height of the forward maximum and the absolute cross sections agree qualitatively with experimental stripping results for a wide variety of nuclei. A typical experimental S-wave stripping result[7] is shown in Fig. 4. It should be emphasized that our model is not designed to describe a particular nucleus and we have chosen an example which resembles our curves for dramatic effect. Already at 1.78 MeV, a sharp diffraction peak with a pronounced minimum is developing; at 11.1 MeV, we have typical stripping pattern.

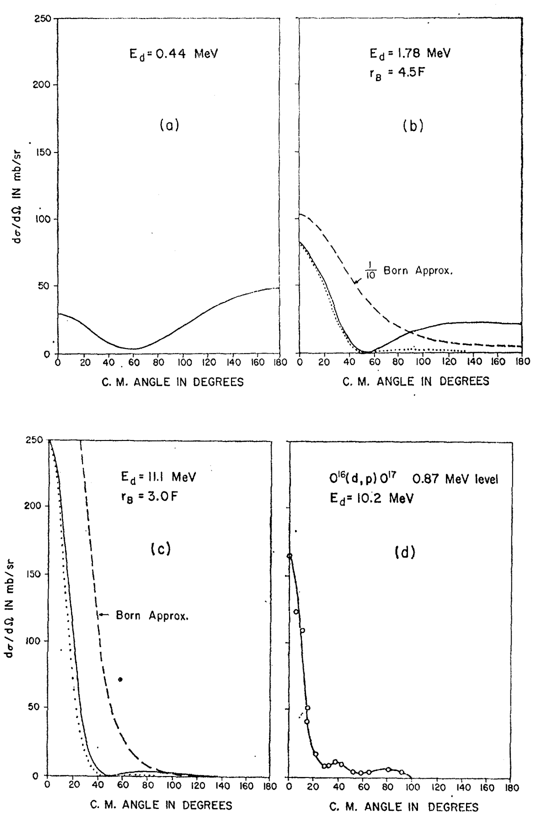

FIG. 4. In (a), (b), and (c) we show stripping angular distributions for $\epsilon_B=8.90$ MeV and $\beta_{BnA}=4.0$, as given by our exact theory (——) at 0.44, 1.78, and 11.1 MeV, respectively. The Butler (····) and Born (– – – –) approximations are shown when possible. The Butler radius r_B is noted. In (d) a typical experimental stripping pattern is shown for the reaction $O^{16}(d,p)O^{17}$,[7] leading to the 0.87-MeV level of O^{17}.

The stripping angular distributions at 11.1 MeV for different choices of parameters are shown in Fig. 5. Technical difficulties associated with our present programs prevent us from choosing parameters such that either $\epsilon_B\sim\epsilon_d$ and/or $r_{\mathrm{rms}}>\sim2.25$ F. The latter condition corresponds to $\beta_{BnA}\sim\alpha_B$. As can be seen from Eq.

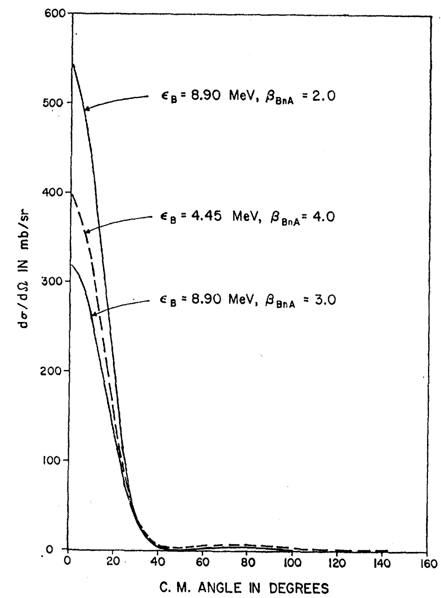

FIG. 5. Stripping angular distributions for different choices of parameters. Full curves are for $\epsilon_B=8.90$ MeV with $\beta_{BnA}=2.0$ giving $r_{\mathrm{rms}}=2.24$ F, and for $\beta_{BnA}=3.0$ giving $r_{\mathrm{rms}}=1.88$ F. The broken curve is for $\epsilon_B=4.45$ MeV, $\beta_{BnA}=4.0$ leading to $r_{\mathrm{rms}}=2.14$ F.

(8), in the limit $\beta_{BnA}\rightarrow\alpha_B$, the wave function has incorrect asymptotic behavior. In future calculations, we plan to remove some of the above restrictions, and also to use more realistic nuclear wave functions such as, for example, the eigenfunctions of the Woods-Saxon potential.

It is instructive to examine some of the standard approximation schemes used for stripping reactions in the context of our model. In particular, we calculate our stripping amplitudes in Born approximation, Butler theory, and DWBA. The Born approximation is considered for pedagogical reasons and not because it has

[7] E. W. Hamburger, Phys. Rev. **123**, 619 (1961). It should be stressed here that our model is presently *not* designed to describe a specific nucleus.

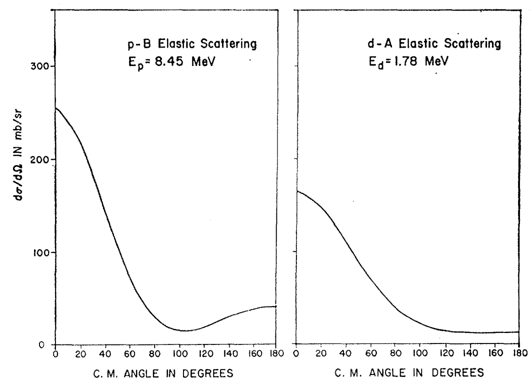

FIG. 6. Elastic d-A and p-B angular distributions at arbitrary energies.

been a useful tool in studying stripping reactions. The results are as follows:

(1) Born approximation: As can be seen from Figs. 3(b) and 3(c), the Born Approximation is much too large, and can be shown to violate S-wave unitarity by roughly an order of magnitude.

(2) Butler theory (cut-off Born approximation): One of the first reasonably successful theories of nuclear stripping was proposed by Butler.[8] In this theory, the effects of nuclear absorption are simulated by cutting off integrals over nuclear coordinates in the usual Born approximation at an arbitrary radius r_B, which becomes an adjustable parameter. A common feature of the Butler calculation is that the radius r_B is generally larger than the nuclear radius found by other means. We find the same situation to hold in our model as can be seen in Figs. 4b and 4c, that is, the Butler radii required to fit our cross sections are much larger than the rms radius of nucleus B. The Butler calculations were performed at 1.78 and 11.1 MeV and normalized to the exact cross sections at $\theta=0°$, the Butler cross section being smaller than the exact result.

(3) Distorted-wave Born approximation: Finally, we have sent our exact d-A and p-B elastic scattering angular distributions (Fig. 6) to Oak Ridge. Woods-Saxon potentials have been constructed to describe these processes, and along with binding energies, etc., fed into the DWBA code. Preliminary results have just been obtained. The stripping amplitude calculated at several energies are in reasonable agreement with those given by our model. The success of the DWBA in describing our model nucleus, at least for the parameters chosen and the energies considered, is rather amusing. We certainly do not claim to provide understanding for the current DWBA model. Rather, we are presenting an alternative model which gives similar results—hopefully the relation between the two can be made clear in the future. A program toward this end is proceeding in collaboration with R. Bassel.

IV. CONCLUSIONS

We now discuss the significance of the above results. Let us first point out implications about the general structure of stripping amplitudes. It is clear that many of the qualitative features of deuteron stripping amplitudes are contained in our model—the reason for this can be made plausible if we assume that the basic mechanism responsible for a "real" $A(d,p)B$ reaction is the exchange of a neutron to form either a deuteron or nucleus B (see Fig. 1). Recall that the Born approximation is much too large and violates S-wave unitarity by roughly an order of magnitude. Clearly, the Born term itself is a poor approximation to the true stripping amplitude. What we request is a sum of diagrams involving our basic exchange mechanism which gives amplitudes in all channels satisfying three-body unitarity. Solving the Schrödinger equation with separable potentials gives exactly this unitary sum of ladder diagrams. While we have not laid bare the mechanism responsible for stripping, our results do imply that nucleon exchange is an extremely important ingredient. Since a direct p-A interaction occurs in nature and is necessary in the DWBA model, it would be satisfying to understand the effect of its inclusion in our model. Such an investigation is, however, beyond the scope of the present work.

We are considering extensions of this model to describe real nuclei. It is straightforward to include spin in our model, although this inclusion gives additional coupled integral equations to solve. (See Ref. 2.) More important for the description of real stripping would be the use of a form factor $f_{BnA}(q^2)$ corresponding to a real nuclear wave function. For example, in the case of the $O^{16}(d,p)O^{17*}$, where capture takes place into a $2S$ state, at the very least, one would want a position space wave function with one node. The above can be done with little additional effort, the resulting new difficulty being that the integral equations must now be decomposed into partial waves by machine. The inclusion of further effects, such as the singlet state of the deuteron, nuclear excited states, higher partial waves in the two-body amplitudes, etc., all result in additional coupled integral equations and are not being considered presently.

[8] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).