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Configuration Interaction and the hfs of the *sl* Configuration

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Accurate hfs measurements in both the singlet and the triplet states of the first excited configuration of the group II elements have recently been obtained. These results have shown that the hfs of the singlet state cannot be reliably predicted from a knowledge of the hfs of the triplet states. A modification of the Breit-Wills theory for the hfs of the *sl* configuration is developed which takes into account the fact that the *l* electron has a different radial wave function in the singlet and the triplet state. The theory introduces a new parameter λ (the ratio of the off-diagonal to the diagonal matrix element of the spin-orbit operator) which can be estimated from the fine structure. This modified Breit-Wills theory is compared to recent experimental data for first excited *sp* configuration of Cd, Ba, and Hg. For Hg and Ba excellent agreement between all the hfs of the *sp* configuration is obtained. For Cd a significant improvement is obtained but a discrepancy still remains which, however, can be explained.

I. INTRODUCTION

IN recent years a considerable amount of accurate experimental data has been obtained on the atomic properties of the first excited *sp* configuration of the group II elements. These data (hfs, lifetimes, g_J values) have been obtained principally by optical resonance and atomic-beam spectroscopy. A complete theory of the fine structure of the *sl* configuration whose term levels are ${}^3L_{l+1,l,l-1}$ and ${}^1L_l$ has been given by Araki.[1] More recently, Vainshtein and Poleuktov[2] (referred to as VP) have redone the calculation using Racah techniques, and in addition, have permitted the *l* electron to have a different radial wave function in the singlet and triplet states. This extra degree of freedom in the *l*-electron wave function takes partially into account the effect of configuration interaction. The theory of the hfs of the *sl* configuration in intermediate coupling has been given by Breit and Wills.[3] In their work, however, the radial wave function for the *l* electron is taken to be the same in both the singlet and the triplet terms. In this paper, we will extend the theory of Breit and Wills to permit the *l*-electron wave function to be different in the singlet and triplet states. We will give a simple procedure for estimating the extra parameter introducted into the theory and will compare theory and experiment for the cases of Hg, Ba, and Cd.

II. DISCUSSION OF THE BREIT-WILLS THEORY

The theory of Breit and Wills expresses the hfs of the singlet and the triplet states in terms of the individual *s*- and *l*-electron hyperfine interaction constants. These constants are the dipole interaction constants $a_{l+1/2}$, $a_{l-1/2}$, a_s and the quadrupole interaction constants $b_{l+1/2}$ and $b_{l-1/2}$, where the ratios $(a_{l+1/2})/(a_{l-1/2})$ and $(b_{l+1/2})/(b_{l-1/2})$ are known from theory. In addition to these constants, the hfs of the ${}^1L_l$ and the ${}^3L_l$ states also depends on the coefficients c_1 and c_2 which are related to the degree of intermediate coupling of the two electrons.

Only for the first excited *sp* configuration of the group II elements is there sufficient experimental data to test the theory. Our object is to study how well the constants obtained from data in the triplet state can be used to predict the hfs of the singlet state.

In Table I are given the parameters used in extracting the individual electron coupling constants from the measured dipole and quadrupole interaction constants in the ${}^3P_2$ and ${}^3P_1$ states. The symbols are defined in the paper of Lurio, Mandel, and Novick[4] (referred to as LMN). The coefficients c_1 and c_2 (or α and β) can be determined in a number of different ways (see LMN, p. 1766) and are given in the literature. The values of c_1 and c_2 in Table I are the best estimate in the opinion of the author and give preference to the lifetime method of the determination.

In Table II are given the experimental values of the dipole coupling constants in the triplet and singlet states and also the calculated value of $A({}^1P_1)$ using the data from the triplet state. In the case of Ba there is no accurate value for $A({}^3P_2)$ so that one cannot calculate $A({}^1P_1)$ in this manner. For Cd the measured and predicted values are of opposite sign. For Hg the agreement is fair.

In Table III is given information for the quadrupole coupling constants. Since only $b_{3/2}$ enters the theory, one can calculate both $B({}^3P_1)$ and $B({}^1P_1)$ from the

TABLE I. Experimental and theoretical parameters necessary to calculate the individual electron coupling constants. (See LMN for a definition of the symbols.)

Element	c_1	c_2	α	β	ξ	θ	η
Cadmium	0.5401[a]	0.8416	0.9989	−0.0449	1.034	1.29	1.094
Barium	0.4969[b]	0.8677	0.9951	−0.0989	1.038	1.30	1.109
Mercury	0.4285[c]	0.9035	0.9851	−0.1717	1.094	2.38	1.354

[a] A. Lurio and R. Novick, Phys. Rev. **134**, A608 (1964); F. W. Byron, M. N. McDermott, and R. Novick, *ibid*. **134**, A615 (1964).
[b] A. Lurio, Phys. Rev. **136**, A376 (1964).
[c] A. Lurio, Phys. Rev. **140**, A1505 (1965).

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[1] G. Araki, Proc. Phys. Math. Soc. Japan **19**, 128 (1937); **19**, 592 (1937).

[2] L. A. Vainshtein and I. A. Poluektov, Opt. i Spektroskopiya **12**, 460 (1962) [English transl.: Opt. Spectry. (USSR) **12**, 254 (1962).

[3] G. Breit and L. A. Wills, Phys. Rev. **44**, 470 (1933).

[4] A. Lurio, M. Mandel, and R. Novick, Phys. Rev. **126**, 1758 (1962).

TABLE II. Comparison of experimental and calculated value of $A\,(^1P_1)$. Given below are also the experimentally measured values of the dipole coupling constants in the triplet state and the derived single-electron coupling constants obtained from the Breit-Wills theory.

Element	Experimental values (Mc/sec) $A\,(^3P_2)$	$A\,(^3P_1)$	$A\,(^1P_1)$	Calculated values (Mc/sec) $A\,(^1P_1)$	a_s	$a_{3/2}$
$\mathrm{Cd}^{111}(I=\frac{1}{2})$[a]	−3392.936	−4123.814	186	−148	−12 383.7	−262.7
$\mathrm{Ba}^{137}(I=\frac{3}{2})$[b]	~720	1150.59	−113.2	...	...	...
$\mathrm{Hg}^{201}(I=\frac{3}{2})$[c]	−3352.029	−5454.569	1316	1093	−12 938.5	−156.6

[a] W. Faust, M. McDermott, and W. Lichten, Phys. Rev. **120**, 469 (1960); P. Thaddeus and R. Novick, *ibid.* **126**, 1774 (1962); A. Lurio and R. Novick, *ibid.* **134**, A608 (1964); J. Lehmann, J. Phys. Radium **25**, 809 (1964).

[b] G. zu Putlitz, Ann. Physik **11**, 248 (1963); A. Lurio, Phys. Rev. **136**, A376 (1964).

[c] J. Blaise and H. Chantrell, J. Phys. Radium **18**, 193 (1957); M. N. McDermott and W. Lichten, Phys. Rev. **119**, 134 (1960); R. Kohler, *ibid.* **121**, 1104 (1961).

measured $B(^3P_2)$. This has been done for Hg. For Ba, $B(^3P_2)$ has not been measured so we have calculated $B(^1P_1)$ using the experimental value of $B(^3P_1)$. In both cases the agreement between the experimental and the calculated value of $B\,(^1P_1)$ is quite poor. One should note that there exists a choice of c_1 and c_2 which will predict the correct value of $A\,(^1P_1)$. For Hg, this choice ($c_1 \cong 0.4135$, $c_2 = 0.9105$) not only lies well outside the range of c values obtained in other ways but also still leaves the calculated value of $B(^1P_1)$ (453 Mc/sec) in poor agreement with the experimental value. The conclusion to be drawn from the above discussion is that the Breit-Wills theory is inadequate to predict the singlet-state hfs constants from a knowledge of the triplet-state hfs constants.

III. MODIFIED BREIT-WILLS THEORY

We shall assume as a starting point that we have (or could obtain) the Hartree-Fock solutions for the external s and p electron in both the ^1P and the ^3P states. The radial wave function for the p electron in these states is known to be quite different, as, for example, in beryllium[5] and calcium,[6] while the s-electron wave function is quite similar. In order to have a theory with less parameters than experimental results we choose to take the s-electron wave function to be identical in both states. This approach takes into account some configuration interaction, but the validity of the approximation will be in the comparison of theory and experiment. We now introduce spin-orbit and spin-other-orbit interactions which mix the $^1P_1$ and $^3P_1$ states. While the size of the spin-orbit interaction is not much smaller than the configuration interaction, we will assume that its introduction does not change our starting Hartree-Fock radial functions since otherwise the problem is not tractable. Our wave functions for the hfs calculations are thus those of Vainshtein and Poleuktov. Although our present interest is in the sp configuration, we will generalize the calculation to the sl configuration since this does not increase the complexity of the problem. Our starting wave functions are (the superscript 0 means pure Russell-Saunders coupling)

$$\begin{aligned} \psi(^3L_{l+1}) &= \psi(^3L^0{}_{l+1}), \\ \psi(^3L_{l-1}) &= \psi(^3L^0{}_{l-1}), \\ \psi(^3L_l) &= \alpha\psi(^3L^0{}_l)+\beta\psi(^1L^0{}_l), \\ \psi(^1L_l) &= -\beta\psi(^3L^0{}_l)+\alpha\psi(^1L^0{}_l), \end{aligned} \tag{1}$$

where the notation is the same as in LMN except that here the p-electron radial wave function in the singlet and triplet states differ. From LMN we have

$$\mathcal{H}_h=\sum_k\sum_i T_e{}^{(k)}(i)\cdot T_n{}^{(k)}=\sum_k \mathcal{H}^{(k)}, \tag{2}$$

so that

$$\begin{aligned} \langle\psi(^3L_l)|\mathcal{H}^{(1)}|\psi(^3L_l)\rangle &= \tfrac{1}{2}K[\alpha^2A^{TT}+2\alpha\beta A^{ST}+\beta^2A^{SS}] \\ &= \tfrac{1}{2}K\left\{\alpha^2\left[\frac{a_s{}^{TT}}{2l(l+1)}+\frac{2l+3}{(2l+1)(2l+2)}\left(l-\frac{\xi}{2l}\right)a_{l+1/2}{}^{TT}+\frac{(l+1)(2l-1)}{(2l+1)2l}a_{l-1/2}{}^{SS}\right]\right. \\ &\quad +2\alpha\beta\frac{[l(l+1)]^{1/2}}{2l+1}\left[-\frac{2l+1}{l(2l+1)}a_s{}^{ST}+\frac{2l+3}{2l+2}\left(1-\frac{\xi}{4l^2(l+1)}\right)a_{l+1/2}{}^{ST}-\frac{2l-1}{2l}a_{l-1/2}{}^{ST}\right] \\ &\quad \left.+\beta^2\left[\frac{2l+3}{2(2l+1)}\left(1+\frac{\xi}{l(2l+2)}\right)a_{l+1/2}{}^{SS}+\frac{2l-1}{4l+2}a_{l-1/2}{}^{SS}\right]\right\}, \end{aligned} \tag{3}$$

where $K=F(F+1)-I(I+1)-J(J+1)$.

These matrix elements are obtained from LMN, Eq. (12), where c_1 and c_2 are set equal to their values for pure RS coupling. The superscripts on the single-electron dipole coupling constants refer to triplet-triplet, singlet-triplet, and singlet-singlet values of $\langle 1/r^3\rangle$. $\langle 1/r^3\rangle$ is related to the coupling constants through the usual

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[5] D. R. Hartree and W. Hartree, Proc. Roy. Soc. (London) **A154**, 588 (1936).

[6] D. R. Hartree and W. Hartree, Proc. Roy. Soc. (London) **A164**, 167 (1938).

equations, e.g.,

$$a^{ST}{}_{j}=2\mu_I\frac{l(l+1)}{j(j+1)}\left\langle\frac{1}{r^3}\right\rangle^{ST}F_j. \tag{4}$$

As stated previously, we will take $a_s{}^{TT}=a_s{}^{ST}$; $a_s{}^{SS}$ does not enter the problem. For the quadrupole coupling constant, we obtain in a similar manner

$$\langle\psi(^3L_l)|\mathcal{H}^{(2)}|\psi(^3L_l)\rangle=B\frac{3[K(K+1)-\frac{4}{3}I(I+1)J(J+1)]}{4I(2I-1)2J(2J-1)}, \tag{5}$$

and

$$\begin{aligned}B=\alpha^2B^{TT}+2\alpha\beta B^{ST}+\beta^2B^{SS}=&\frac{\alpha^2}{2l+1}\left\{\left(\frac{l(2l-1)(2l+4)}{(2l+2)(2l+1)}-\frac{6\eta}{2l+1}\right)b_{l+1/2}{}^{TT}+(l+1)b_{l-1/2}{}^{TT}\right\}\\&+\frac{2\alpha\beta}{2l+1}\left[\left(\frac{(l(l+1))^{1/2}}{2l+1}\frac{(2l-1)(2l+4)}{2l+2}-\frac{3\eta}{(2l+1)(l(l+1))^{1/2}}\right)b_{l+1/2}{}^{ST}+[l(l+1)]^{1/2}b_{l+1/2}{}^{ST}\right]\\&+\frac{\beta^2}{2l+1}\left\{\left(\frac{(2l-1)(l+2)}{2l+1}+\frac{3\eta}{2l+1}\right)b_{l+1/2}{}^{SS}+lb_{l-1/2}{}^{SS}\right\},\end{aligned} \tag{6}$$

and as before we have for the single-electron singlet-triplet quadrupole coupling constant

$$b_j{}^{ST}=\frac{2j-1}{2j+2}eQ\left\langle\frac{1}{r^3}\right\rangle^{ST}R_j. \tag{7}$$

For the $^1P_1$ state, $\langle\psi(^1L_l)|\mathcal{H}^{(k)}|\psi(^1L_l)\rangle$ is obtained from $\langle\psi(^3L_l)|\mathcal{H}^{(k)}|\psi(^3L_l)\rangle$ by replacing α with $-\beta$ and β with α in Eqs. (3) and (6).

IV. DISCUSSION OF THE MODIFIED BREIT-WILLS THEORY

The modified Breit-Wills theory in addition to the usual parameters requires a knowledge of three different values of $\langle 1/r^3\rangle$, namely, $\langle 1/r^3\rangle^{TT}$, $\langle 1/r^3\rangle^{ST}$, and $\langle 1/r^3\rangle^{SS}$. $\langle 1/r^3\rangle^{TT}$ is obtained as usual from the $^3P_2$ and $^3P_1$ hfs. The most obvious way to estimate $\langle 1/r^3\rangle^{ST}$ is from the fine structure. Vainshtein and Poluektov have derived all the matrix elements for the magnetic interactions in the sl configuration. While they have taken the l-electron wave function to be different in the singlet and triplet states for evaluating the magnetic matrix elements, they take them to be the same in evaluating the electrostatic interactions. This inconsistency is easily removed and one obtains for the matrix elements of the interaction in the sl configuration:

$$\begin{aligned}&H_{11}=E_0+F_0-G_l+l\delta+c_1M,\\&H_{22}=E_0+F_0-G_l-\delta,\\&H_{33}=E_0'+F_0'+G_l',\\&H_{44}=E_0+F_0-G_l-(l+1)\delta-c_2M,\\&H_{23}=H_{32}=\kappa.\end{aligned} \tag{8}$$

The quantities δ, κ, c_1, c_2, and M are defined by Vainshtein and Poluektov. [The c_1 and c_2 used in Eq. (8) are not the intermediate coupling coefficients of LMN.] The quantities E_0, F_0, and G_l are defined by Condon and Shortley,[7] where the unprimed letters refer to the triplet state and the primed letters refer to the singlet state. Except for the lightest elements, the spin-spin interaction M is negligible and we shall drop this term for all subsequent discussion. With this simplification the eigenvalues of the energy matrix become

$$\begin{aligned}&\epsilon_1\equiv E(^3L_{l+1})=E_0+F_0-G_l+l\delta,\\&\epsilon_4\equiv E(^3L_{l-1})=E_0+F_0-G_l-(l+1)\delta,\\&\epsilon_{2,3}\equiv E(^{3,1}L_l)=\tfrac{1}{2}(E_0+F_0-G_l+E_0'+G_l')\\&\qquad\mp[\tfrac{1}{4}(E_0'+F_0'+G_l'-E_0-F_0+G_l+\delta)^2+\kappa^2]^{1\ 2}.\end{aligned} \tag{9}$$

There are four experimental energies and four unknowns so we can solve for the quantities $E_0+F_0-G_l$, $E_0'+F_0'+G_l'$, δ, and κ. It should be noted that we cannot obtain E_0, F_0, G_l, E_0', F_0', or G_l' individually.

One can now define a parameter λ after the suggestion of King and Van Vleck[8] which in the above notation is given by $\lambda=\kappa/\delta[l(l+1)]^{1/2}\cong\xi'_l/\xi_l$. ξ and ξ' are, respectively, the diagonal and the off-diagonal values of the spin-orbit operator. In the treatment of King and Van Vleck, $\lambda=\xi'_l/\xi_l$, whereas in the more complete treatment

TABLE III. Comparison of experimental and theoretical values of the quadrupole coupling constant using Breit-Wills theory.

	Experimental values (Mc/sec)			Calculated values (Mc/sec)	
Element	$B(^3P_2)$	$B(^3P_1)$	$B(^1P_1)$	$B(^3P_1)$	$B(^1P_1)$
Ba^{137} [a]	⋯	41.61	58.2	⋯	80.3
Hg^{201} [b]	399.1	−280.107	260	−259	459

[a] G. zu Putlitz, Ann. Physik 11, 248 (1963); D. A. Jackson and D. H. Tuan, Phys. Rev. Letters 11, 209 (1963).

[b] See footnote c, Table II.

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[7] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935), Chaps. 6, 7.

[8] G. W. King and J. H. Van Vleck, Phys. Rev. 56, 464 (1939).

of VP, λ also includes the effect of spin-other-orbit interaction. Values of λ for the *sp* configuration are tabulated by King and Van Vleck and may be obtained from the following expression

$$\lambda=\frac{2l+1}{[(l+1)l]^{1/2}}\,\frac{[\Delta_0(\epsilon_{32}-\Delta_0)]^{1/2}}{\epsilon_{14}},\tag{10}$$

where $\epsilon_{32}=\epsilon_3-\epsilon_2$, etc., and $\Delta_0=[l/(2l+1)]\epsilon_{14}-\epsilon_{24}$. From the values of λ for the heavier group II elements and Watari's[9] calculations for Mg and Be it seems that λ for the *sp* configuration should always be of the order of 0.8 although values of $\lambda>1$ may be possible in other configurations. From VP we have

$$\xi_l=\tfrac{1}{2}\alpha^2\int\frac{Z(r)}{r^3}[R^T{}_l(r)]^2dr,\tag{11}$$

and (α is the fine structure constant)

$$\xi_l'=\tfrac{1}{2}\alpha^2\int\frac{Z(r)}{r^3}[R^S{}_l(r)R_l{}^T(r)]dr.\tag{12}$$

Making the usual approximation we can write

$$\xi_l\cong 2\mu_0{}^2Z_i\langle 1/r^3\rangle^{TT},\tag{13}$$

$$\xi_l'\cong 2\mu_0{}^2Z_i\langle 1/r^3\rangle^{ST},\tag{14}$$

so that we now obtain the useful relation $\lambda=\langle 1/r^3\rangle^{ST}/\langle 1/r^3\rangle^{TT}$. We may carry this idea one step further and write

$$\left\langle\frac{1}{r^3}\right\rangle^{SS}\Big/\left\langle\frac{1}{r^3}\right\rangle^{TT}=\lambda^2.\tag{15}$$

These two expressions relating the different values of $\langle 1/r^3\rangle$ follow immediately if one can write $R_l{}^s=\lambda R_l{}^T$ in the region where the contribution to $\langle 1/r^3\rangle$ occurs. This is certainly not true in general. We know, however, that only the region near the nucleus is important in calculating $\langle 1/r^3\rangle$, so that if the above relation holds reasonably well in this region, our expressions relating $\langle 1/r^3\rangle$ are valid. This type of approximation has been made by Fermi[10] for the alkalis. An examination of the Hartree-Fock radial function for calcium[6] shows that all the 4*p* nodes, and all the maxima except the one furthest from the nucleus occur at the same place for both the singlet and the triplet radial function. In addition, the *p*-state radial functions are proportional out to a radius of 3 atomic units.

Physically, one expects the *l*-electron wave function in both the singlet and triplet states to have the same shape near the nucleus since the nuclear Coulomb field and the core-electron screening are the dominant terms in the Hamiltonian. In the outer region the singlet and triplet functions differ because of the exchange interaction between the *s* and the *l* electron. The exchange interaction in the triplet state keeps the *s* and *l* electrons apart so that their Coulomb repulsion is minimized. Thus, when normalization is taken into account, the *l*-electron wave function in the triplet state has a greater density near the nucleus than it does in the singlet state. The opposite effect, although much less pronounced, occurs for the *s*-electron radial function which lies closer to the nucleus in the singlet state.

TABLE IV. Calculated value of $A(^1P_1)$ and derived single-electron coupling constants using the modified Breit-Wills theory.

		Calculated values (Mc/sec)			Experimental values (Mc/sec)
Element	λ	a_s	$a_{3/2}$	$A(^1P_1)$	$A(^1P_1)$
Cd^{111}	0.7618	−12 372.0	−266.6	75.4	186
Ba^{137}	0.8414	...	...	...	...
Hg^{201}	0.7578	−12 908.9	−166.4	1244.7	1316

V. COMPARISON OF MODIFIED BREIT-WILLS THEORY WITH EXPERIMENT

The additional parameter λ needed in the modified theory is given in Table IV. All other parameters have been given in Table I. The values of a_s and $a_{3/2}$ obtained from the modified theory are listed in Table IV together with the predicted value of $A(^1P_1)$. It can be seen that there is a significant improvement in the agreement between the predicted and the experimental value of $A(^1P_1)$. In both cases the predicted values are less than the experimental value. This is not surprising, since from our previous discussion we know that a_s is actually larger in the singlet state although our theory does not provide a method of estimating this increase. In Table V is given the comparison for the quadrupole interaction constants. Here we find that the predicted and measured values are in very good agreement for both Ba and Hg.

In the case of cadmium the remaining discrepancy between the experimental and the theoretical value of $A(^1P_1)$ can be ascribed to two effects: (1) the extreme sensitivity of $A(^1P_1)$ on the degree of intermediate coupling and (2) the failure to take into account the fact that a_s is different in the singlet and the triplet states.

VI. CONCLUSIONS

The hyperfine structure of the singlet state of the first excited *sp* configuration of the group II elements

TABLE V. Calculated value of quadrupole coupling constants using modified Breit-Wills theory.

		Calculated values (Mc/sec)			Experimental values (Mc/sec)	
Element	λ	$b_{3/2}$	$B(^3P_1)$	$B(^1P_1)$	$B(^3P_1)$	$B(^1P_1)$
Ba^{137}	0.8414	73.0		55.5	41.61	58.2
Hg^{201}	0.7578	399.1	−268.2	257.7	−280.107	260

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[9] Wataro Watari, Progr. Theoret. Phys. (Kyoto) 4, 314 (1949).

[10] E. Fermi, Z. Physik **59**, 680 (1930).

cannot be calculated reliably from the hfs of the triplet states if one requires the p electron to have the same radial wave function in both states. A modification of the Breit-Wills theory can be developed which permits the p electron to have a different radial wave function in the singlet and triplet states. This theory introduces an additional parameter λ which can be evaluated from the fine structure. For Hg and Ba the modified theory brings the quadrupole moments estimated from the singlet and triplet states into very good agreement. The theory also greatly improves the agreement for the dipole coupling constants.

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Electron Bremsstrahlung from Proton-Excited Targets

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Formulas are derived for continuous x-ray quantum yields from proton-bombarded targets due to electron acceleration during collision and due to subsequent bremsstrahlung as the electrons stop. Measurements with a proportional counter of yields from a beryllium target bombarded with 1.6-MeV protons are in agreement within a factor of two with calculated bremsstrahlung yields, and they are in qualitative agreement with calculated acceleration yields, which fall in a band at roughly one-quarter of the bremsstrahlung quantum energies.

WHILE studying soft x-ray emission from targets bombarded with protons in the range of energies from 1.3 to 1.6 MeV, we have observed a rapid increase in intensity of the continuous spectrum at quantum energies below the maximum energy which can be transferred to an electron by collision with a proton. We attribute this spectrum to electron bremsstrahlung produced by acceleration and subsequent deceleration of target electrons. That such radiation must exist has been suggested before,[1] but it appears that no observations or calculations of the intensity of such a spectrum have been reported previously.[2] In this article, formulas for spectral yield are derived from well-known electron-bremsstrahlung yield formulas. Computations based on these formulas are in agreement with observed spectral yields from beryllium targets, and within the limitations imposed by competing characteristic radiation yields they agree also with observations from targets of carbon and heavier elements.

THEORY

The cross section for bremsstrahlung production from collisions between electrons moving with velocity v and stationary nuclei was derived in a classical calculation by Kramers,[3] who obtained the quantum yield

$$\sigma(\nu)d\nu=\frac{16\pi q^2\alpha v^2}{3\sqrt{3}c^2}\frac{d\nu}{\nu} \tag{1}$$

for $h\nu<\frac{1}{2}m_ev^2$ and zero for $h\nu>\frac{1}{2}m_ev^2$. Here α is the fine-structure constant and q is the Rutherford characteristic distance, which for proton-electron collisions can be represented in terms of the first Bohr orbit radius, a_0, as $q=4a_0(13.59/E_0)$, with $E_0=2m_ev^2$ expressed in electron volts. This result is confirmed by available experimental evidence from thin targets.[4] Viewed in center-of-mass coordinates, the interaction by which electrons are accelerated by a moving proton is this type of collision, and the above formula therefore must apply.

Consider a beam of protons of initial energy E_{p0} to be incident on a target surface at angle θ_p with the surface normal. X-ray quanta emerging at angle θ_q from a depth where the proton energy is E_p must penetrate a thickness of target material

$$m=(\cos\theta_p/\cos\theta_q)\int_{E_p}^{E_{p0}}\{s(\epsilon_p)\}^{-1}\,d\epsilon_p\,,$$

in which $s(\epsilon_p)$ is the stopping power as a function of proton energy ϵ_p. Let Z' represent the effective number of electrons of a target atom available for scattering and A the atomic weight. The primary quantum yield

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[1] Č. Zupančič and T. Huus, Phys. Rev. **94**, 205 (1954).

[2] Recent relevant review articles are by S. Town Stephenson, in *Encylopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 30, p. 337, and by E. Merzbacher and H. W. Lewis, *ibid.* (1958), Vol. 34, p. 190.

[3] H. A. Kramers, Phil. Mag. **46**, 836 (1923).

[4] H. Kulenkampff, Ann. Physik **87**, 597 (1928).