# Cluster Decomposition of S-Matrix Elements*

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This paper discusses the physical basis for the cluster-decomposition properties of momentum-space $S$-matrix elements. The starting point is the proposition that in the limit of wide separation the probability for two separate processes should factor as the product of the individual probabilities. It is shown that factorization of the probabilities implies factorization of the corresponding $S$-matrix elements; i.e., any phase factors which might appear are at most constants which can be adjusted to unity. This provides the basis for the proof given recently by Wichmann and Crichton of the momentum-space cluster properties. It is shown that the resulting decomposition equations may contain unwanted factors which depend on the phase of the one-particle $S$-matrix elements. In order to minimize the effect of these factors the decomposition equations can be rewritten so that the unitarity equations for the "connected parts" (which are supposed to be the basis of a dynamical $S$-matrix theory) take their usual form without spurious phase factors.

## I. INTRODUCTION

THE cluster decomposition properties of momentum-space $S$-matrix elements seem to be characteristic of almost all scattering theories. They appear in potential and field theories, and in both cases have been proved in particular instances.[1] In $S$-matrix theory they are considered essential and are introduced as postulates.[2] However, it is certainly not the case that the momentum-space decomposition properties are directly related to experimental fact. (No properties of momentum eigenstates are.) Thus, in order to understand their physical origin, it seems desirable to find some proposition which is itself a statement of physical experience and from which the decomposition properties can be derived, without reference to any particular theory. The establishment of such a physical basis for the decomposition equations would be of interest in any scattering theory, but would be especially valuable to $S$-matrix theory where it is hoped to use only physically verifiable postulates and where the decomposition properties play such a basic role.

It has always been felt that the $S$-matrix decomposition properties should be a consequence of the approximate locality, or short range, of particle interactions. In 1963 Wichmann and Crichton[3] proposed that a suitable expression of this approximate locality could be made by stating the observed fact that experiments sufficiently separated in space or time are mutually independent; which implies that the probability of observing two well separated events will be the product of their separate probabilities. From this proposition it should be possible to derive the momentum-space decomposition equations. But the derivation given by Wichmann and Crichton is incomplete; in particular, they assume without proof that factorization of probabilities implies factorization—without any phase factors—of the corresponding $S$-matrix elements. In this paper I shall complete the derivation.

In Sec. II, I present a mathematical statement of the momentum-space decomposition equations (which are to be derived) and of the independence of well-separated experiments (which is the starting point of the derivation). The section concludes with an outline of the derivation, whose details are given in Secs. III–V. For convenience Sec. VI contains a brief summary of the assumptions used and conclusions reached.

## II. BASIC FORMULATION

The momentum-space decomposition equations, which it is my aim to derive, are rather cumbersome when written in complete generality. Fortunately, it will suffice for the present to give two simple examples, which can be conveniently written in diagrammatic form. Accordingly the decompositions of two simple $S$-matrix elements connecting states of distinct particles are illustrated in Fig. 1, in the form usually assumed for

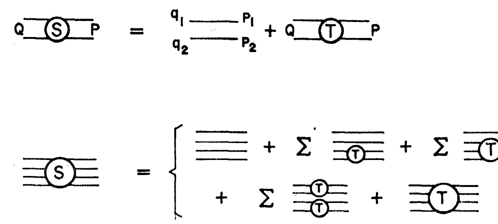


FIG. 1. The decomposition into their connected parts of the momentum-space $S$-matrix elements for the processes (2 particles ← 2 particles) and (4 ← 4).

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[1] Any of the proofs of the Mandelstam representation in potential theory provide a proof of the cluster properties (and much more) of the $S$-matrix element for the process (2 particles ← 2 particles). In Feynman perturbation theory the properties are almost obvious, but a careful discussion has been given by Crichton, [J. H. Crichton, University of California, thesis, University of California Radiation Laboratory UCRL-11961 (unpublished). In the L.S.Z. framework they were proved by W. Zimmermann, Nuovo Cimento 13, 503 (1959).

[2] See H. P. Stapp, Institute of Mathematical Sciences, Madras, Report 26 (unpublished); D. I. Olive, Phys. Rev. 135, B745 (1964); or J. R. Taylor, J. Math. Phys. 7, 181 (1966).

[3] E. H. Wichmann and J. H. Crichton, Phys. Rev. 132, 2788 (1963).

analytic $S$-matrix theory. In these diagrams a bubble labeled $S$ between sets of lines $Q$ and $P$ denotes the momentum-space matrix element $\langle Q|S|P\rangle$, where $P=(p_1,\cdots,p_n)$ labels the initial $n$-particle momentum eigenstate[4] and $Q$ the corresponding final state. A straight line with ends $q$ and $p$ denotes a factor[5] $\omega_p\delta_3(\mathbf{q}-\mathbf{p})$ [where $\omega_p=+(\mathbf{p}^2+m^2)^{1/2}$], and a bubble labeled $T$ denotes the "connected part" $\langle Q|T|P\rangle$. Thus the first equation in Fig. 1 represents the familiar decomposition of the $(2\leftarrow 2)$ $S$-matrix element,

$$\langle q_1,q_2|S|p_1,p_2\rangle=\omega_1\delta_3(\mathbf{q}_1-\mathbf{p}_1)\omega_2\delta_3(\mathbf{q}_2-\mathbf{p}_2)+\langle q_1,q_2|T|p_1,p_2\rangle.$$

In fact, as will be seen clearly in Sec. V, equations of the form shown in Fig. 1 can always be written down and used to *define* the functions $\langle Q|T|P\rangle$. The physical content of the decomposition properties, therefore, lies not in the form of the equations but in the statement that, when defined in this way, the connected part $\langle Q|T|P\rangle$ has a factor $\delta_4(\sum q_i-\sum p_j)$ corresponding to conservation of over-all energy momentum but contains *no other $\delta$ functions* of this type (which would represent conservation of energy momentum for some subset of the particles involved).

The physical interpretation of these decomposition properties is very simple. The first example represents the two possibilities for elastic scattering: the incident particles may miss one another entirely and proceed undeflected or they may interact. In the second example the $S$-matrix element for the process (4 particles $\leftarrow$ 4 particles) has five possibilities: all four particles may go through undeflected; any two may go through while two interact; any one may go through while three interact; the four particles may interact in two pairs; or all four may interact together. These possibilities obviously arise from the fact that interactions are more or less local and hence that two particles can pass one another at a distance without interacting.

While this interpretation seems entirely plausible it is certainly not a matter of experimental experience; no experiments are ever done with pure momentum eigenstates, and even if they were, it is hard to imagine how one could make precise the notion of two plane waves missing one another. It is for this reason that one would like an alternative more physical proposition from which to derive these momentum-space equations.[6]

The starting point proposed by Wichmann and Crichton is based on the observed independence of experiments which are sufficiently separated in space or time. Thus, given two free wave-packet states $a$ and $b$, one considers the process $(b\leftarrow a)$, with probability $P(b\leftarrow a)$, leading from the initial asymptotic state $a$ to the final state $b$. Since the wave packets are more or less localized, there are approximately defined intervals in time and space within which any interactions occur. Thus one could say, for example, that the experiment $(b\leftarrow a)$ occurs in New York today. One may similarly consider a second experiment $(g\leftarrow f)$ which occurs either in New York tomorrow or London today. In either case, the two processes $(b\leftarrow a)$ and $(g\leftarrow f)$ could be viewed as a single process $(b,g\leftarrow a,f)$ and experience tells one that the probability for this composite process is just the product of the separate probabilities $P(b\leftarrow a)$ and $P(g\leftarrow f)$; i.e.,

$$P(b,g\leftarrow a,f)=P(b\leftarrow a)P(g\leftarrow f).$$

Since the separation necessary to achieve this factorization obviously depends on the range of forces and the size of the wave packets, one should expect such factorization only in the limit of infinite separation. Therefore, given wave packets $f$ and $g$, one defines new wave packets $f_x$ and $g_x$ which are obtained from $f$ and $g$ by rigid translation through the four-vector $x=(t,\mathbf{x})$. Then the precise formulation of our basic proposition is[7]

$$P(b,g_x\leftarrow a,f_x)\xrightarrow[|x|\to\infty]{}P(b\leftarrow a)P(g\leftarrow f) \qquad (1)$$

for all wave packets $a$, $b$, $f$, $g$, where

$$|x|=+(t^2+\mathbf{x}^2)^{1/2}.$$

It should be noted that the rate at which this limit is approached will obviously depend on the size and shape of the wave packets considered; also that the second probability on the right is written as $P(g\leftarrow f)$ rather than $P(g_x\leftarrow f_x)$ since translational invariance ensures that these two are equal. Finally the class of wave packets allowed in Eq. (1) must be specified; for definiteness I shall suppose that the states $a$, $b$, $f$, $g$ are defined by momentum-space wave functions in the space $\mathfrak{D}$ of testing functions (i.e., wave functions which are infinitely differentiable and of compact support).

The first task in deriving the momentum-space decomposition equations is to rewrite Eq. (1) in terms of

[4] Strictly speaking, an $n$-particle state should be labeled by its $n$ spin states and $n$ particle types as well as the $n$ momenta. For simplicity I shall consider only spinless particles and assume the normal connection between spin and statistics; although spin introduces no essential complications unless one wishes to prove this connection. I shall not restrict myself to one type of particle—on the contrary, it is usually simpler to consider all distinct particles. However, no ambiguity arises from leaving the particle-type labels implicit.

[5] The factor $\omega_p$ arises from the normalization of the momentum eigenstates; namely, $\langle q|p\rangle=\omega_p\delta_3(\mathbf{q}-\mathbf{p})$.

[6] Even if one were to insist that the detailed structure implied by the momentum-space equations is a matter of experience, it would be necessary to prove that arbitrary phase factors did not appear in front of each term in each equation. This problem has been discussed by Stapp (footnote 2) on the basis of the postulated analyticity of the connected parts. In this way he can prove the connection between spin and statistics. The derivation of the momentum-space decomposition properties given here is independent of any such powerful postulate (which is its *raison d'être*) and naturally does not produce such a result.

[7] It is satisfactory that since this characterization of short range forces was first discussed by Wichmann and Crichton the condition (1) has been proved in field theory by K. Hepp [Helv. Phys. Acta **37**, 659 (1964) for the case that $x$ is space-like, and J. Math. Phys. **6**, 1762 (1965) for $x$ time-like]. In potential theory the condition has been proved for the case that $x$ is space-like [W. Hunziker, J. Math. Phys. **6**, 6 (1965)].

$S$-matrix elements. To do this, one must, of course, assume that an $S$ matrix exists. Specifically, I shall assume that the experimental correspondence between possible initial and possible final states is represented by a unitary operator $S$, mapping the Hilbert space of asymptotic free states onto itself, and that $S$ is invariant under translations.[8] My only other general assumption will be that the superposition principle applies in the usual way; namely, that the Hilbert space of asymptotic states is the Fock space of all stable particles and is divided into disjoint superselection subspaces, within which all vectors represent physically realizable and identifiable states.[9]

The unit vector representing the wave packet $a$ is denoted $|a\rangle$ and, as already mentioned, is characterized by its momentum-space wave function

$$a(\mathbf{P})=a(\mathbf{p}_1,\cdots,\mathbf{p}_n)$$

in $\mathfrak{D}$. The probability for the process $(b\leftarrow a)$ is

$$P(b\leftarrow a)=|\langle b|S|a\rangle|^2\equiv|S_{ba}|^2,\quad \text{say.}$$

If the state $f$ is represented by the wave function $f(\mathbf{P})$ then the wave function for the displaced state $f_x$ is

$$f_x(\mathbf{P})=f(\mathbf{P})\exp(i\sum_{j=1}^{n}p_j\cdot x),$$

$$[p_j^0\equiv(\mathbf{p}_j^2+m_j^2)^{1/2}]$$

and, finally, the composite initial state $(a,f_x)$ is represented by the tensor product[10]

$$|a,f_x\rangle=|a\rangle\otimes|f_x\rangle.$$

With this notation the appropriate restatement of Eq. (1) is

$$|\langle b,g_x|S|a,f_x\rangle|\xrightarrow[|x|\to\infty]{}|\langle b|S|a\rangle\langle g|S|f\rangle|\equiv|S_{ba}S_{gf}|.\quad(2)$$

The next step is to show that the modulus signs can be removed from this equation. This is done in two stages. In Sec. III the superposition principle is used to show that

$$\langle b,g_x|S|a,f_x\rangle\xrightarrow[|x|\to\infty]{}\alpha_{A,F}S_{ba}S_{gf},\quad(3)$$

where $\alpha_{A,F}$ is a constant phase factor ($|\alpha|=1$) depending at most on the superselection classes $A$ and $F$ of the states $a$ and $f$. To prove this one needs a statement of the independence of well separated experiments which is slightly more general than that given in Eqs. (1) and (2) above. This stronger statement is formulated in Eq. (10).

In Sec. IV it is shown that by adjusting the over-all phase of the $S$ operator on each superselection subspace all factors $\alpha_{A,F}$ in Eq. (3) can be reduced to unity. Since the number of factors $\alpha_{A,F}$ is in general greater than the number of arbitrary phases in the definition of $S$, it is necessary to show here that not all factors $\alpha_{A,F}$ are independent. This is done by considering the separation of a composite process into three separate parts.

Finally the limit (3), with $\alpha_{A,F}=1$, is used to derive the usual cluster properties for momentum-space $S$-matrix elements. This derivation, which is due to Wichmann and Crichton, is briefly sketched in Sec. V. This allows me to introduce a somewhat simplified notation and to analyze one remaining difficulty. This difficulty arises when Wichmann and Crichton, in order to derive the momentum-space cluster equations, assume that the phase of the one-particle $S$-matrix elements is zero; i.e., that

$$\langle q|S|p\rangle=\omega_p\delta_3(\mathbf{q}-\mathbf{p}).$$

It is not obvious that the phase of $S$ can always be adjusted to make this true—particularly since it has already been adjusted to absorb the phase factors $\alpha$ in Eq. (3). This means that the decomposition equations cannot in general be written in the simple form shown in Fig. 1. However, it is possible to rewrite the equations (i.e., redefine the connected parts) so that the unitarity equations for the connected parts themselves take their usual form without spurious phase factors. This is discussed in the remainder of Sec. V.

## III. REDUCTION OF POSSIBLE PHASE FACTORS TO A CONSTANT

The first step in the proof of the momentum-space decomposition equations is to apply the superposition principle in Eq. (2),

$$|\langle b,g_x|S|a,f_x\rangle|\xrightarrow[|x|\to\infty]{}|S_{ba}S_{gf}|\quad(2)$$

(for any wave packets $a$, $b$, $f$, $g$) to derive Eq. (3),

$$\langle b,g_x|S|a,f_x\rangle\xrightarrow[|x|\to\infty]{}\alpha_{A,F}S_{ba}S_{gf},\quad(3)$$

where $\alpha_{A,F}$ is a constant phase factor depending only

[8] Since I am considering only the Hilbert space $\mathcal{H}_0$ of asymptotic free states, the generator of time translations is the *free* Hamiltonian $H_0$. I shall not commit myself to the existence of a Hilbert space $\mathcal{H}$ of interacting states and therefore make no claim for the existence of a full Hamiltonian $H$ as generator of time translations on $\mathcal{H}$.

[9] In most formulations of analytic $S$-matrix theory the language of Hilbert space is not explicitly used. It can be shown, however, that such formulations contain sufficient information to reconstruct the Hilbert space of asymptotic states with all the usual properties [J. R. Taylor, Phys. Rev. **140**, B187 (1965)]. Under these circumstances, it seems foolish not to use the convenient formalism which Hilbert space provides.

[10] In the event that some type of particle is present in both $a$ and $f$, the vector $|a,f_x\rangle$ defined here is not normalized. However, as $x\to\infty$ it can be shown that its norm approaches one. (See Wichmann and Crichton, Ref. 3.) Thus, in Eq. (2) below any normalization factor can be ignored.

on the superselection classes $A$ and $F$ of the states $a$ and $f$.

Since the superposition principle applies only within each superselection subspace it is clear that the constant $\alpha_{A,F}$ will, in general, be different according to the superselection classes $A$ of $a$ (or $b$) and $F$ of $f$ (or $g$). However, for the remainder of this section, I shall for convenience suppose that $a$ and $b$ belong to a *fixed* superselection class and similarly $f$ and $g$. No further mention of superselection rules need then be made.

It follows from Eq. (2) that for any set of wave packets $a$, $b$, $f$, $g$ one can find a phase factor $\alpha(a,b,f,g;x)$ such that

$$\langle b,g_x|S|a,f_x\rangle=\alpha(a,b,f,g;x)S_{ba}S_{gf}+r(a,b,f,g;x), \quad (4a)$$

where $|\alpha|=1$ and

$$r(a,b,f,g;x)\rightarrow 0, \quad \text{as} \quad |x|\rightarrow\infty. \quad (4b)$$

The phase factor $\alpha$ is to some extent arbitrary. For definiteness I shall fix $\alpha$ such that all terms in Eq. (4a) have the same phase.

In order to prove Eq. (3) one must now show that $\alpha(a,b,f,g;x)$ approaches a constant limit independent of $a$, $b$, $f$, $g$; and to this end I shall first prove that[11]

$$[\alpha(a,b,f,g;x)-\alpha(a',b',f',g';x)]\xrightarrow[|x|\to\infty]{} 0 \quad (5)$$

for any two sets of states $a\cdots$ and $a'\cdots$. This limit will be proved by inserting into Eq. (2) an appropriate superposition of the primed and unprimed states. The existence of the limit (2) for this superposition of states will be shown to imply that the factors $\alpha$ and $\alpha'$ must approach one another as $|x|\rightarrow\infty$.

I shall next prove for any given set of states $a$, $b$, $f$, $g$ that $\alpha(a,b,f,g;x)$ approaches a constant. [It is at this point that the generalization of Eq. (2) is needed.] This result combined with Eq. (5) shows that all factors $\alpha(a,b,f,g;x)$ approach a limit independent of $a$, $b$, $f$, $g$; and Eq. (3) follows immediately.

The rather tedious details of the two steps in this proof are as follows.

### Proof of Eq. (5)

It is sufficient to prove that

$$[\alpha(a,b,f,g;x)-\alpha(a',b',f,g;x)]\rightarrow 0 \quad (6)$$

for fixed $f$ and $g$, since the argument leading to this result gives the same limit for

$$[\alpha(a',b',f,g;x)-\alpha(a',b',f',g';x)]$$

and from these two results Eq. (5) follows. Accordingly I shall fix $f$ and $g$ and assume that $S_{gf}\neq 0$. (Otherwise there is nothing to prove, since the factors $\alpha$ are undefined.)

[11] It is perhaps worth noting that since $|\alpha|=1$ the condition $(\alpha-\alpha')\rightarrow 0$ is equivalent to the condition $\alpha/\alpha'\rightarrow 1$.

Now let us consider any four states $a$, $b$ and $a'$, $b'$ with $S_{ba}\neq 0\neq S_{b'a'}$. It is convenient to assume also that either $S_{ba'}$ or $S_{b'a}$ (the former, say) is nonzero. (In the event $S_{ba'}=S_{b'a}=0$ the same proof works, but with one intermediate step using the state $|a''\rangle=|a\rangle+|a'\rangle$.) It suffices to prove that

$$[\alpha(a,b,f,g;x)-\alpha(a',b,f,g;x)]\rightarrow 0;$$

then the same argument gives the same result for

$$[\alpha(a',b,f,g;x)-\alpha(a',b',f,g;x)]$$

and Eq. (6) follows.

Replacing the state $|a\rangle$ in Eq. (2) by the superposition $|a\rangle-\lambda|a'\rangle$, one finds that

$$|\langle b,g_x|S|a-\lambda a',f_x\rangle|\rightarrow|\langle b|S|a-\lambda a'\rangle\langle g|S|f\rangle| = |(S_{ba}-\lambda S_{ba'})S_{gf}|. \quad (7)$$

Now by choosing $\lambda=S_{ba}/S_{ba'}$ one can make this limit zero. Meanwhile the left-hand side can be rewritten using the identity

$$|a-\lambda a',f_x\rangle=|a,f_x\rangle-\lambda|a',f_x\rangle.$$

Substitution of Eq. (4) for both terms on the left-hand side then brings the limit (7) into the form

$$|[\alpha(a,b,f,g;x)-\alpha(a',b,f,g;x)]S_{ba}S_{gf}+r-\lambda r'|\rightarrow 0. \quad (8)$$

Now in this equation $r$ and $r'$ tend to zero as $x\rightarrow\infty$, and $S_{ba}S_{gf}\neq 0$; it follows that

$$[\alpha(a,b,f,g;x)-\alpha(a',b,f,g;x)]\rightarrow 0.$$

Q.E.D.

### Proof that $\alpha(a,b,f,g;x)$ Approaches a Constant

In Eq. (5) one can choose $a'$, $b'$, $f'$, $g'=a$, $b$, $f_\xi$, $g_\xi$, where $\xi$ is any constant four-vector, and then, since

$$\alpha(a,b,f_\xi,g_\xi;x)=\alpha(a,b,f,g;x+\xi),$$

one obtains the result

$$[\alpha(x)-\alpha(x+\xi)]\rightarrow 0 \quad (\text{any fixed } \xi).$$

This condition has a superficial resemblance to the Cauchy condition for the convergence of $\alpha(x)$. It is insufficient, however, because it holds only for any *fixed* $\xi$; for example,

$$\alpha(x)=\exp(i\ln|x|) \quad (9)$$

would satisfy this condition but obviously has no limit. In fact, since such a behavior for $\alpha$ is consistent both with the original formulation of the independence of well separated experiments [Eqs. (1) and (2)] and with the superposition principle, an extra assumption is needed to eliminate it. In an earlier version of this paper,[12] I proved that $\alpha(x)$ has a limit by assuming that for any direction in which $x\rightarrow\infty$ there are at least some states for which the limit (2) is reached

[12] J. R. Taylor, University of Wisconsin report, 1965 (unpublished).

faster than $|x|^{-1}$. Although this condition is amply fulfilled in field theory,[13] it is not particularly satisfactory in the present context where the aim is to find a statement of approximate locality which is more a matter of experimental experience. A more satisfactory procedure is to exploit the superposition principle more strongly as follows.[14]

Let $x$ and $x'$ be any two four-vectors. Then if $f$ and $f'$ are in the same superselection class, so are $f_x$ and $f'_{x'}$. Thus the superposition principle gives meaning to initial and final states of the form $(f_x+f'_{x'})$ and $(g_x+g'_{x'})$. Then if $a$, $b$, $f$, $f'$, $g$, $g'$ are any given states and both $x$ and $x'$ are made sufficiently large the processes $(b\leftarrow a)$ and $(g_x+g'_{x'}\leftarrow f_x+f'_{x'})$ should be independent. Thus, precisely the reasoning which led to Eq. (2) allows one to state that, because interactions are more or less local,

$$|\langle b,\, g_x+g'_{x'}|S|a,\, f_x+f'_{x'}\rangle| - |S_{ba}\langle g_x+g'_{x'}|S|f_x+f'_{x'}\rangle|\to 0\,, \quad (\text{as } |x| \text{ and } |x'|\to\infty). \tag{10}$$

Mathematically, Eq. (10) is of course more general than Eq. (2)—Eq. (2) implies Eq. (10) only in the case that $|x|\to\infty$ with $(x'-x)$ fixed, whereas Eq. (10) implies some uniformity for this limit—but physically both are expressions of the same idea, the short range of particle interactions.

If we now consider any given states $a$, $b$, $f$, $g$ we can use Eq. (10) to show that $\alpha(a,b,f,g;x)=\alpha(x)$, say, approaches a limit. Let us consider the matrix element

$$M(x,x')=\langle b,\, g_x+g_{x'}|S|a,\, f_x-f_{x'}\rangle.$$

Using Eq. (10) we can write

$$|M(x,x')|=|S_{ba}\langle g_x+g_{x'}|S|f_x-f_{x'}\rangle|+R_1(x,x')\,,$$

where $R_1\to 0$ as $|x|$ and $|x'|\to\infty$. Expansion of the matrix element on the right-hand side of this equation gives four terms, of which two cancel and two vanish as $|x-x'|\to\infty$. Therefore,

$$M(x,x')\to 0\,,\quad (\text{as } |x|,\ |x'| \text{ and } |x-x'|\to\infty). \tag{11}$$

Returning to the original expression for $M(x,x')$ and expanding, we find that

$$M(x,x')=[\alpha(x)-\alpha(x')]S_{ba}S_{gf}+R_2(x,x')\,, \tag{12}$$

where our previous results, combined with Eq. (10), imply that $R_2\to 0$ as $|x|$, $|x'|$ and $|x-x'|\to\infty$. Eqs. (11) and (12) together show that

$$[\alpha(x)-\alpha(x')]\to 0\,,\quad (\text{as } |x|,\ |x'| \text{ and } |x-x'|\to\infty).$$

This is equivalent to the Cauchy condition and establishes that each $\alpha(a,b,f,g;x)$ has a limit as $|x|\to\infty$.

[13] Hepp (Ref. 7) has shown that as $x\to\infty$ along any direction in space-time there are states for which the limit (2) is approached faster than any power of $|x|^{-1}$.

[14] I am indebted to Dr. Van Hove for pointing out how this can be done.

Combining this result with the result (5) that any two factors $\alpha(a,b,f,g;x)$ and $\alpha(a',b',f',g';x)$ approach one another as $|x|\to\infty$, we conclude that all $\alpha(a,b,f,g;x)$ have a common finite limit.

## IV. ADJUSTMENT OF THE PHASE FACTORS $\alpha$ TO UNITY

Our next task is to show that the phase factors $\alpha_{A,F}$ in Eq. (3),

$$\langle b,g_x|S|a,f_x\rangle\to\alpha_{A,F}S_{ba}S_{gf}\,, \tag{3}$$

can be adjusted to unity by exploiting the arbitrariness in the over-all phase of $S$.

Let us briefly recall the extent of this arbitrariness. The unitary $S$ operator is related to the physical correspondence between initial and final free states by Wigner's theorem.[15] This theorem establishes that, if the physical correspondence of states is one-to-one, "onto" and preserves superpositions, then it can be represented by a unitary or anti-unitary operator $S$ on the Hilbert space of free states and that $S$ is unique up to one phase factor on each superselection subspace. (That $S$ is actually unitary follows because the relation between configuration and momentum spaces is the same for initial and final states.) Thus if there are $n$ different superselection classes, one has available just $n$ phases which can be freely adjusted. On the other hand, there are $n^2$ phase factors $\alpha_{A,F}$ which have to be absorbed. This means that, except in the case $n=1$ (where there are no superselection rules), one must establish some constraints among the factors $\alpha_{A,F}$. In practice, of course, when $n$ is greater than one it is actually infinite, and it turns out that the procedure for removing all factors $\alpha_{A,F}$ depends not only on the number or superselection classes but on their relationship one to another. It is therefore impossible to give a general prescription for eliminating the $\alpha_{A,F}$ and instead I shall consider just three cases:

(A) The simple case where there are no superselection rules ($n=1$).

(B) A world with one superselection rule, such as charge. (Here there is a single infinity of superselection classes characterized by their charge number $N=0$, $\pm1$, $\pm2$, $\cdots$.)

(C) The actual world as it seems to be, with superselection classes uniquely defined by three integers—charge, baryon, and lepton numbers.

### A. The Case of No Superselection Rules

When no superselection rules operate (e.g., in a world made up of chargeless pions) there is one arbitrary phase in $S$ and one phase factor $\alpha$ to be absorbed. Inspection of Eq. (3) shows that replacement of $S$ by $\beta S$ ($|\beta|=1$) changes $\alpha$ to $\alpha\beta$. There is therefore a

[15] I have in mind the theorem originally stated for rotations in the Appendix to Chap. 20 of E. P. Wigner, *Group Theory* (Academic Press Inc., New York, 1959).

unique phase for $S$, given by $\beta=\alpha^*$, such that the factor $\alpha$ in Eq. (3) is unity.

### B. The Case of One Additive Superselection Rule

If there were just one additive superselection rule (e.g., that on charge in a world of charged pions or of protons and antiprotons), there would be a single infinity of superselection classes which could be labeled by their charge number $N=0, \pm1, \pm2, \cdots$. Since the number of phase factors $\alpha$ is the square of the number of superselection classes, the first task is to find constraints among the factors $\alpha$.

The first such contraint arises from the physical equivalence of the state vectors $|a,f\rangle=|a\rangle\otimes|f\rangle$ and $|f,a\rangle=|f\rangle\otimes|a\rangle$, which means that the two vectors differ at most by a phase factor. Substitution in Eq. (3) shows immediately that

$$\alpha_{A,F}=\alpha_{F,A};$$

i.e., that $\alpha$ is symmetric.

The second constraint arises when one considers the separation of a composite process into three well separated parts. Physically, one obviously expects a result analogous to Eq. (1); namely

$$P(d_x, f, g_y \leftarrow a_x, b, c_y) \to P(d\leftarrow a)P(f\leftarrow b)P(g\leftarrow c), \quad (\text{as } |x|, |y|, |x-y| \to \infty). \tag{13}$$

Just as in the case of two separate experiments, this property implies the existence of a phase factor $\alpha(a\cdots g; x,y)$ which relates the corresponding $S$-matrix elements, and by arguments analogous to those of Sec. III one can show that this phase factor has a constant limit as $|x|$, $|y|$ and $|x-y|\to\infty$.

However, the two special cases where first $x\to\infty$ ($y$ fixed) and then $y\to\infty$, or vice versa, are already covered by the results of Sec. III which imply that

$$\begin{aligned}\langle d_x,f,g_y|S|a_x,b,c_y\rangle &\to \alpha_{A,BC}\alpha_{B,C}S_{da}S_{fb}S_{gc}, \quad (x\to\infty, y\to\infty)\\ &\to \alpha_{AB,C}\alpha_{A,B}S_{da}S_{fb}S_{gc}, \quad (y\to\infty, x\to\infty),\end{aligned}$$

where $BC$ denotes the superselection class of the state $|b,c\rangle$. It follows that these two limits must be the same and hence that

$$\alpha_{A,BC}\alpha_{B,C}=\alpha_{AB,C}\alpha_{A,B}. \tag{14}$$

Having obtained the two constraints $\alpha_{A,F}=\alpha_{F,A}$ and Eq. (14), one can return to the problem of absorbing all the factors $\alpha$. The first task is to rewrite Eq. (14) in the form appropriate to the superselection rule under consideration, namely, charge. In this case, if the states $|a\rangle$ and $|b\rangle$ have charge $N$ and $M$, respectively, then the charge number of the state $|a,b\rangle$ is just $N+M$. Thus, Eq. (14) takes the form

$$\alpha_{N,M+L}\alpha_{M,L}=\alpha_{N+M,L}\alpha_{N,M}. \tag{15}$$

One can now reduce all of the factors $\alpha_{N,M}$ to unity in the five following steps:

(i) I consider first the vacuum superselection class $O$. (It is not, of course, necessary to insist that experiments be done with the vacuum *state* in order to discuss the vacuum superselection class. States such as $\pi^0$, $\pi^+\pi^-$, $p\bar{p}$ all belong to $O$ and quite generally a state $|o\rangle$ is in $O$ if the states $|a\rangle$ and $|a\rangle\otimes|o\rangle$ belong to the same superselection class for any $|a\rangle$.) Substitution of $M=0$ in Eq. (15) gives

$$\alpha_{N,L}\alpha_{0,L}=\alpha_{N,L}\alpha_{N,0}$$

and since $\alpha_{N,M}$ is symmetric this implies that $\alpha_{N,0}=\alpha_{0,N}$ is the same[16] for all $N$.

Now if $o$ and $o'$ are any two states in $O$, a particular case of Eq. (3) is

$$\langle a',o_x'|S|a,o_x\rangle \to \alpha_{A,0}\langle a'|S|a\rangle\langle o'|S|o\rangle$$

and from this it is clear that by adjusting the phase of $S$ on $O$ [i.e., $\text{ph}(\langle o'|S|o\rangle)$] one can remove $\alpha_{A,0}$. Furthermore, since the states $a$ and $(a,o)$ belong to the same superselection class the adjustment to $\alpha_{A,0}=1$ is independent of the phase of $S$ on all superselection classes other than that of the vacuum.

(ii) Next, the factors $\alpha_{N,1}$ ($N>0$) are adjusted in succession. A special case of Eq. (3) is (in much simplified notation)

$$S_{22}\to\alpha_{1,1}S_{11}S_{11}.$$

Thus to set $\alpha_{1,1}=1$ it is only necessary to fix the phase $\text{ph}(S_{22})$ in terms of $\text{ph}(S_{11})$. Similarly,

$$S_{N+1,N+1}\to\alpha_{N,1}S_{NN}S_{11}$$

and adjustment of $\text{ph}(S_{N+1,N+1})$ relative to $\text{ph}(S_{11})$ leads to $\alpha_{N,1}=1$, with $\text{ph}(S_{11})$ still arbitrary.

By the same process one can arrange that $\alpha_{-N,-1}=1$ ($N>0$) leaving $\text{ph}(S_{-1,-1})$ free.

(iii) Using Eq. (15) one can now show by induction that all remaining factors $\alpha_{N,M}$ ($NM\geqslant0$) are automatically unity. The first step is

$$\alpha_{2,2}\alpha_{1,1}=\alpha_{3,1}\alpha_{2,1},$$

whence $\alpha_{2,2}=1$, and so on.

(iv) Another special case of Eq. (3) is[17]

$$S_{00}\to\alpha_{1,-1}S_{11}S_{-1,-1}.$$

Thus, by fixing $\text{ph}(S_{-1,-1})$ relative to $\text{ph}(S_{11})$ one can arrange that $\alpha_{1,-1}=1$.

(v) Finally, all remaining factors $\alpha_{N,M}$ ($NM<0$)

[16] In fact, if one wishes to consider the vacuum state itself—and the formalism certainly gives it a well-defined meaning—it is easy to show for any $N$ that $\alpha_{N,0}=\langle\text{vac.}|S|\text{vac.}\rangle$.

[17] Written more fully, an example of this equation is

$$\langle\pi^+,\pi^-|S|\pi^+,\pi^-\rangle\to\alpha_{1,-1}\langle\pi^+|S|\pi^+\rangle\langle\pi^-|S|\pi^-\rangle.$$

are automatically unity, as one can show by induction. The elements $\alpha_{N,-1}$ follow from Eq. (15) in the form

$$\alpha_{N,-1}\alpha_{N-1,1}=\alpha_{N-1,0}\alpha_{1,-1}.$$

The remaining factors follow as in (iii) above.

In conclusion, steps (i)–(v) above establish, for the case that the only superselection rule is that on charge (or baryon number or lepton number), that all $\alpha_{N,M}$ can be adjusted to unity leaving one arbitrary phase in the definition of $S$ (that of $S$ on the superselection class $N=+1$, say).

### C. The Case of Three Additive Superselection Rules

At the present time it is generally believed that there are four superselection rules in all—conservation of charge, baryon number and lepton number and the separation of states of integral and half-odd-integral angular momentum—of which the last is redundant since the baryons and leptons all have half-odd-integral spin and appear to be the only such particles. It is therefore worth considering briefly the extension of case (B) above to a world where superselection classes are characterized by three integers—charge number $Q$, baryon number $B$ and lepton number $L$. As representative states one could consider states of the form $(Q,B,L)=$ ($Q$ positive pions, $B$ neutrons, $L$ neutrinos).

The procedure of paragraph (B) can be used to absorb all factors $\alpha$ which involve only one type of particle. This still leaves free the phase of $S$ on states of mixed-particle type and these can be used, much as in (B) above, to absorb all the remaining factors $\alpha$. The detailed procedure is tedious but straightforward, so can safely be omitted. The result is that all factors $\alpha$ can be absorbed, leaving *three* arbitrary phases in $S$; namely, those of $S$ on the three superselection classes $(Q,B,L)=(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

## V. PROPERTIES OF MOMENTUM-SPACE MATRIX ELEMENTS

The $S$-matrix elements which have been discussed in the previous two sections are those for physically observable processes, i.e., matrix elements between normalized wave packets. However, since the latter are defined by momentum-space wave functions in $\mathfrak{D}$ it follows that one can define $S$-matrix elements between momentum eigenstates as distributions[18] in $\mathfrak{D}'$. In Sec. II the notation $\langle Q|S|P\rangle=\langle q_1\cdots q_m|S|p_1\cdots p_n\rangle$ was introduced for these distributions.

The wave-packet matrix elements can be written as integrals of the appropriate wave functions times $\langle Q|S|P\rangle$. In particular,

$$\langle b,g_x|S|a,f_x\rangle=\int d\Omega(\mathbf{Q}',\mathbf{Q}'',\mathbf{P}',\mathbf{P}'')\times b^*(\mathbf{Q}')g_x{}^*(\mathbf{Q}'')a(\mathbf{P}')f_x(\mathbf{P}'')\langle Q',Q''|S|P',P''\rangle,$$

where

$$d\Omega(\mathbf{P})=\prod(d^3p/\omega_p).$$

Since

$$f_x(\mathbf{P})=f(\mathbf{P})\exp(i\textstyle\sum p_j\cdot x)$$

the factorization derived in the previous two sections

$$\langle b,g_x|S|a,f_x\rangle \xrightarrow[|x|\to\infty]{} \langle b|S|a\rangle\langle g|S|f\rangle$$

can be written

$$\int d\Omega(\mathbf{Q}',\mathbf{Q}'',\mathbf{P}',\mathbf{P}'')b^*(\mathbf{Q}')g^*(\mathbf{Q}'')a(\mathbf{P}')f(\mathbf{P}'')\times\exp[i(\textstyle\sum p''-\sum q'')\cdot x]\langle Q',Q''|S|P',P''\rangle \xrightarrow[|x|\to\infty]{} \displaystyle\int d\Omega(\mathbf{Q}',\mathbf{Q}'',\mathbf{P}',\mathbf{P}'')b^*(\mathbf{Q}')g^*(\mathbf{Q}'')a(\mathbf{P}')f(\mathbf{P}'')\times\langle Q'|S|P'\rangle\langle Q''|S|P''\rangle.$$

It can be shown[19] that this implies

$$\exp[i(\textstyle\sum p''-\sum q'')\cdot x]\langle Q',Q''|S|P',P''\rangle \xrightarrow[|x|\to\infty]{} \langle Q'|S|P'\rangle\langle Q''|S|P''\rangle \quad (16)$$

the convergence here being convergence in the space $\mathfrak{D}'$ of distributions.

Equation (16) is the essential tool in deriving the $\delta$-function structure of the $S$-matrix elements. The procedure is first to use the usual decomposition equations to define the connected parts $\langle Q|T|P\rangle$ and then to show that $\langle Q|T|P\rangle$ contains no momentum-conserving $\delta$ functions apart from the over-all factor $\delta_4(\sum q-\sum p)$. The role of Eq. (16) in this demonstration may be indicated by the following two obvious facts: First, if any matrix element contains the factor $\delta_4(\sum q-\sum p)$, then multiplication by $\exp[i(\sum p-\sum q)\cdot x]$ leaves the matrix element unchanged, whatever the value of $x$. Second, if

$$\exp[i(\textstyle\sum p-\sum q)\cdot x]R\to 0 \quad \text{as} \quad |x|\to\infty$$

then the distribution $R$ contains no factor $\delta_4(\sum p-\sum q)$ nor any of its derivatives.

It will become clear that the decomposition equations as written in Fig. 1 are not quite correct. A correct form is shown in Fig. 2. In this figure the straight lines which denoted the factor $\omega_p\delta_3(\mathbf{q}-\mathbf{p})$ in Fig. 1 have been replaced by bubbles representing the actual $S$-matrix element $\langle q|S|p\rangle$ [for which Lorentz invariance dictates the form $\gamma\omega_p\delta_3(\mathbf{q}-\mathbf{p})$, $|\gamma|=1$]. The form of the general

[18] For $\langle q_1\cdots q_m|S|p_1\cdots p_n\rangle$ to be defined as a distribution in $\mathfrak{D}'[3(m+n)]$, $\langle b|S|a\rangle$ must define a continuous linear functional on $\mathfrak{D}[3(m+n)]$. Using the fact that $S$ is unitary, one can easily show that this is so.

[19] See Wichmann and Crichton (Ref. 3).

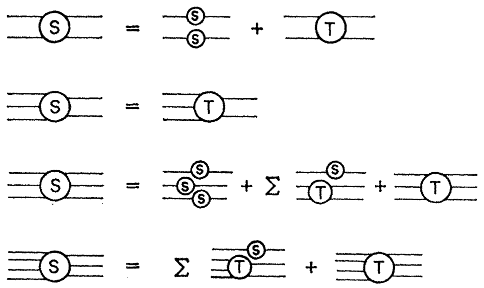

FIG. 2. The first four cluster equations, which define the connected parts $\langle Q|T|P\rangle$. The form of these differs from that of Fig. 1 only in the single-line bubbles $S$, which denote the $S$-matrix element $\langle q|S|p\rangle=\gamma\omega_p\delta_3(\mathbf{q}-\mathbf{p})$. The equations for the processes (2 particles $\leftarrow$ 3 particles), $(2\leftarrow 4)$ and $(4\leftarrow 2)$ are similar to that for $(3\leftarrow 2)$, and are omitted.

decomposition equation can be easily written down from inspection of Fig. 2. It is

$$\langle Q|S|P\rangle=\sum_{\text{partitions }\pi}\{\prod_{i=1}^{n_\pi}\langle Q_i{}^\pi|T|P_i{}^\pi\rangle\}+\langle Q|T|P\rangle. \quad (17)$$

Here the sum is over all partitions $\pi$ of $Q$ and $P$ in the form

$$Q=(Q_1{}^\pi,\cdots,Q_{n_\pi}{}^\pi),$$
$$P=(P_1{}^\pi,\cdots,P_{n_\pi}{}^\pi).$$

The trivial partition $Q=(Q)$, $P=(P)$ is not included in the sum but is written explicitly. The one-particle matrix elements are defined separately as

$$\langle q|T|p\rangle\equiv\langle q|S|p\rangle=\gamma\omega_p\delta_3(\mathbf{q}-\mathbf{p}) \quad \text{(same particle)}$$
$$=0 \quad \text{(different particles)}$$

and

$$\langle q|T|p_1\cdots p_n\rangle=\langle q_1\cdots q_m|T|p\rangle=0 \quad (m,n>1).$$

These equations are used to define the distributions $\langle Q|T|P\rangle$ by induction on the total number of particles. Clearly as this number increases each $\langle Q|T|P\rangle$ appears for the first time in the equation for the corresponding $\langle Q|S|P\rangle$ and is therefore well defined in terms of $\langle Q|S|P\rangle$ and connected parts $\langle Q_i{}^\pi|T|P_i{}^\pi\rangle$ with a smaller number of particles. Furthermore, since each $\langle Q|S|P\rangle$ contains a factor $\delta_4(\sum q-\sum p)$, arising from conservation of energy momentum, it follows by induction from Eq. (17) that $\langle Q|T|P\rangle$ must also contain such a factor.

The proof that $\langle Q|T|P\rangle$ contains no further $\delta$-function factors also proceeds by induction. For each $S$-matrix element $\langle Q|S|P\rangle$ one considers all nontrivial partitions $Q=(Q',Q'')$, $P=(P',P'')$. Inspection of Eq. (17) or Fig. 2 shows that for each such partition one can write

$$\langle Q',Q''|S|P',P''\rangle=\langle Q'|S|P'\rangle\langle Q''|S|P''\rangle+R, \quad (18)$$

where $R$ contains all terms from the right-hand side of Eq. (17) not contained in $\langle Q'|S|P'\rangle\langle Q''|S|P''\rangle$, and in particular, contains $\langle Q',Q''|T|P',P''\rangle$. Now Eq. (16) shows that, as $|x|\to\infty$, the left-hand side of Eq. (18) times $\exp[i(\sum p''-\sum q'')\cdot x]$ approaches $\langle Q'|S|P'\rangle\langle Q''|S|P''\rangle$. Since the latter has a factor $\delta_4(\sum q''-\sum p'')$, this is precisely the limit of the first term on the right-hand side. Therefore, the second term must tend to zero; i.e.,

$$R\exp[i(\sum p''-\sum q'')\cdot x]\to 0 \quad \text{as} \quad |x|\to\infty.$$

Thus, the term $R$ in Eq. (18) does not contain any $\delta$ functions or derivatives of $\delta$ functions in $(\sum q''-\sum p'')$.

Now in the case (2 particles $\leftarrow$ 2 particles) the only nontrivial partition is $Q=(q_1,q_2)$ and $P=(p_1,p_2)$ and in this case $R$ is just $\langle Q|T|P\rangle$ itself. It follows that $\langle q_1,q_2|T|p_1,p_2\rangle$ cannot contain a factor $\delta_3(\mathbf{q}_1-\mathbf{p}_1)$ nor any of its derivatives. [If the two particles are identical or have the same mass, one must also consider the term $\delta_3(\mathbf{q}_1-\mathbf{p}_2)$.] This is the desired result for the process $(2\leftarrow 2)$. In the general case the remainder $R$ in Eq. (18) may contain other terms besides $\langle Q',Q''|T|P',P''\rangle$, but the inductive hypothesis ensures that these do not contain any $\delta$ functions in $(\sum q''-\sum p'')$. Thus, since $R$ itself does not, nor does $\langle Q',Q''|T|P',P''\rangle$.

This completes the proof of the decomposition properties for momentum-space matrix elements $\langle Q|S|P\rangle$. As already noted, their final form [Fig. 2 and Eq. (17)] differs from the usual form shown in Fig. 1 in that the factors $\omega_p\delta_3(\mathbf{q}-\mathbf{p})$ of Fig. 1 are replaced by the matrix elements $\langle q|S|p\rangle$. Now, Lorentz invariance implies that

$$\langle q|S|p\rangle=\gamma\omega_p\delta_3(\mathbf{q}-\mathbf{p}), \quad (|\gamma|=1), \quad (19)$$

but there is no physical principle which ensures that the phase factor $\gamma$ is actually unity. It is customary to suppose that the phase of $S$ can be adjusted to absorb it but this can only be justified with some additional assumptions.[20] Almost all arbitrariness in the definition of $S$ has already been used to remove the phase factors $\alpha$ from Eq. (3). In fact, in the case of no superselection rules the phase of $S$ is already uniquely determined by the requirement $\alpha=1$, while in more general cases the number of remaining free phases is certainly much less than the number of one-particle states. Since there is no reason to expect the factor $\gamma$ in Eq. (19) to be the same for all particles one is forced to accept Eq. (19) as it stands with $\gamma$ not necessarily unity.

[20] The physical basis for the supposition that the one-particle matrix elements have zero phase is as follows: One believes that the actual time dependence of the one-particle states (and the vacuum), as determined by the full Hamiltonian $H$, is identical to that given by the free Hamiltonian $H_0$. Since in the absence of interactions the one-particle and vacuum matrix elements all have the same phase—within each superselection class—one concludes that the same is true even with interactions. In this case all phases can be adjusted to unity.

Obviously these ideas go far beyond what is assumed in this paper and what is usually assumed in $S$-matrix theory. In potential and field theories they appear to be valid but their general validity seems to be an open question.

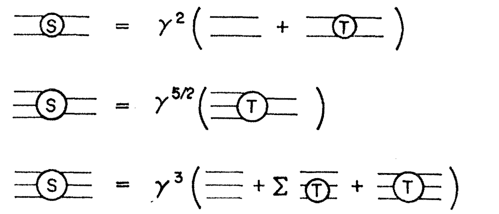

FIG. 3. The decomposition equations rewritten to ensure that unitarity for the connected parts $\langle Q|T|P\rangle$ takes its usual form.

One might hope to restore the decomposition equations to their traditional form by relaxing the requirement that $\alpha=1$ in Eq. (3) and adjusting $S$ such that some suitable combination of $\alpha$ and $\gamma$ was equal to one, but this too is easily seen to be impossible. For example, in the simplest case with no superselection rules the two factors $\alpha$ and $\gamma$ can be seen to enter the decomposition equations in the combinations $\alpha\gamma^2$, $\alpha^2\gamma^3$, $\cdots$ and obviously these cannot all be absorbed unless one can guarantee that $\alpha=\gamma=1$, which one cannot.

Under these circumstances the most one can hope for is to redefine the connected parts $\langle Q|T|P\rangle$ so that, in any particular theory, their properties are independent of the factors $\gamma$. I shall consider the situation in analytic $S$-matrix theory, whose mathematical structure is supposed to be determined by the unitarity equations for the connected parts. These equations can be cast in their usual form as follows.

One proceeds as in Sec. IV to absorb all factors $\alpha$, which justifies the decomposition equations of Fig. 2 and Eq. (17). In the case of a world with only one type of particle (e.g., $\pi^0$) one now redefines the connected parts as shown in Fig. 3. The general form for the decomposition of the process ($m$ particles $\leftarrow n$ particles) becomes

$$\langle Q|S|P\rangle=\gamma^{(m+n)/2}(\textstyle\sum_\pi \prod_i \langle Q_i^\pi|T|P_i^\pi\rangle + \langle Q|T|P\rangle), \quad (20)$$

where now one defines

$$\langle q|T|p\rangle=\omega_p\delta_3(\mathbf{q}-\mathbf{p}).$$

It is easy to see that this new decomposition equation arises from the consistent redefinition of $\langle Q|T|P\rangle$ by

$$\langle Q|T|P\rangle\to\gamma^{(m+n)/2}\langle Q|T|P\rangle.$$

While the content of Eq. (20) is exactly the same as that of Eq. (17), the unitarity equations for the newly defined connected parts are free of all factors $\gamma$. If one writes unitarity for $S$, somewhat symbolically, as

$$\sum_m \int d\Omega(\mathbf{Q})\langle P,n|S|Q,m\rangle\langle P',n'|S|Q,m\rangle^* =\delta_{nn'}\delta(\mathbf{P}-\mathbf{P}'),$$

where $n$, $m$, and $n'$ label the number of particles, then substitution of Eq. (20) gives as unitarity for the connected parts

$$\gamma^{(n-n')/2}\sum_m \int d\Omega(\mathbf{Q})(\textstyle\sum\prod\langle T\rangle)(\sum\prod\langle T\rangle^*) =\delta_{nn'}\delta(\mathbf{P}-\mathbf{P}').$$

Since the right-hand side of this equation is zero unless $n=n'$ the factor $\gamma^{(n-n')/2}$ can simply be omitted; i.e., unitarity for the newly defined connected parts takes the traditional form without any spurious phase factors.

The generalization to a theory with many particles is perfectly straightforward. The factor $\gamma^{(m+n)/2}$ in Eq. (20) is replaced by a product with one factor $\gamma_t^{1/2}$ for each particle of type $t$. This is easily seen to produce the required result.

Since the mathematical structure of analytic $S$-matrix theory is based on the analytic properties of the connected parts and their unitarity equations, our result means that this structure is quite independent of the existence of the factors $\gamma$. In fact, since all observable probabilities have the form $|\langle b|S|a\rangle|^2$, it is clear that, so long as experiments detect only one set of particles at a time, the presence of the factors $\gamma$ will have no observable consequences whatever. On the other hand, it can be seen from Eq. (20) that, as soon as experiments measure interference between states of different types or numbers of particles, the phase factors $\gamma$ will become observable numbers.

## VI. CONCLUSION

The principal goal of this paper has been the derivation of the momentum-space decomposition properties. Since these properties are supposed to be linked with the short range of interactions, the aim was to start with a physically reasonable statement of the short range of particle interactions. And since the decomposition properties appear to be valid in all scattering theories, the derivation given was independent of any particular theory (such as potential theory, field theory or analytic $S$-matrix theory).

The assumptions used were:

### General Assumptions

(1) The existence of a unitary $S$ operator which maps the Hilbert space of asymptotic free states onto itself and which is invariant under translations.[21]

(2) The superposition principle; namely, that the Hilbert space of asymptotic states is divided into disjoint superselection subspaces, within which all vectors represent physically realizable states.

[21] Invariance under the Lorentz group has also been used in writing Eq. (19) for $\langle q|S|p\rangle$. This is inessential to the main result, which holds equally well in a Galilean theory.

### Statement of Approximate Locality

(3) The factorization of the probability for two separate events Eq. (1)

$$P(b,g_x \leftarrow a,f_x) \xrightarrow[|x|\to\infty]{} P(b \leftarrow a)P(g \leftarrow f).$$

This was strengthened in Eq. (10) by replacing the state $f_x$ by $(f_x+f'_{x'})$ and $g_x$ by $(g_x+g'_{x'})$, and requiring that the corresponding limit hold as $|x|$ and $|x'| \to \infty$ [uniformly with respect to $(x-x')$ of course].

(4) The analogous factorization for three separate experiments as in Eq. (13).

### Superselection Rules

(5) Superselection rules defined by discrete additive quantum numbers; namely, charge, baryon, and lepton numbers.[22]

These assumptions led first to factorization of $S$-matrix elements and thence, using the argument of Wichmann and Crichton, to the momentum-space equations, whose final form is shown in Fig. 3 and Eq. (20). While this form may differ from the usual one shown in Fig. 1—a difference which would show up in interference experiments between states of different particle type—nonetheless the unitarity equations for the connected parts take precisely the usual form assumed in analytic $S$-matrix theory.

### ACKNOWLEDGMENTS

I would like to thank Dr. K. McVoy for hospitality at the National Science Foundation Summer Institute at the University of Wisconsin, and several colleagues there, especially Dr. T. F. Jordan and Dr. L. van Hove, for helpful discussions and suggestions. Also, in Princeton I would like to thank Professor A. S. Wightman, Dr. K. Hepp, and Dr. G. Velo for interesting conversations.

[22] In fact the method of Sec. IV can accommodate any number of quantum numbers of this kind, including those with infinitely many eigenvalues, such as charge, and those with a finite number of eigenvalues, such as the separation of integral and half-odd-integral angular momenta.

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# Application of the Padé Approximant to Scattering Theory†

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The Padé approximant is shown to yield exactly unitary $S$ matrices in scattering theory. The method is used to construct a unitary $S$ matrix for $n$-$d$ scattering above the threshold for inelastic scattering. Thus the effect of inelastic scattering on elastic-scattering total cross sections and angular distributions is calculated. No cusps are found at the threshold.

## I. INTRODUCTION

The Padé approximant[1] has been applied to summing series which occur in calculations based on the Ising and Heisenberg models of various types of critical phenomena.[2] The Padé approximant may also be applied to the Born series which occur in scattering theory; recently Tani[3] has made a study of this application.

In this paper we show that the Padé approximant method may be used to construct exactly unitary $S$ matrices. In problems in which states with more than two free particles are energetically possible (the example we are concerned with is $n$-$d$ scattering above the threshold for inelastic scattering; however, there are many others of great interest, for example, nucleon-nucleon or pion-nucleon scattering above the threshold for pion production) the problem of constructing unitary $S$ matrices has not been satisfactorily solved previously.

In calculations of $n$-$d$ elastic scattering based on the no-distortion approximation,[4] inelastic scattering is ignored even above the threshold for inelastic scattering. Because of this neglect of inelastic scattering, the matrix elements for transitions between states in which three particles are free are not required because in the no-distortion approximation three-body states are precisely what are left out. Our work represents an attempt to go beyond the no-distortion approximation. In calculations of $n$-$d$ inelastic cross sections based on

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[1] For a review, see G. A. Baker, Jr., in *Advances in Theoretical Physics*, edited by K. A. Brueckner (Academic Press Inc., New York, 1965), Vol. 1.
[2] See the surveys of M. Fisher and C. Domb in the Proceedings of the Conference on Phenomena in the Neighborhood of Critical Points, Washington, D. C., 1965 (unpublished).
[3] S. Tani, Phys. Rev. **139**, B1011 (1965).
[4] See, for example, R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1953).