Solution of Scattering and Bound-State Problems by Construction of Approximate Dynamical Symmetries

DANIEL I. FIVEL*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland
(Received 26 May 1965; revised manuscript received 11 October 1965)

A concept of "approximate dynamical symmetry" is formulated by means of which we are able to produce approximate solutions to scattering and bound-state problems in a purely group-theoretic manner for essentially any isotropic potential and to any desired degree of accuracy. This concept forms a natural generalization to arbitrary potentials of the familiar Runge-Lenz symmetry of the Coulomb problem and the SU_3 symmetry of the harmonic oscillator. The method consists in transforming the dynamical Lie algebra, i.e., the smallest algebra generated by the Hamiltonian and a complete set of dynamical variables, which is usually infinite and simple, into a finite, simple Lie algebra as follows. We first transform the dynamical algebra into another infinite Lie algebra, but one which contains a "large" ideal, by a process similar to Inonu-Wigner contraction, the angular momentum acting as the contraction parameter. The factor algebra modulo this ideal turns out to be a finite, simple Lie algebra which still contains some of the dynamical information. The above two-stage process will be called "truncation." We are able to develop a sequence of such truncations leading to successively higher dimensional simple Lie algebras whereby we obtain successively better approximations to energy levels and phase shifts. The solutions obtained involve all powers of the coupling constant, and the Nth-order approximation is at least as good as the Nth-order W.K.B. approximation and probably better. The special role of the angular momentum in the contraction process enables us to relate these group-theoretic methods to the Regge formalism in a natural way. In fact we are able, thereby, to produce exact solutions to dispersion relations for Regge trajectories obtained by including an arbitrarily large but finite set of trajectories in the unitarity relation. As an illustration of the methods developed we include a calculation to the first nontrivial order of energy levels of an anharmonic oscillator and the well-depth parameter and phase shifts for a Yukawa potential.

I. INTRODUCTION

IT is well known that the soluble problems of quantum mechanics, such as the harmonic-oscillator and Coulomb problems, owe their simplicity to the presence of dynamical symmetries[1] whereby the determination of energy levels is reduced to finding the representations of certain Lie groups. For problems where no recognizable dynamical symmetry exists, such as the Yukawa potential or the anharmonic oscillator, one is normally compelled to use perturbative methods with the familiar limitations. It is natural to inquire, therefore, whether these more general potentials exhibit dynamical symmetries in some approximate sense, by which approximate solutions could be obtained in a purely group-theoretical manner. Such approximations would relate to the *algebraic form* of the interaction rather than its strength and might provide a useful tool where perturbation methods fail.

In this paper we shall show that it is in fact possible, for essentially all interesting potentials, to find a sequence of approximate dynamical symmetries with the property that the sequence of approximate solutions to the bound-state and scattering problems obtained therefrom converge rapidly to the exact solution. The method that we shall employ is a quantum-mechanical analog of a method first described for classical systems in the thesis of Poincaré[2] and subsequently developed for such systems by Birkhoff, Sternberg, Lewis, and others.[3]

A rather remarkable by-product will be that the approximate solutions obtained by this method automatically provide closed expressions for the solution of approximate Regge dispersion relations, i.e., dispersion relations obtained under the assumption that only a finite number of the trajectories are coupled by unitarity.

As the concept of approximate dynamical symmetry which we are going to develop bears some resemblance to the so-called "saturation approximations" presently being employed in connection with strongly interacting current algebras,[4] it is possible that the machinery here developed may throw some light on the meaning of such approximations.

II. DYNAMICAL SYMMETRY VERSUS SYMMETRY OF THE HAMILTONIAN

The term "dynamical symmetry" is currently used in two different ways which it is well to differentiate from the outset. Consider, for example, a two-dimensional, isotropic harmonic oscillator:

$$H_0=pp^\dagger+qq^\dagger, \quad q=\tfrac{1}{2}(q_1+iq_2), \quad p=\tfrac{1}{2}(p_1+ip_2).$$

H_0 commutes with

$$J_0=i(pq^\dagger-p^\dagger q), \quad J_+=i(p^{\dagger 2}+q^{\dagger 2}), \quad J_-=-i(p^2+q^2).$$

Also

$$[J_0,J_\pm]=\pm J_\pm, \quad [J_+,J_-]=2J_0,$$

* Supported in part by National Science Foundation, Contract No. GP3221.

[1] For instance, the Runge-Lenz symmetry for the Coulomb problem.

[2] H. Poincaré, *Oeuvres 1* (Gautier-Villars, Paris, 1928).

[3] S. Sternberg, J. Math. Mech. **10**, 451 (1961). (References to other work on the classical problem may be found in this paper.)

[4] B. W. Lee, Phys. Rev. Letters **14**, 676 (1965).

and

$$H_0^2-\tfrac{1}{4}=J_0^2+\tfrac{1}{2}(J_+J_-+J_-J_+).$$

The $J_\pm$, J_0 have the algebra of the generators of SU_2 and since $J_0=J_0^\dagger$ and $J_+=J_-^\dagger$, one may obtain H_0 from the allowed values of the Casimir operator for unitary representations of SU_2. We shall refer to the presence of a group with the Hamiltonian as its Casimir operator as a "symmetry of the Hamiltonian," and if the generators, as above, involve transformations in phase space rather than coordinate space, we shall speak of a "dynamical symmetry of the Hamiltonian."

On the other hand, the symmetry of the oscillator can be viewed somewhat differently. Consider the operators

$$\xi_+=(p+iq)(p^\dagger+iq^\dagger),\quad \xi_-=(q+ip)(q^\dagger+ip^\dagger).$$

Then

$$[H_0,\xi_\pm]=\pm\xi_\pm \quad\text{and}\quad [\xi_+,\xi_-]=2H_0,$$

while

$$J_0^2-\tfrac{1}{4}=H_0^2+\tfrac{1}{2}(\xi_+\xi_-+\xi_-\xi_+).$$

Thus H_0, $\xi_\pm$ generate the *algebra* of SU_2 but since $\xi_+=-\xi_-^\dagger$ we see that they are appropriate generators for a unitary representation, not of SU_2, but of the Lorentz group with two-space and one-time dimension. Moreover, H_0 now is one of the generators while the angular momentum perpendicular to the plane of motion plays the role of Casimir operator. In this form one solves the problem by selecting a representation of the $(+,+,-)$ Lorentz group from Bargmann's catalog[5] for which H_0 is diagonal and the Casimir operator has one of the allowed angular momentum values. Since, for the physical case, H_0 must have a lowest eigenvalue ≥ 0, then the allowed values of J_0 are also determined. If in any problem we find that the Hamiltonian together with a complete set of dynamical variables generate a *finite* Lie algebra we shall say simply that the problem possesses a "dynamical symmetry."

As it will develop, no problems other than the Coulomb problem and the harmonic oscillator (up to canonical equivalence) exhibit a dynamical symmetry in this sense. We shall, however, be able to weaken the definition somewhat and obtain a useful concept of "approximate dynamical symmetry" that applies to a wide class of problems.

Because the existence of such approximations becomes most clearly manifest in connection with the so-called normal form problem in classical mechanics, the following section is devoted to a brief exposition of this problem.

III. THE NORMAL-FORM PROBLEM IN CLASSICAL MECHANICS

The frequency $\nu(E)$ for a classical particle of mass $M=\frac{1}{2}$ and energy E in a one-dimensional potential $V(q)$ is given by

$$\nu(E)=\left(\oint\frac{dq}{(E-V)^{1/2}}\right)^{-1}, \tag{1}$$

where the integration is carried out around the branch cut between a pair of turning points. If $V(q)$ is transcendental there will be an infinite number of (complex) turning points and one cannot extract via Cauchy's theorem a very clear picture of the relationship between ν and E. One may, however, approach the problem somewhat differently:

Suppose that the one-particle Hamiltonian $H(q,p)$ is an entire function of p and q and that $H(x,x)$ is even, i.e., we can write

$$H(q,p)=\sum_{m,n=1}^{\infty}\beta_{mn}q^mp^n,\quad \beta_{mn}=0 \quad \text{unless } m+n \text{ is even.} \tag{2}$$

If $H(q,p)$ happens to have the special form

$$H(q,p)=\sum_{s=1}^{\infty}\alpha_s(H_0)^s,\quad H_0\equiv p^2+q^2, \tag{3}$$

then, since H_0 is a constant of motion, the solution of Hamilton's equation is

$$p+iq=C\exp[2\pi i\nu(E)t],$$

and

$$\nu(E)=-\frac{1}{\pi}\left(\frac{dH}{dH_0}\right)_{H=E}=\sum_{s=1}^{\infty}s\alpha_s[H_0(E)]^{s-1}, \tag{4}$$

where H_0 is one of the roots of (3) for $H=E$. The form (3) is called a "normal form."

For the general case (2) one may attempt to bring H closer and closer to normal form by a succession of canonical transformations $A_N(q,p)$. Thus, defining

$$f^{(A)}(q,p)=e^{\{A,\}}f\equiv\sum_{n=0}^{\infty}\frac{1}{n!}\{A^{(n)},f\}, \tag{5}$$

where

$$\{A^{(0)},f\}\equiv f,\quad \{A^{(1)},f\}=\frac{\partial A}{\partial q}\frac{\partial f}{\partial p}-\frac{\partial A}{\partial p}\frac{\partial f}{\partial q}\equiv\{A,f\},$$

and

$$\{A^{(n+1)},f\}\equiv\{A^{(1)},\{A^{(n)},f\}\},$$

we seek $A_N(q,p)$ such that

$$H^{(A_N)}=\sum_{s=1}^{N}\alpha_{Ns}(q^2+p^2)^s+\epsilon_{N+1}(q,p), \tag{6}$$

where the "error" term ϵ_{N+1} consists of terms q^np^m with $n+m\geq 2(N+1)$. We shall refer to

$$H_{nf}{}^{(N)}\equiv\sum_{s=1}^{N}\alpha_{Ns}(q^2+p^2)^s,$$

as the Nth approximant to the normal form and the

[5] V. Bargmann, Ann. Math. 48, 569 (1947).

frequencies of $H_{nf}{}^{(N)}$ given by (4), as the Nth approximants to the frequencies of H.

We shall restrict (2) slightly further and assume that the quadratic part of H is already in the form p^2+q^2. This can be accomplished by a trivial canonical transformation provided only that H has a "little" harmonic oscillator in it to begin with. That is, the quadratic term may be $p^2+\alpha q^2$ for any $\alpha\neq 0$. In this case H will be said to be "regular," the significance of this restriction to be made clear later. The transformations A_N which bring H into the form

$$H^{(A_N)}=q^2+p^2+\sum_{s=2}^{N}\alpha_{Ns}(q^2+p^2)^s+\epsilon_{N+1}(q,p),$$

are obtained by the following algorithm: Let $A_1=1$ and we have $\alpha_{11}=1$. Suppose that A_{N-1} has been found and that $\alpha_{N-1,1}=1$ and that A_{N-1} is an even function. Then

$$H^{(A_{N-1})}=q^2+p^2+\sum_{s=2}^{N-1}\alpha_{N-1s}(q^2+p^2)^s+\epsilon_N(q,p),$$

and $\epsilon_N(q,p)$ is even. Hence, one may write

$$\epsilon_N(q,p)=\sum_{l=-N}^{N}\gamma_{Nl}(p+iq)^{N+l}(p-iq)^{N-l}+\epsilon_{N+1}(q,p),$$

with ϵ_{N+1} even. Let

$$A_N(q,p)\equiv\frac{i}{4}\sum_{\substack{l=-N\\ l\neq 0}}^{N}\frac{1}{l}\gamma_{Nl}(p+iq)^{N+l}(p-iq)^{N-l}.$$

Then under A_N, $H^{(A_{N-1})}$ is transformed into

$$q^2+p^2+\sum_{s=2}^{N-1}\alpha_{N-1,s}(q^2+p^2)^s+\gamma_{N0}(q^2+p^2)^N$$
$$+\text{even terms of degree}>2N.$$

Thus

$$\alpha_{Ns}=\alpha_{N-1,s}\quad\text{for}\quad s=1,2,\cdots,N-1,$$
$$\alpha_{NN}=\gamma_{N0}.$$

By induction H can be put into normal form to any order, and obviously one can drop the subscript N on the coefficients α_{Ns}. The simplicity of this procedure belies the tedium actually involved. To calculate some $H_{nf}{}^{(N)}$ one must calculate every $H_{nf}{}^{(M)}$, $M=1,2\cdots,N-1$ together with all terms of degree $\leq 2N$ in the error terms. The algorithm thus gives little insight into the convergence of the method. We shall, however, show in the following two sections that the process of finding approximate normal forms for both the classical and the quantum-mechanical problems has a simple group-theoretic interpretation which both facilitates the determination and suggests the conditions under which rapid convergence may be expected.

IV. FACTOR ALGEBRAS OF CANONICAL TRANSFORMATIONS

In the following f_s, g_s, h_s, etc., denote any power series of the form

$$\sum_{m,n=0}^{\infty}\alpha_{mn}q^m p^n,\quad \alpha_{mn}=0$$
$$\text{unless } m+n=2s,\ 2s+2,\ 2s+4,\ \cdots$$

The set of all such series with complex coefficients will be denoted R_s. R_s is closed under the formation of linear combinations of its elements as well as under taking the Poisson bracket of its elements. R_s is thus an infinite Lie algebra. Further one observes that for *any* s and k,[6]

$$\{f_s,g_{s+k}\}\in R_{s+k},$$

so that R_{s+k} is an *ideal* in R_s for $k=0,1,2\cdots$. Let

$$R_{s,k}\equiv R_s/R_{s+k},$$

i.e., the homomorphic image of R_s modulo the ideal R_{s+k}. The $R_{s,k}$ are clearly finite Lie algebras but with the exception of $R_{1,1}$ they are not semisimple. In fact if l is any integer for which

$$s+k>l>\max[s,\ (s+k)/2],$$

then R_l/R_{s+k} is an Abelian ideal of $R_{s,k}$. In the following we will be solely concerned with the factor algebras of R_1 and we denote by $f_{(N)}$ the image in $R_{1,1+N}$ of any $f\in R_1$.

Since the Lie bracket is preserved under homomorphism it follows that if for $H\in R_1$, there exists $A\in R_1$ such that

$$e^{\{A,\ \}}H=H^{(A)}=\sum_{s=1}^{\infty}\alpha_s(q^2+p^2)^s,$$

then

$$H_{(N)}{}^{(A)}=\sum_{s=1}^{N}\alpha_s(q^2+p^2)^s=H_{nf}{}^{(N)},$$

with the *same* coefficients α_s. Thus, to compute the approximate normal form of order N, we have only to calculate canonical transformations in the sense of the Poisson bracket modulo R_{N+1}. Thus, in the given H, we delete all terms of degree $>2N$ and H is now to be treated as an element of the finite algebra $R_{1,N+1}$. The elements $(q^2+p^2)^s$, $s=1,2,\cdots,N$ generate an Abelian subalgebra of $R_{1,N+1}$ and it is not difficult to see that this is a Cartan, subalgebra of $R_{1,N+1}$. The existence of the algorithm of Part III is a consequence of the following theorem proved in Ref. 3:

A regular element of a Lie algebra is conjugate via inner automorphism to some element of any Cartan subalgebra. (A regular element is one which itself generates a Cartan subalgebra and it can be shown that

[6] If one wishes to include odd powers $q^m p^n$, one must exclude $m+n=1$ to obtain this result.

this requires us to have at least a "little" harmonic-oscillator component in H in the sense described in Sec. III).

In order now to compute the α_s explicitly, we consider the representations of $R_{1,N+1}$. For example we may examine the "adjoint" representation in which $f_{(N)}\in R_{1,N+1}$ operates on the elements of $R_{1,N+1}$ as a linear operator f_N':

$$f_{(N)}'\cdot g_{(N)}\equiv\{f_{(N)},g_{(N)}\},$$

all Poisson brackets now being defined modulo R_{N+1}. Then clearly

$$\{f_{(N)},g_{(N)}\}'=[f_{(N)}',g_{(N)}']\equiv f_{(N)}'g_{(N)}'-g_{(N)}'f_{(N)}',$$

and

$$(e^{\{A_{(N)},\}}f_{(N)})'=e^{A_{(N)}'}f_{(N)}'e^{-A_{(N)}'}.$$

Thus, the approximate normal form problem is solved if we can find a similarity transformation B' such that

$$B'H_{(N)}'B'^{-1}=\sum_{s=1}^{N}\alpha_s((p^2+q^2)^s)'.$$

It is important to point out here that

$$((p^2+q^2)^s)'\neq((p^2+q^2)')^s.$$

In fact, even though all of the $((p^2+q^2)^s)'$, $s=1, 2, \cdots, N$, mutually commute and, e.g., $(p^2+q^2)'$ can be taken to be diagonal, we may not even conclude that $((p^2+q^2)^s)'$ for $s\neq1$ are diagonal. The reason is that $R_{1,N+1}$ is not semisimple. For semisimple algebras it is always possible to diagonalize the elements of the Cartan subalgebra simultaneously. If it *were* possible to treat the $((p^2+q^2)^s)'$ as a complete set of diagonal matrices, our problem would then simply be to solve the eigenvalue problem for the finite dimensional matrix $H_{(N)}'$ and express the diagonal form of $H_{(N)}'$ as a linear combination of the diagonal matrices $((p^2+q^2)^s)'$. The classical problem is thus *not* of this type.

One observes, however, that in quantum mechanics the prescription for assigning operators to functions *is* such that if $(p^2+q^2)^s$ are made into quantum mechanical operators $[(p^2+q^2)^s]$, then for all s these operators are simultaneously diagonalizable. Thus, we guess that it may be possible to understand the normal form problem as an eigenvalue problem in quantum mechanics with $\hbar$ then set equal to zero to obtain the classical answer. We shall thus examine next the quantum analog of the normal form problem. We will see that indeed we do obtain an algebra which is semisimple, in fact simple, but for this very reason one cannot directly carry out the factoring procedure used above to find a homomorphic finite algebra which required the presence of ideals.

V. THE DYNAMICAL LIE ALGEBRA OF QUANTUM MECHANICS

The usual prescription for forming a quantum-mechanical operator $[f]$ from a polynomial $f(p,q)$ may be stated[7]:

(i) $[\alpha f+\beta g]=\alpha[f]+\beta[g]$ (α, β complex numbers),

(ii) $[\exp(\alpha q+\beta p)]=\exp(\alpha[q]+\beta[p])$,

(iii) $[[q],[p]]=i\hbar$,

from which

$$[\exp(\alpha p+\beta q)\exp(\gamma p+\delta q)]=\exp[-\tfrac{1}{2}i\hbar(\beta\gamma-\alpha\delta)]\times\exp(\alpha[p]+\beta[q])\exp(\gamma[p]+\delta[q]),$$

and matching powers in an obvious way one obtains:

$$[[f],[g]]=i\hbar\left\{\left[\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right]+\frac{1}{3!}\left(\frac{i\hbar}{2}\right)^2\left[\frac{\partial^3 f}{\partial q^3}\frac{\partial^3 g}{\partial p^3}-3\frac{\partial^3 f}{\partial q^2\partial p}\frac{\partial^3 g}{\partial q\partial p^2}+3\frac{\partial^3 f}{\partial q\partial p^2}\frac{\partial^3 g}{\partial q^2\partial p}-\frac{\partial^3 f}{\partial p^3}\frac{\partial^3 g}{\partial q^3}\right]+\cdots\right\}. \quad (7)$$

It is thus evident that the quantum-mechanical algebra is not isomorphic to the Poisson bracket algebra except for $\hbar\to0$. Further, the ideals formed by the even polynomials do not go over into ideals in the quantum-mechanical algebra. For example consider: $f=q^4$, $q=p^4$. Then $\{q^4,p^4\}$ is of degree 6 (i.e., >4) while

$$[[q^4],[p^4]]=16i\hbar([q^3p^3]-\tfrac{3}{2}\hbar^2[qp])$$

and a term of degree 2 has now crept in. In fact, one may show that the quantum-mechanical algebra has no non-trivial ideals, i.e., it is simple. We are therefore confronted with the problem of finding some analog of the factoring process used in IV to obtain a finite algebra.

We must first cast the quantum-mechanical algebra into a different form. We shall henceforth drop the [ ] notation, it being understood that f means $[f]$ for any $f(p,q)$.

Consider the operators

$$J_0=\frac{1}{4\hbar}(p^2+q^2),\quad J_+=\frac{1}{4\hbar}(p+iq)^2,\quad J_-=-\frac{1}{4\hbar}(p-iq)^2, \quad (8a)$$

which satisfy

$$[J_0,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=2J_0, \quad (8b)$$

$$J_0^2+\tfrac{1}{2}(J_+J_-+J_-J_+)=-3/16. \quad (8c)$$

We are interested in the set of even-order polynomials (R_1) which is spanned, e.g., by the functions $(p-q)^{k-m}(p+iq)^{k+m}$, $k=1, 2, \cdots$; $m=-k, -k+1, \cdots, k$.

[7] (ii) is the Weyl prescription of symmetrization, e.g., it implies $[pq]=\frac{1}{2}([p][q]+[q][p])$.

From (7):

$$(p-iq)^{k-m}(p+iq)^{k+m}=\alpha_{km}\hbar^k[J_-{}^{(k-m)},J_+{}^k], \quad (9)$$

where α_{km} are constants (independent of $\hbar$).[8] Thus, the $[J_-{}^{(k-m)},J_+{}^k]$ generate the algebra of the operators associated with the polynomials of R_1. We denote this algebra $\mathfrak{L}^{QM}$.

It will be very useful now to consider $\mathfrak{L}^{QM}$ as the special case $\mathfrak{L}_{-3/16}$ of a family of algebras, denoted $\mathfrak{L}_{j(j+1)}$, generated by the $[J_-{}^{(k-m)},J_+{}^k]$ where J_0, $J_\pm$ satisfy (8b) *but* instead of (8c) satisfy

$$J_0{}^2+\tfrac{1}{2}(J_+J_-+J_-J_+)=j(j+1), \quad (8c')$$

with an arbitrary complex j. It is also convenient to take for the generators of $\mathfrak{L}_{j(j+1)}$

$$T_j(k,m)\equiv\left(\frac{(k+m)!}{(2k)!(k-m)!}\right)^{1/2}[J_-{}^{(k-m)},J_+{}^k], \quad (10)$$

for which we have the property

$$\begin{aligned}&[J_\pm,T_j(k,m)]=[(k\mp m)(k\pm m+1)]^{1/2}T_j(k,m\pm1),\\ &[J_0,T_j(k,m)]=mT_j(k,m).\end{aligned} \quad (11)$$

Thus, if $j=0, \pm\frac{1}{2}, \pm1, \cdots, T_j(k,m)$ behaves like the mth component of a spherical tensor of rank k.

It is not difficult to see from the tensor-like property (11) that the commutation relations of the $T_j(k,m)$ are of the form

$$[T_j(k_1,m_1),T_j(k_2,m_2)]=\sum_k C_{k_1m_1k_2m_2}{}^{k,m_1+m_2}(j(j+1))T_j(k,m), \quad (12)$$

where $m=m_1+m_2$ and where the summation index runs over

$$k=k_1+k_2-1, k_1+k_2-3, \cdots \begin{cases}1 & \text{if } k_1+k_2 \text{ is even}\\ 2 & \text{if } k_1+k_2 \text{ is odd},\end{cases}$$

and that the structure constants $C_{k_1m_1k_2m_2}{}^{km}(j(j+1))$ of $\mathfrak{L}_{j(j+1)}$ are polynomials in $j(j+1)$ of degree $\frac{1}{2}(k_1+k_2-k-1)$. Hence, if one can find a polynomial expression in $j(j+1)$ for the structure constants which is known to be valid, e.g., at all sufficiently large integer j, then it will be valid at all complex j. Thus the spherical tensor property at the integers may be employed to obtain, via the Wigner-Eckart theorem, an expression for the C's valid at all $j(j+1)$ and in particular for the quantum mechanical case $j(j+1)=-3/16$. This calculation is carried out in Appendix A.

We now come to the reason for considering the whole family of algebras $\mathfrak{L}_{j(j+1)}$ in the first place. In the following it is convenient to use the notation j_0 whenever we want to restrict j to the values $0, \pm\frac{1}{2}, \pm1, \pm\frac{3}{2}, \cdots$.

[8] The iterated commutator is defined for any A,B as:
$[A^{(0)},B]=B$, $[A^{(1)},B]=[A,B]$, $[A^{(n)},B]=[A,[A^{(n-1)},B]]$.

Suppose j takes on one of the values j_0. Then (11) has a finite dimensional representation obtained by putting for the $J_{\pm,0}$ in (10) the familiar spin j_0 representation of (8b) and (8c'). In this representation, denoted $T_{j_0}'(k,m)$, $T_{j_0}'(k,m)=0$ for $k>2j_0$ but not for $k\leq 2j_0$. Hence we must have

$$C_{k_1m_1k_2m_2}{}^{k,m_1+m_2}(j_0(j_0+1))=0 \quad \text{whenever } k_1 \text{ and/or } k_2>2j_0 \text{ and } k\leq 2j_0. \quad (13)$$

But this simply means that the $T_{j_0}(k,m)$ with $k=2j_0$, $2j_0+1, \cdots$ generate an ideal in $\mathfrak{L}_{j_0(j_0+1)}$. Thus, while $\mathfrak{L}^{QM}$ was simple, the algebra obtained by going from $j(j+1)=-3/16$ to a value $j_0(j_0+1)$ is not simple. The factor algebra modulo the ideal is represented by $T_{j_0}'(k,m)$, $k=1, 2, \cdots, 2j_0$, $m=-k, \cdots, k$. From the tensor property and (10) we see that these $T_{j_0}'(k,m)$ constitute a $(2j_0+1)^2-1$ member set of linearly independent, traceless matrices. Hence, they generate the fundamental representation of the familiar simple Lie algebra A_{2j_0} (the algebra of generators of the group SU_{2j_0+1}). Thus, the factor algebra is isomorphic to A_{2j_0} and will be denoted $\mathfrak{L}_{j_0(j_0+1)}'$.

Our program will now be as follows: By taking the factor algebra modulo the ideals that appear at $j(j+1)=j_0(j_0+1)$ we will be able to construct a $2j_0$th-order approximate normal form at $j=j_0$ just as we did in the classical case. Moreover, since the factor algebras A_{2j_0} are simple, the determination of the approximate normal form will reduce to an ordinary $(2j_0+1)$-dimensional eigenvalue problem. We will then develop an interpolation scheme in order to obtain approximate normal forms for arbitrary $j(j+1)$ and in particular for $j(j+1)=-3/16$ (i.e. $\mathfrak{L}^{QM}$).

VI. THE QUANTUM-MECHANICAL NORMAL FORM PROBLEM

Let a Hamiltonian be given of the form

$$H=p^2+q^2+V(q^2); \quad V(x)=\sum_{k=1}^{\infty}\alpha_k x^k, \quad \alpha_1\neq-1. \quad (14)$$

Then from the identity

$$e^{J-}J_+{}^ke^{-J-}=(e^{J-}J_+e^{-J-})^k=(J_+-2J_0=J_-)^k=(-q^2/\hbar)^k,$$

we may expand H in terms of the generators of $\mathfrak{L}^{QM}$ as

$$H=4\hbar J_0+e^{J-}V(-\hbar J_+)e^{-J-}=-2\sqrt{2}\hbar T(1,0)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\alpha_k(-\hbar)^k\left(\frac{(2k)!}{(k-m)!(k+m)!}\right)^{1/2}T(k,m). \quad (15)$$

As in Sec. V we can generalize from $\mathfrak{L}^{QM}$ to $\mathfrak{L}_{j(j+1)}$ and define a generalized $H_j\in\mathfrak{L}_{j(j+1)}$ as

$$H_j=-2\sqrt{2}\hbar T_j(1,0)+\sum_{k=1}^{\infty}\sum_{m=-k}^{k}\alpha_k(-\hbar)^k\times\left(\frac{(2k)!}{(k-m)!(k+m)!}\right)^{1/2}T_j(k,m). \quad (16)$$

Suppose that there exists an element $B_j \in \mathfrak{L}_{j(j+1)}$ such that

$$e^{B_j} H_j e^{-B_j} = \sum_{k=1}^{\infty} \beta_k(j) \hbar^k T_j(k,0). \tag{17}$$

Then (17) will be called a normal form of H_j. Suppose now that j takes on one of the values $j_0(\frac{1}{2},1,\frac{3}{2},\cdots)$ and let T' denote the image of any element $T \in \mathfrak{L}_{j_0(j_0+1)}$ under the homomorphism of $\mathfrak{L}_{j_0(j_0+1)}$ into the factor algebra $\mathfrak{L}_{j_0(j_0+1)}'$ described in Sec. V. Then the homomorphic property implies

$$\begin{aligned}[\exp(B_{j_0}) H_{j_0} \exp(-B_{j_0})]' &= \exp(B_{j_0}') H_{j_0}' \exp(-B_{j_0}') \\ &= \sum_{k=1}^{2j_0} \beta_k(j_0) \hbar^k T_{j_0}'(k,0), \quad [\text{by (17)}] \end{aligned} \tag{18}$$

with the same coefficients $\beta_k(j_0)$ as in (17). The matrices $T_{j_0}'(k,0)$, $k=1, 2, \cdots, 2j_0$ form a complete set of diagonal, traceless matrices of rank $2j_0+1$. Thus, to determine the $\beta_k(j_0)$ in (18) it suffices to diagonalize

$$H_{j_0}' = -2\sqrt{2}\hbar T_{j_0}'(1,0) + \sum_{k=1}^{2j_0} \sum_{m=-k}^{k} \alpha_k (-\hbar)^k \times \left(\frac{(2k)!}{(k-m)!(k+m)!} \right)^{1/2} T_{j_0}'(k,m), \tag{19}$$

and express the resulting diagonal form as a linear combination of the $T_{j_0}'(k,0)$, $k=1, 2, \cdots, 2j_0$ and read off the coefficients. Thus, for any $j=j_0$, we may define a "normal form of order $2j_0$" as the first $2j_0$ terms of (17), and this can be obtained from the solution of an eigenvalue problem of a $2j_0+1$ rank matrix. This $2j_0$th-order normal form has the same structure as that in the classical case for it is easy to see that the matrices $T_{j_0}'(k,0)$ can be expressed as a linear combination of the matrices $(H_{0j_0}')^n$, $n=0, 1, \cdots, 2j_0$ where H_{0j_0}' is the image of the harmonic-oscillator Hamiltonian.[9]

The matrices $T_{j_0}'(k,0)$ span the Cartan subalgebra of $\mathfrak{L}_{j_0(j_0+1)}'$ (i.e. A_{2j_0}) so that the existence of the normal form (18) in fact follows from the conjugacy theorem remarked in Sec. IV provided that we know H_{j_0}' to be regular. This is simply proved as follows: Let x be an arbitrary, nonvanishing complex number and consider the algebra $\mathfrak{L}_{j(j+1)}^*$ obtained from $\mathfrak{L}_{j(j+1)}$ by replacing $T_j(k,m)$ by $x^k T_j(k,m)$. Then $\mathfrak{L}_{j(j+1)}$ and $\mathfrak{L}_{j(j+1)}^*$ are isomorphic and the image of H_j under the isomorphism is obtained by simply replacing $\hbar$ in (16) by $x\hbar$. Thus, for our purposes $\hbar$ can be regarded as arbitrarily small ($\neq 0$). Hence, H_{j_0}' can be brought uniformly arbitrarily close to its terms linear in $\hbar$ (since it is a finite matrix), i.e., to the terms:

$$-\hbar 2\sqrt{2} T_{j_0}'(1,0) + \alpha_1 \sum_{m=-1}^{+1} \left(\frac{2}{(1-m)!(1+m)!} \right)^{1/2} T_{j_0}'(1,m). \tag{20}$$

In Appendix B it is shown that for $\alpha_1 \neq -1$ Eq. (20) may be diagonalized by a similarity transformation and will moreover have distinct eigenvalues. This suffices to prove regularity.

Before proceeding further it is important to remark the following byproduct of the regularity proof: According to a well-known theorem on finite matrices, if a matrix has elements depending analytically on a parameter λ and eigenvalues which are nondegenerate for some value λ_0 of the parameter, then the eigenvalues and eigenmanifolds perturb analytically in any neighborhood of λ_0 in which they remain nondegenerate.[10] We may therefore conclude that the quantities $\beta_k(j_0)\hbar^k$, $j_0=k/2, (k+1)/2, \cdots$ are analytic functions of $\hbar$ in some neighborhood of $\hbar=0$. As we shall see below the special form of the algebras $\mathfrak{L}_{j(j+1)}$ will imply the stronger conclusion that $\beta_k(j_0)$ is itself analytic near $\hbar=0$.

We now turn to the problem of obtaining approximate normal forms of any order for arbitrary j. As we have seen $\mathfrak{L}_{j(j+1)}$ has a finite factor algebra only for $j=j_0$ and even in this case we can only find a factor algebra by which the coefficients of the $2j_0$th-order normal form may be calculated. In order to generate finite algebras of arbitrary order and for arbitrary j we therefore resort to the following device. The Hamiltonian H_j is a linear superposition of the quantities $\hbar^k T_j(k,m) k=1, 2, \cdots; m=-k, \cdots, k$. Heretofore, we have regarded $\hbar$ merely as some given number and considered the algebra $\mathfrak{L}_{j(j+1)}$ with generators $T_j(k,m)$. We now choose to regard $\hbar$ as an indeterminate and consider a new algebra which we denote $\mathfrak{L}_{j(j+1)}(\hbar)$ with the generators:

$$(km,n) \equiv \hbar^{k+n-1} T_j(k,m), \tag{21}$$

where $k=1, 2, \cdots; m=-k, \cdots, k; n=0, 1, 2, \cdots$. The fact that $\mathfrak{L}_{j(j+1)}(\hbar)$ is an algebra, i.e., that commutators of elements (21) are again of the form (21), is a consequence of the fact previously noted that the structure constants $C_{k_1 m_2 k_1 m_2}{}^{km}(j(j+1))$ of $\mathfrak{L}_{j(j+1)}$ vanish unless $k=k_1+k_2-1, k_1+k_2-3, \cdots,$ (1 or 2). The algebra $\mathfrak{L}_{j(j+1)}(\hbar)$ has a number of features in common with the dynamical algebra associated with the classical problem. In particular, we note that the set of elements I_N of the form (21) with $k+n-1>N$, where N is any nonnegative integer, generate an ideal in $\mathfrak{L}_{j(j+1)}(\hbar)$. The factor algebra

$$\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar) \equiv \mathfrak{L}_{j(j+1)}(\hbar)/I_N$$

[9] This is due to the fact that $H_{0j_0}' = 4\hbar \begin{pmatrix} -j_0 & & 0 \\ & \ddots & \\ 0 & & j_0 \end{pmatrix}$ has distinct eigenvalues.

[10] This theorem is discussed, e.g., in L. Brown, D. Fivel, B. W. Lee, and R. Sawyer, Ann. Phys. (N. Y.) 23, 187 (1963).

is a finite algebra generated by the elements (21) with $k+n-1\leq N$. Moreover, we have

$$\mathfrak{L}_{j(j+1)}{}^{(0)}(\hbar)\subset\mathfrak{L}_{j(j+1)}{}^{(1)}(\hbar)\subset\mathfrak{L}_{j(j+1)}{}^{(2)}(\hbar)\subset\cdots. \quad (22)$$

Thus any of the algebras $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$, $N>0$, have nontrivial ideals and it is quite easy to see that there are always Abelian ideals so that these finite algebras are not semisimple.

Referring now to (16) we see that $(1/\hbar)H_j$ is an element of $\mathfrak{L}_{j(j+1)}(\hbar)$. Suppose that $(1/\hbar)H_j$ possesses a normal form with respect to $\mathfrak{L}_{j(j+1)}(\hbar)$, i.e., suppose that there exists an element $B_j(\hbar)$ of $\mathfrak{L}_{j(j+1)}(\hbar)$ such that

$$\exp(B_j(\hbar))\frac{H_j}{\hbar}\exp(-B_j(\hbar))=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\rho_{kn}(j)\hbar^{k-1+n}T_j(k,0). \quad (23)$$

Denoting by a superscript (N) the image of any element of $\mathfrak{L}_{j(j+1)}(\hbar)$ under the homomorphism into the factor algebra $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$ we see then that

$$\exp B_j{}^{(N)}(\hbar)\left(\frac{H_j}{\hbar}\right)^{(N)}\exp(-B_j{}^{(N)}(\hbar))=\sum_{k=1}^{N+1}\sum_{n=0}^{N+1-k}\rho_{kn}(j)(\hbar^{k-1+n}T_j(k,0))^{(N)}. \quad (24)$$

The elements of the form

$$\hbar^{k-1+n}T_j(k,0) \quad \text{and} \quad (\hbar^{k-1+n}T_j(k,0))^{(N)}$$

generate the Cartan subalgebras of $\mathfrak{L}_{j(j+1)}(\hbar)$ and $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$, respectively. Thus, as in the classical case, if the normal form (23) exists the normal form coefficients $\rho_{kn}(j)$ for $k+n-1\leq N$ may be computed by finding the normal form with respect to the finite algebra $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$. Moreover, because of the property (22) it is a simple matter to modify the algorithm described in Sec. III used in the classical problem to obtain the $\rho_{kn}(j)$ for $k+n-1\leq N$ in a finite number of steps. This algorithm, described in Appendix C, also shows that because the structure constants $C_{k_1m_1k_2m_2}{}^{km}(j(j+1))$ are polynomials in $j(j+1)$ of degree $(k_1+k_2-k-1)/2$ that $\rho_{kn}(j)$ are polynomials in $j(j+1)$ of degree at most $n/2$. Thus, if (23) exists, by returning to the point of view of (16) and (17), i.e., treating $\hbar$ as just a given number, we see that we may conclude that $\beta_k(j)$ has the form

$$\beta_k(j)=\sum_{n=0}^{\infty}\hbar^n\rho_{kn}(j), \quad (25)$$

where $\rho_{kn}(j)$ is a polynomial of degree at most $n/2$ in $j(j+1)$. The properties of the finite algebra $\mathfrak{L}_{j_0(j_0+1)}{}'$ insure us that the series (25) converges in some neighborhood of $\hbar=0$ for $j=k/2$, $(k+1)/2$, $\cdots$. In the general case the existence of (25) is contingent on the assumed existence of (23).[11] If we now write (25) as

$$\beta_k(j)=\sum_{n=0}^{N+1-k}\hbar^n\rho_{kn}(j)+o(\hbar^{N+1-k}), \quad (26)$$

we see that $\beta_k(j)$ is determined to order $\hbar^{N+1-k}$ by the solution of the normal form problem with respect to the finite algebra $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$. Thus, by the solution of this problem it is always possible to find $B_j(\hbar)\in\mathfrak{L}_{j(j+1)}(\hbar)$ such that

$$\exp(B_j(\hbar))\frac{H_j}{\hbar}\exp(-B_j(\hbar))=\sum_{k=1}^{N+1}\sum_{n=0}^{N+1-k}\rho_{kn}(j)\hbar^{n+k-1}T_j(k,0)+o(\hbar^N), \quad (27)$$

to which we will refer as a "normal form of H_j modulo $\hbar^{N+2}$."

Because the algebras $\mathfrak{L}_{j(j+1)}{}^{(N)}(\hbar)$ are not semisimple, the algorithm for finding the $\rho_{kn}(j)$ does not directly reduce to an eigenvalue problem as did the problem of finding the coefficients of the $2j_0$th-order normal form for $j=j_0$ wherein we encountered the simple algebra A_{2j_0}. However, the fact that we have discovered the $j(j+1)$ dependence of $\rho_{kn}(j)$ as a polynomial of degree $\leq n/2$ enables us to compute the $\rho_{kn}(j)$ in (27) by determining $\beta_k(j)$ for $k=1, 2, \cdots, N+1$ at the points $j=k/2$, $(k+1)/2$, $\cdots$, $(N+1)/2$ and expressing the result in powers of $\hbar$ as in (26). Thus a knowledge of the $2j$th order normal forms for $j=\frac{1}{2}, 1, \cdots, j_0$ enables one to extract the normal forms at arbitrary j modulo $\hbar^{2j_0+1}$. Thus defining

$$\beta_{j_0k}(j)=\sum_{j_1=k/2}^{j_0}\beta_k(j_1)\prod_{\substack{j'=k/2\\ j'\neq j_1}}^{j_0}\left(\frac{j(j+1)-j'(j'+1)}{j_1(j_1+1)-j'(j'+1)}\right), \quad (28)$$

we see that

$$\beta_{j_0k}(j)=\beta_k(j)\begin{cases}k=1, 2, \cdots, 2j_0\\ j=k/2, (k+1)/2, \cdots, j_0\end{cases} \quad (29)$$

$$\beta_{j_0k}(j)=\beta_k(j)+o(\hbar^{2j_0-k}) \quad \text{all } j \text{ and } k.$$

There remains one great inconvenience in the computation (28), namely that to obtain $\beta_k(j_1)$ for $j_1=\frac{1}{2}, \cdots, j_0$ and $k=1, \cdots, 2j_1$ we have to solve $2j_0$ different eigenvalue problems for matrices of rank 2, 3, $\cdots$, $2j_0+1$, respectively. We shall now, however, show that it is possible to solve all of these simultaneously by solving a single eigenvalue problem associated with a $2j_0+1$ rank matrix. To do this we first note that we may, using the Wigner-Eckart theorem, express the matrix elements of the spin-j_0 representation of $T_{j_0}{}'$ in

[11] It should be remarked that the knowledge of the convergence of (25) for the infinite set of j values $k/2$, $(k+1)/2$, $\cdots$ does not insure its convergence at all complex j without some information concerning the asymptotic behavior of (25) for large $j(j+1)$.

the form

$$\langle j_0\nu|T_{j_0}'(k,m)|j_0\mu\rangle = \delta_{\nu,\mu+m}\sigma_{j_0k}(kmj_0-j_0+\mu|kj_0j_0-j_0+\mu+m), \quad (30)$$

where $\mu, \nu=0, 1, \cdots, 2j_0$, $k=1, 2, \cdots, 2j_0$ and[12]

$$\sigma_{jk}\equiv\left(\frac{(k!)^2(2j+k+1)!}{(2j+1)(2j-k)!}\right)^{1/2}.$$

We next note that if j is an arbitrary complex number we can define a $2j_0+1\times 2j_0+1$ matrix with elements

$$\langle j_0\nu|T_{jj_0}'(k,m)|j_0\mu\rangle \equiv \delta_{\nu,\mu+m}\sigma_{jk}(kmj-j+\mu|kjj-j+\mu+m), \quad \mu,\nu=0,1,\cdots,2j_0, \quad k=1,2,\cdots,2j_0, \quad (31)$$

where the Clebsch-Gordan coefficient and σ_{jk} are defined for complex j by replacing factorials by gamma functions. The matrices $T_{jj_0}'(k,m)$ now have the following useful properties:

(1) $T_{jj_0}'(k,m)=T_{j_0}'(k,m)$ for $j=j_0$.

(2) For $j\neq 0, \frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$ there exists a non-singular matrix $U^{(m)}$ with elements $U_{kk'}{}^{(m)}$ such that

$$T_{jj_0}'(k,m)=\sum_{k'=1}^{2j_0}U_{kk'}{}^{(m)}T_{j_0}'(k,m),$$

i.e., the matrices $T_{jj_0}'(k,m)$ generate an algebra isomorphic to A_{2j_0} for almost all j. The matrix $U^{(m)}$ may be constructed in an obvious way by using the orthogonality properties of the Clebsch-Gordan coefficients in (30).

(3) When j takes on one of the values $0, \frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$ the matrices $T_{jj_0}'(k,m)$ "fragment" in the following manner: Consider for example $T_{jj_0}'(k,k)$, for which we have from (31)

$$\langle j_0\nu|T_{jj_0}'(k,k)|j_0\mu\rangle = \delta_{\mu,\nu+k}\left(\frac{\Gamma(2j-\nu+1)\Gamma(\nu+k+1)}{\Gamma(2j-\nu-k+1)\Gamma(\nu+1)}\right)^{1/2}, \quad (32)$$

so that $\langle j_0\nu|T_{jj_0}'(k,k)|j_0\mu\rangle=0$ when $j=\frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$ unless both μ, ν are greater than $2j$. Moreover, if k exceeds $2j$ the matrix elements vanish for $\mu, \nu\leq 2j$ also.

Making use of (10) a similar conclusion is obtained for all $T_{jj_0}'(k,m)$. Thus, at these j values, the $(2j_0+1)\times(2j_0+1)$ matrices have the form:

$$T_{jj_0}'(k,m)=\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}, \quad (33)$$

where the A block is a $(2j+1)\times(2j+1)$ matrix and the B block is a $2(j_0-j)\times 2(j_0-j)$ matrix and the A block is also zero for $k=2j+1, \cdots, 2j_0$. One observes, in fact, that the A block is identical with the spin-j representation of $T_j'(k,m)$. Thus, for these j values the elements $T_{jj_0}'(k,m)$ with $k=2j+1, \cdots, 2j_0$ generate an ideal in the algebra of the matrices $T_{jj_0}'(k,m)$ and the factor algebra modulo this ideal is the algebra $\mathfrak{L}_{j(j+1)}'$ which is represented by the A blocks. Thus, while the $T_{jj_0}'(k,m)$ generate a simple algebra isomorphic to A_{2j_0} for $j\neq\frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$, the algebra changes form when j takes one of these values. (This process of passing from A_{2j_0} to a non-isomorphic algebra by continuing in j from j_0 to one of the "singular points" $j=\frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$ is a type of contraction related to Inonu-Wigner contraction.)

Suppose then that we consider the normal form problem for a matrix H_{jj_0}' obtained from H_{j_0}' by replacing every T_{j_0}' by T_{jj_0}', i.e.,

$$H_{jj_0}'\equiv -2\sqrt{2}\hbar T_{jj_0}'(1,0)+\sum_{k=1}^{2j_0}\sum_{m=-k}^{k}\alpha_k(-\hbar)^k \times\left(\frac{(2k)!}{(k-m)!(k+m)!}\right)^{1/2}T_{jj_0}'(k,m) \quad (34)$$

so that $H_{jj_0}'=H_{j_0}'$ for $j=j_0$ and fragments at $j=\frac{1}{2}, 1, \cdots, j_0-\frac{1}{2}$ with an A block identical to H_j'. For arbitrary complex j we may as above diagonalize H_{jj_0}' and express the diagonal form as a linear combination of the matrices $T_{jj_0}'(k,0)$, i.e., we obtain the normal form analogous to (18):

$$H_{jj_0}'^{(\mathrm{NF})}=\sum_{k=1}^{2j_0}\beta_k(j_0,j)\hbar^kT_{jj_0}'(k,0). \quad (35)$$

Because of the fragmentation noted above we see that

$$\beta_k(j_1)=\beta_k(j_0,j_1) \quad (36)$$

for $j_1=\frac{1}{2}k, \cdots, j_0$, $k=1, 2, \cdots, 2j_0$ and hence (28) becomes

$$\beta_{j_0k}(j)=\sum_{j_1=k/2}^{j_0}\beta_k(j_0,j_1) \times\prod_{\substack{j'=k/2\\ j'\neq j_1}}^{j_0}\left(\frac{j(j+1)-j'(j'+1)}{j_1(j_1+1)-j'(j'+1)}\right). \quad (37)$$

The approximate energy levels are then computed as follows: In the approximate normal form

$$\sum_{k=1}^{2j_0}\beta_{j_0k}(j)\hbar^kT_j(k,0) \quad (38)$$

we insert the expressions (37) for β_{j_0k} with $j(j+1)=-3/16$. For $T_j(k,0)$ we insert the diagonal matrices defined by (10) and (8a), i.e., the matrices $T(k,0)$ constructed from (10) using the usual quantum-mechanical representation of J_+ and J_-. The diagonal elements of the diagonal matrix (38) are then the approximate

[12] Clebsch-Gordan notation follows A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957). Note that one usually labels the matrix elements by $\mu, \nu=-j_0, \cdots, +j_0$. It is more convenient here to redefine μ, ν relative to $-j_0$ so as to have integer values of μ and ν.

energy levels, and as we have seen are accurate at least to order $\hbar^{2j_0}$. The method thus yields a result at least as good as $2j_0$th-order WKB approximation in the sense that it contains the correction to the corresponding approximate classical normal form to order $\hbar^{2j_0}$.

In Sec. X we shall indicate a number of tricks whereby the above calculation can be carried out quite simply and an example will also be worked out.

VII. THE CONCEPT OF APPROXIMATE DYNAMICAL SYMMETRY

We shall now collect the ideas developed in the previous section into a formal definition of approximate dynamical symmetry.

Let the Hamiltonian and a complete set of dynamical variables generate an algebra $\mathfrak{L}$. We shall say that $\mathfrak{L}$ admits a "natural analytic extension" if the structure constants $C_{ij}{}^k(i, j, k=1, 2, \cdots)$ of $\mathfrak{L}$ possess the following property: For each $i=1, 2, \cdots$ there exists an integer $s(i)$ such that

$$\begin{aligned} &\text{(a)}\quad 0\leq s(1)<s(2)<s(3)<\cdots. \\ &\text{(b)}\quad s(i)+s(j)\geq s(k)\quad \text{whenever}\quad C_{ij}{}^k\neq 0. \end{aligned} \tag{39}$$

If we then consider the one parameter family of Lie algebras $\mathfrak{L}_x$ with structure constants:

$$C_{ij}{}^k(x)\equiv x^{s(i)+s(j)-s(k)}C_{ij}{}^k,$$

we see that $\mathfrak{L}_x$ is isomorphic to $\mathfrak{L}$ for all $x\neq 0$ and that $C_{ij}{}^k(x)$ are analytic in x for all x. One may regard the family $\mathfrak{L}_x$ as a single Lie algebra by treating x as an indeterminate, i.e., we consider an algebra $\mathfrak{L}(x)$ with generators:

$$(n,k)\equiv x^{n+s(k)}L_k,\quad n=0, 1, 2, \cdots; k=1, 2, \cdots, \tag{40}$$

in which the L_k are generators of $\mathfrak{L}$. Commutators of the elements (40) are then of the form:

$$[(n_1,k_1),(n_2,k_2)]=\sum_{k,n} R_{k_1n_1,k_2n_2}{}^{kn}(n,k),$$

where

$$R_{k_1n_1,k_2n_2}{}^{kn}=\delta_{n,(n_1+n_2+s(k_1)+s(k_2)-s(k))}C_{k_1k_2}{}^k. \tag{41}$$

The algebra $\mathfrak{L}(x)$ is mapped homomorphically into any $\mathfrak{L}_{x_0}$ by

$$(n,k)\rightarrow x_0{}^{n+s(k)}L_k, \tag{42}$$

i.e., that mapping which returns us to the point of view of x as a fixed number x_0. Thus, $\mathfrak{L}(x)$ may be regarded as the "parent" algebra of the family $\mathfrak{L}_x$. The importance of $\mathfrak{L}(x)$ is that it possesses a sequence of *finite*, homomorphic images. For, letting N be any positive integer, the elements (40) for which $n+s(k)>N$ generate an ideal $I_N(x)$ in $\mathfrak{L}(x)$ and the factor algebra

$$\mathfrak{L}^{(N)}(x)\equiv\mathfrak{L}(x)/I_N(x) \tag{43}$$

is a *finite* algebra generated by the elements with $n+s(k)\leq N$. Note that (39) was essential in obtaining this result. The sequence of algebras $\mathfrak{L}^{(N)}(x) N=1, 2, \cdots$ are "nested" in that each is contained in and is in fact a factor algebra of its successors. Moreover,

$$\mathfrak{L}(x)=\text{"}\lim_{N\rightarrow\infty}\text{"}\ \mathfrak{L}^{(N)}(x), \tag{44}$$

in the sense that any finite set of elements of $\mathfrak{L}(x)$ is contained in $\mathfrak{L}^{(N)}(x)$ for all sufficiently large N.[13] The approximation of $\mathfrak{L}(x)$ by $\mathfrak{L}^{(N)}(x)$ consists in neglecting powers of x greater than the Nth. That is, the occurrence of a natural analytic extension enables one to introduce a natural expansion parameter for approximating the infinite algebra $\mathfrak{L}$ by a sequence of finite algebras. In the case of $\mathfrak{L}^{(\mathrm{QM})}$ with the structure constants $C_{k_1m_1k_2m_2}{}^{km}$ we found the function $s(k,m)=k$ satisfying (39) while the role of x above was played by $\hbar$. We may conjecture that natural analytic extensions are always associated with fundamental physical constants which then may be used as natural expansion parameters. This is the algebraic structure underlying the WKB method.

The difficulty in using $\mathfrak{L}^{(N)}(x)$ as approximations to $\mathfrak{L}(x)$ arises from the fact that they are not, in general, semisimple and hence do not have a very intelligible Cartan geometry. (This is because the natural metric, the Killing form, becomes degenerate.) A practical consequence was that we could not regard the normal form problem, the solution to which is required to find approximate energy levels, as a diagonalization problem.[14] We should like, therefore, to relate $\mathfrak{L}^{(N)}(x)$ to finite simple Lie algebras in such a way that all calculations can be done using these algebras which have a complete geometry. Suppose then that we can find a set of finite, simple Lie algebras $S_N(\xi)$ (ξ indexes the set) and mappings $\lambda_N(\xi)$ of $\mathfrak{L}$ onto $S_N(\xi)$ such that Cartan subalgebras of $S_N(\xi)$ are the images of a Cartan subalgebra in $\mathfrak{L}$ and such that the normal form of any regular $\mathfrak{IC}\in\mathfrak{L}$ with respect to $\mathfrak{L}^{(N)}(x)$ [i.e., the normal form in $\mathfrak{L}^{(N)}(x)$ of the image of $\mathfrak{IC}$ in $\mathfrak{L}^{(N)}(x)$ under the homomorphism of $\mathfrak{L}$ into $\mathfrak{L}^{(N)}(x)$] is determined by the normal forms in $S_N(\xi)$ of the image of $\mathfrak{IC}$ in $S_N(\xi)$ under $\lambda_N(\xi)$. (By "determined" we mean of course that the rule for determination be independent of the particular element $\mathfrak{IC}$.) If the $S_N(\xi)$ are all isomorphic to a single semisimple algebra S_N we shall say that $\mathfrak{L}^{(N)}(x)$ is "geometrically completed" by S_N.

We are now in a position to state our definition of approximate dynamical symmetry.

Let the Hamiltonian $\mathfrak{IC}$ be a regular element of a Lie algebra $\mathfrak{L}$ generated by $\mathfrak{IC}$ and a complete set of dynamical variables. If $\mathfrak{L}$ admits a natural analytic extension $\mathfrak{L}(x)$ the factors $\mathfrak{L}^{(N)}(x)$ of which are geometrically completed by simple algebras S_N we shall say that $\mathfrak{IC}$

[13] Perhaps one of the most important problems is to topologize this notion effectively.

[14] It should be remarked that the problem does reduce to a triangularization problem followed by expansion in terms of the triangular matrices of the Cartan subalgebra. This, however, complicates matters considerably.

admits S_N as an approximate dynamical symmetry to order x^N.

In this language we have proved that one-dimensional Hamiltonians of the form (14) in $\mathfrak{L}^{(QM)}$ admit A_N as approximate dynamical symmetries to order $\hbar^N$ for $N=1, 2, \cdots$.[15] We also found that these algebras were generated from $\mathfrak{L}^{(QM)}$ by embedding $\mathfrak{L}^{(QM)}$ in a family of Lie algebras $\mathfrak{L}_\lambda(\lambda=j(j+1))$, analytically continuing in λ to certain singular points ($j=\frac{1}{2}, 1, \cdots$) and taking the factor algebras modulo the ideals which appeared thereat. This process, which we shall call "truncation," bears certain striking similarities to the process of "contraction" which has been discussed extensively in the case of finite algebras.[16] We are thus tempted to conjecture that truncation can be understood in a completely algebraic manner forming some sort of generalization of the classical algebraic process of homomorphism. In the next section we shall see that the truncation parameter has a simple physical interpretation.

VIII. EXTENSION TO TWO AND THREE DIMENSIONS

We shall now show that the results of Sec. VI carry over to two and three dimensions with the result that the angular momentum l incorporates itself into the Casimir operator of the dynamical algebra, and that analytic continuation in l to the singular points may serve to generate the truncations. In n dimensions we set

$$J_0=\frac{1}{4\hbar}\sum_{s=1}^{n}(p_s^2+q_s^2), \quad J_\pm=\pm\frac{1}{4\hbar}\sum_{s=1}^{n}(p_s\pm iq_s)^2. \quad (45)$$

Then J_0, $J_\pm$ still obey (8b) while (8c) becomes

$$J_0^2+\tfrac{1}{2}(J_+J_-+J_-J_+)=\frac{n(n-4)}{16}+\tfrac{1}{4}L^2, \quad (46)$$

where

$$L^2=\tfrac{1}{2}\sum_{\substack{s,t=1\\ s\neq t}}^{n}L_{st}^2, \quad L_{st}=(q_sp_t-p_sq_t),$$

and L^2 commutes with J_0, J_+, J_-.

Any potential of the form $V(q_1^2+q_2^2+\cdots+q_n^2)$ in n dimensions can then be expanded in terms of the tensor operators (10) where the J_0, $J_\pm$ still satisfy (8b) but (8c) is replaced by (46). The coefficients in the expansion then depend only on the functional form of V and not on the number of dimensions which shows up only in determining the allowed values of the Casimir operator. Thus, for 2 dimensions, the angular momentum L is half-odd integral while for 3 dimensions it is integral. Thus if for 2 dimensions we put $L^2=(l+\frac{1}{2})^2$ and for 3 dimensions put $L^2=l(l+1)$ with $l=0, 1, 2, \cdots$, then (46) in either case becomes

$$J_0^2+\tfrac{1}{2}(J_+J_-+J_-J_+)=\tfrac{1}{4}l(l+1)-3/16=j(j+1), \quad (47)$$

where

$$j=\tfrac{1}{2}l-\tfrac{1}{4} \quad \text{or} \quad j=-\tfrac{1}{2}l-\tfrac{3}{4}.$$

The one-dimensional case is simply $l=0$.

In the method of Sec. VI we were able to give a prescription for obtaining approximate energy levels of H for all complex values of j. Hence for the two- and three-dimensional problems we need only take as the physical case the algebra corresponding to $j(j+1)=\frac{1}{4}l(l+1)-3/16$ where l is fixed by the angular momentum of interest. Note that for l integral or half-integral, j does *not* take on integral or half-integral values. That is, the physical algebras are free of ideals, i.e., are simple just as in one dimension.

The method of Sec. VI involved an extrapolation in j from its physical value to the integral and half-integral values. In virtue of (47) we see that this is equivalent to extrapolating in the angular momentum. Thus already we begin to see the basis of a connection with Regge methods which will be elucidated later on.

IX. EXTENSION OF THE APPROXIMATION TECHNIQUE TO YUKAWA-LIKE POTENTIALS

We shall now modify the ideas developed above to deal with spherically symmetric potentials with a $1/r$ singularity at the origin. Thus we write

$$V(r)=g/r+(G/r)u(r), \quad (48)$$

where

$$u(r)=\sum_{n=1}^{\infty}\gamma_n r^n.$$

If in the radial Schrödinger equation

$$-\frac{\hbar^2}{2m}\psi''+\left\{V+\frac{\hbar}{2m}\frac{l(l+1)}{r^2}\right\}\psi=E\psi, \quad (49)$$

we make the substitutions

$$r=y^2, \quad \psi(r)=y^{1/2}\varphi(y),$$

we obtain

$$-\frac{\hbar^2}{2m}\varphi''+\left\{\tfrac{1}{2}Ky^2+4Gu(y^2)+\frac{\hbar^2}{2m}\frac{\lambda(\lambda+1)}{y^2}\right\}\varphi=\epsilon\varphi, \quad (50)$$

where

$$K=-8E, \quad \epsilon=-4g, \quad \text{and} \quad \lambda=2l+\tfrac{1}{2}.$$

Thus, by this substitution, the dynamical problem for a potential of the type (48) changes into that for a potential of the type (14) in which: (i) the coupling constant becomes the constant (energy) term except for a factor, (ii) the energy becomes a spring constant,

[15] It is amusing to note the parallelism between the nesting of the $\mathfrak{L}^{(N)}(x)$ and that of the Dynkin diagrams o, o—o, o—o—o, $\cdots$ of the algebras A_N which geometrically complete them.

[16] E. J. Saletan, J. Math. Phys. 2, 1 (1960). (Other references may be found in this paper.)

and (iii) the angular momentum value changes as indicated. Thus, a solution of the energy eigenvalue problem of (50) to yield ϵ as a function of the other parameters in particular K, will give an expression for the coupling constant g as a function of the energy for the problem (49) at the shifted angular momentum. We shall refer to Eq. (50) as the "conjugate" to (49). Thus, e.g., the harmonic-oscillator problem is conjugate to the Coulomb problem.[17] With the scale change $y=(2/K)^{1/4}z$ (assuming $E\neq 0$) the $\frac{1}{2}Ky^2$ term becomes just z^2, and the potential in (50) is exactly of the form (14). The regularity condition then takes the form

$$\gamma_1 \neq E/G. \tag{51}$$

Thus, e.g., for a Yukawa potential $ge^{-\mu r}/r$, $\mu>0$, this means

$$E\neq -g\mu. \tag{52}$$

Since we are here considering only the bound-state problem where $E<0$ and $g<0$, (52) is automatically satisfied.

Now we apply the methods of Sec. VI to the potential in (50). The only essential change is in the relation between the Casimir operator value j and the given angular momentum l. From (47)

$$j=\tfrac{1}{2}\lambda-\tfrac{1}{4} \quad \text{or} \quad -\tfrac{1}{2}\lambda-\tfrac{3}{4}, \tag{53}$$

i.e.,

$$j=l, \quad \text{or} \quad -l-1. \tag{54}$$

That is, the physical algebra $\mathfrak{L}_{j(j+1)}$ for the conjugate potential corresponds to $j(j+1)=l(l+1)$ where l is the angular momentum given in (49). The physically interesting values of l are integers for the three-dimensional problem and half-odd-integers for the two-dimensional problem. Thus, for these problems one is led to an $\mathfrak{L}_{j(j+1)}$ with integer and half-integer j values. But as we have seen, these are precisely the values j at which $\mathfrak{L}_{j(j+1)}$ develops ideals by means of which we generated the truncations. Thus, for angular momentum l, the $2l$th-order approximate normal form has a special property for potentials of the type (48), namely, that we do not have to continue in l to obtain the appropriate truncation—we are already there. Hence, for this case there is no loss of homomorphism in the truncation operation and we therefore will calculate a correct $2l$th-order normal form to *all* powers of $\hbar$. Note that this does not mean that this approximate normal form is the complete solution to the dynamical problem, but merely asserts the complete reliability of the $2l$th-order normal form calculated by the method of Sec. VI *qua* a $2l$th-order normal form. To obtain higher order approximate normal forms we have to extrapolate in l as we did in Sec. VI and will therefore be able to conclude only that the normal forms are accurate to some power of $\hbar$.

[17] The reader will find it amusing to obtain the Balmer formula from the formula for the oscillator levels by making the interchanges described between coupling constant and energy and shifting the angular momentum.

Approximate expressions for the coupling constant are now obtained from the normal forms exactly as the approximate energy levels were computed in Sec. VI (see, e.g., 38ff). This will yield expressions for the values of the coupling constant for which a bound state may appear at a given energy. These must then be inverted to obtain the energies at which a bound state may appear for a given coupling constant. Complex solutions are then resonances. The form of these expressions will become clear in the next section in which an example is worked out.

X. PRACTICAL TECHNIQUES FOR CALCULATION AND EXAMPLES

In this section we will show how the calculation of approximate energy levels by the techniques described above may quite simply be carried out. We shall then apply the technique to the second-order approximation (first nontrivial case) for the anharmonic oscillator and the Yukawa potential.

We first observe that (14) can be written

$$e^{-J-}He^{J-}=4(J_0-J_-)+V(-\hbar J_+)\equiv\tilde{H}. \tag{55}$$

Since the levels are invariant to canonical transformation we may compute using $\tilde{H}$ instead of H. The advantage of $\tilde{H}$ is that its tensorial expansion is quite simple, namely

$$\tilde{H}=2\hbar[2T(1,-1)-\sqrt{2}T(1,0)]+\sum_{k=1}^{\infty}\alpha_k(-\hbar)^kT(k,k). \tag{56}$$

The first step, then, in finding the $2j_0$th-order normal form is to diagonalize

$$\tilde{H}_{jj_0}'=2\hbar[2T_{jj_0}'(1,-1)-\sqrt{2}T_{jj_0}'(1,0)]+\sum_{k=1}^{2j_0}\alpha_k(-\hbar)^kT_{jj_0}'(k,k), \tag{57}$$

the elements of which are, by (21),

$$\langle\mu|\tilde{H}_{jj_0}'|\nu\rangle = 4\hbar\{(\mu-j)\delta_{\mu,\nu}-\delta_{\mu,\nu-1}[\nu(2j-\nu+1)]^{1/2}\} + \sum_{k=1}^{2j_0}\alpha_k(-\hbar)^k\delta_{\mu,\nu+k}\left\{\frac{\Gamma(2j-\nu+1)\Gamma(\nu+k+1)}{\Gamma(2j-\nu-k+1)\Gamma(\nu+1)}\right\}^{1/2}, \quad \mu,\nu=0,1,\cdots,2j_0. \tag{58}$$

Let the eigenvalues be $\lambda_0(j),\cdots,\lambda_{2j_0}(j)$. We choose a "canonical" ordering of these eigenvalues as follows: Consider the $\lambda_\mu(j)$ as a function of the parameters $\alpha_1,\alpha_2,\cdots,\alpha_{2j_0}$ in (58) and then define $\lambda_\mu(j)$ to be that eigenvalue which takes the value $4\hbar(\mu-j)$ when the α's all vanish. Thus when $j=\frac{1}{2},1,\frac{3}{2},\cdots,j_0$ we know that $\lambda_1(j),\lambda_2(j),\cdots,\lambda_{2j}(j)$ are the eigenvalues of $\tilde{H}_j'$.

The next step is to express the matrices

$$\Lambda_j\equiv\begin{bmatrix}\lambda_0(j) & & & 0\\ & \lambda_1(j) & & \\ & & \ddots & \\ 0 & & & \lambda_{2j}(j)\end{bmatrix}\quad j=\tfrac{1}{2},\,1,\,\cdots,\,j_0 \tag{59}$$

as a linear combination of the matrices $T_j'(k,0)$. This calculation is facilitated by an orthonormality property of the $T_j'(k,0)$ obtained from the Wigner-Eckart theorem, namely

$$\mathrm{Tr}\{T_j'(k_1,0)T_j'(k_2,0)\}=\delta_{k_1k_2}\frac{(k_1!)^2(2j+k_1+1)!}{(2k_1+1)!(2j-k_1)!}. \tag{60}$$

From (30), we have

$$\langle\mu|\,T_j'(k,0|\,\nu\rangle=\delta_{\mu\nu}\Phi_\mu'(j,k),$$

where

$$\Phi_\mu'(j,k)\equiv\frac{(-1)^k}{[(2k)!]^{1/2}}\sum_{s=0}^{k}\binom{k}{s}(-1)^s \times\frac{(\mu+s)!}{(\mu+s-k)!}\frac{(2j+k-\mu-s)!}{(2j-\mu-s)!} \tag{61}$$

(the summation over all s for which the factorials have non-negative arguments). We then obtain

$$\hbar^k\beta_k(j_0,j)=\frac{(2k+1)!(2j-k)!}{(k!)^2(2j+k+1)!}\sum_{\mu=0}^{2j}\lambda_\mu(j)\Phi_\mu'(j,k) \quad \text{for}\quad j=\tfrac{1}{2},\,1,\,\cdots,\,j_0, \tag{62}$$

and may then compute the approximate normal form coefficients $\hbar^k\beta_{j_0k}(j)$ by the interpolation process [Eq. (37)].

The final step is the insertion for the $T_j(k,0)$ in (38) of the correct quantum-mechanical representation, namely

$$\langle\mu|\,T_j(k,0)|\,\nu\rangle=\delta_{\mu\nu}\Phi_\mu(j,k),\quad \mu,\,\nu=0,\,1,\,2,\,\cdots,$$

where

$$\Phi_\mu(j,k)=\frac{1}{((2k)!)^{1/2}}\sum_{s=0}^{k}\binom{k}{s}(-1)^s \times\frac{(\mu+s)!}{(\mu+s-k)!}\frac{\Gamma(2j+2+\mu+s)}{\Gamma(2j+2+\mu+s-k)}. \tag{63}$$

One may note that

$$\Phi_\mu(-j-1,k)=\Phi_\mu'(j,k) \tag{64}$$

for $j=\frac{1}{2},\,1,\,\cdots$, and $\mu=0,\,1,\,\cdots,\,2j$. [This is a manifestation of the fact that the structure of $\mathfrak{L}_{j(j+1)}$ is determined by $j(j+1)$ which is invariant under $j\to -j-1$, a fact which will be discussed further in the next section.]

Collecting the above results we now have as a final formula for the $2j_0$th-order approximate energy levels:

$$E_{j_0,\mu}(j)=\sum_{k=1}^{2j_0}\sum_{j'=\frac{1}{2}}^{j_0}\sum_{\mu=0}^{2j'}\frac{(2k+1)!(2j'-k)!}{(k!)^2(2j'+k+1)!} \times\Phi_\mu(j,k)P_{j_0,k}(j,j')\lambda_\mu(j')\Phi_\mu(-j'-1,k), \tag{65}$$

where $\mu=0,\,1,\,2,\,\cdots$, and

$$P_{j_0,k}(j,j')=\prod_{\substack{j''=k/2\\ j''\neq j'}}^{j_0}\frac{j(j+1)-j''(j''+1)}{j'(j'+1)-j''(j''+1)}.$$

(Note that in the above, summations and products indexed by j' and j'' run over both integer and half-integer values.) The value of j is then chosen to be one of the roots $j=-\frac{1}{4},\,-\frac{3}{4}$ of $j(j+1)=-3/16$ in the one-dimensional problem. In the two- and three-dimensional cases we set $j=\frac{1}{2}l-\frac{1}{4}$, $l=0,\,1,\,2,\,\cdots$ for angular momentum L, $L^2=(l+\frac{1}{2})^2$ (2 dimensions) or $l(l+1)$ (3 dimensions).[18]

As a first example consider the anharmonic oscillator

$$H_\alpha\equiv|p|^2+|q|^2+2\alpha|q|^4 \tag{66}$$

for which we compute the second-order ($j_0=1$) approximate energy levels. We must first diagonalize

$$H_{\alpha j_1}=4\hbar\begin{bmatrix}-j & -(2j)^{1/2} & 0\\ 0 & -j+1 & -(2(2j-1))^{1/2}\\ \alpha\hbar(j(2j-1))^{1/2} & 0 & -j-2\end{bmatrix}. \tag{67}$$

Solving the secular cubic equation for the eigenvalues $\lambda_\mu(j)$ we find

$$\lambda_0(\tfrac{1}{2})=-2\hbar,\quad \lambda_1(\tfrac{1}{2})=2\hbar;$$

$$\lambda_0(1)=\frac{8\hbar}{\sqrt{3}}\cos\left(\frac{\varphi+2\pi}{3}\right),\quad \lambda_1(1)=\frac{8\hbar}{\sqrt{3}}\cos\left(\frac{\varphi+4\pi}{3}\right),$$

$$\lambda_2(1)=\frac{8\hbar}{\sqrt{3}}\cos\left(\frac{\varphi}{3}\right), \tag{68}$$

where φ is defined by

$$\cos\varphi=3\sqrt{3}\alpha\hbar.$$

From (65) we then obtain after a slight rearrangement

$$E_{1,\mu}(j)=4\hbar\left\{\left[1+(\tfrac{3}{5}-\tfrac{4}{5}j(j+1))\left(1-\cos\frac{\theta}{3}\right)\right](j+\mu+1) +\sqrt{3}\left(\sin\frac{\theta}{3}\right)[(j+\mu+1)^2-\tfrac{1}{3}j(j+1)]\right\}, \tag{69}$$

where

$$\sin\theta=3\sqrt{3}\hbar\alpha\quad\text{and}\quad\mu=0,\,1,\,2,\,\cdots. \tag{70}$$

[18] The root $j=-\frac{1}{2}l-\frac{3}{4}$ is excluded by the requirement of positive energies.

In particular the $l=0$ ground-state energy E_α differs from the $l=0$ harmonic-oscillator ground-state energy E_0 by[19]

$$\Delta=\frac{E_\alpha-E_0}{\hbar}=\tfrac{9}{2}\sin^2\left(\frac{\theta}{6}\right)+\frac{5\sqrt{3}}{2}\sin\frac{\theta}{3}. \tag{71}$$

We see that real positive energies are obtained only as long as

$$\hbar|\alpha|\leq 1/3\sqrt{3}. \tag{72}$$

If one wishes to consider stronger anharmonic perturbations it is necessary to employ the higher order approximations $j_0=\frac{3}{2}, \cdots$.

As a second example we consider the second-order approximation for Yukawa-like potentials:

$$V(r)=(g/r)+G(1+\tau_1 r+\tau_2 r^2+\cdots), \tag{73}$$

for angular momentum l. By the method of Sec. IX we first go over to the conjugate potential and thus (after a scale change) find that the allowed values of the quantity

$$\epsilon\equiv -2g(G-E)^{-1/2} \tag{74}$$

are the energy levels at angular momentum $2l+\frac{1}{2}$ of the potential:

$$|\mathbf{p}|^2+|\mathbf{r}|^2+\beta_1|\mathbf{r}|^4+\beta_2|\mathbf{r}|^6+\cdots,$$

where

$$\beta_s=4G\tau_s[4(G-E)]^{-1-s/2}.$$

For the second-order calculation ($j_0=1$), this reduces to the same form as the anharmonic oscillator treated above with

$$\alpha=\tfrac{1}{4}G\tau_1(G-E)^{-3/2}. \tag{75}$$

If we suppose then that l, E, G, and τ_1 are given, (69) becomes an equation for the allowed values of g in this approximation, namely

$$g_{1,\mu}=-2\hbar(G-E)^{1/2}\left\{\left[1+(\tfrac{3}{5}-\tfrac{4}{5}l(l+1))\left(1-\cos\frac{\eta}{3}\right)\right]\right.$$
$$\left.\times(l+\mu+1)-\sqrt{3}\left(\sin\frac{\eta}{3}\right)[l+\mu+1)^2-\tfrac{1}{3}l(l+1)]\right\} \tag{76}$$

where

$$\sin\eta=(-3\sqrt{3}\hbar/4)G\tau_1(G-E)^{-3/2}$$

and $\mu=0, 1, 2, \cdots$.

To obtain energy levels for given g one sets $g_{1,\mu}=g$ and then solves for $E=E_{1,\mu}$ as a function of g. In doing this it is important to remember to choose the correct root, namely the one which gives the Balmer formual when $G=0$.[20]

As an illustration we consider a simple Yukawa potential,

$$V(r)=(g/r)e^{-r/r_0}, \tag{77}$$

for which $G=-g/r_0$ and $\tau_1=-1/2r_0$ and find the condition for the existence of at least one bound state. Thus we seek a condition on g, r_0 in order that the $l=0$ ground state has zero energy. Setting $l=\mu=E=0$ in (76) we obtain an equation for $(-gr_0)^{1/2}$. Solving this graphically, one obtains

$$(-gr_0)^{1/2}=1.50\pm 0.01.$$

The first approximation ($j_0=\frac{1}{2}$) for this quantity yields $(-gr_0)^{1/2}=2.0$. Now the exact value of $(-gr_0)^{1/2}$ from numerical solution of the Schrödinger equation may be computed from the well-known well-depth parameter and is

$$(-gr_0)^{1/2}=1.30\pm 0.01.$$

Thus the second approximation has reduced the error in the first approximation considerably.

The third approximation entails the solution of a quartic which can be done but is messy. The fourth and higher orders require the solution of algebraic equations of degree five and higher and so must be done numerically. In a subsequent paper we shall present the results of these computations.

XI. APPLICATION TO SCATTERING

We shall deal with potentials of the form (73) and assume that we have applied the above methods to compute in $2j_0$th approximation the allowed values of g for given energy, angular momentum, G, τ_1, τ_2, $\cdots$. Namely,

$$g=g_{j_0,\mu}(l,G,E), \quad \mu=0, 1, 2, \cdots \tag{78}$$

(the dependence on τ_1, τ_2, $\cdots$ being omitted for convenience). We now show how these may be used to extract corresponding approximations for the phase shifts.

It is convenient first to set

$$G=gG_0$$

in (78) and solve (78) for g to obtain

$$g=\tilde{g}_{j_0,\mu}(l,G_0,E), \quad \mu=0, 1, 2, \cdots \tag{79}$$

the proper root being chosen in each case by the requirement that it coincide at $G_0=0$ with that given by the Balmer formula. That is the $\tilde{g}_{j_0,\mu}(l,G_0,E)$ are the $2j_0$th approximation to the exact values

$$g=\tilde{g}_\mu(l,G_0,E), \quad \mu=0, 1, 2, \cdots$$

for which there is a bound state of given energy E and angular momentum l for the potential:

$$V(r)=g(1/r+G_0(1+\tau_1 r+\tau_2 r^2+\cdots)). \tag{80}$$

Let $|\Psi_\mu\rangle$ be the bound-state wave functions correspond-

[19] Since we know that the levels (69) must be correct to order $\hbar^2$ (since $j_0=1$), one can trivially extract the first-order perturbation-theory result by computing to first order in α.

[20] One may observe that a solution is obtained even for repulsive potentials. These solutions will, however, lie on the second energy sheet. A glance at the Balmer formula for the Coulomb case will make this quite clear.

ing thereto, with the normalization

$$\langle\Psi_\mu|1/r|\Psi_\nu\rangle=\delta_{\mu\nu}, \tag{81}$$

and consider the bound-state Green's function

$$G_l^{(\mathrm{BS})}=\sum_{\mu=0}^{\infty}\frac{|\Psi_\mu\rangle\langle\Psi_\mu|}{\tilde{g}_\mu} \tag{82}$$

for which we have

$$G_l^{(\mathrm{BS})}\frac{1}{r}|\Psi_\mu\rangle=\frac{1}{\tilde{g}_\mu}|\Psi_\mu\rangle. \tag{83}$$

The advantage of fixing E and considering the eigenvalue problem in the coupling constant is that for any $V(r)$ which can be written as a "reasonable" superposition of Yukawa potentials, the operator on the left side of (83) is L^2 and one can apply Fredholm methods. Hence if $|\Psi_{(0)}\rangle$ is an unperturbed ($g=0$) solution then

$$|\Psi_{(g)}\rangle=|\Psi_{(0)}+gG_l^{(\mathrm{BS})}(1/r)|\Psi_{(g)}\rangle \tag{84}$$

is an integral equation giving a solution $|\Psi_{(g)}\rangle$ at any value of g. $G_l^{(\mathrm{BS})}$ is a function of $E=k^2$ and if we analytically continue from the upper imaginary k axis to the positive real k axis we obtain the scattering Green's function:

$$G_l^{(+)}=G_l^{(\mathrm{BS})}(k+i\epsilon). \tag{85}$$

One may then show (c.f. Ref. 10) that the Jost function is given by

$$f_l^{(+)}(k)=\det(1-gG_l^{(\mathrm{BS})}(k+\epsilon)(1/r)). \tag{86}$$

Evaluating the determinant in coordinate system spanned by the $|\Psi_\mu\rangle$ and using (81), one finds

$$f_l^{(+)}(k)=\prod_{\mu=0}^{\infty}\left(1-\frac{g}{\tilde{g}_\mu^{(+)}(l,G_0,E)}\right), \tag{87}$$

where $\tilde{g}_\mu^{(+)}$ indicates that $\tilde{g}_\mu$ is to be computed for positive $E=k^2$ by analytic continuation from the positive imaginary k axis to the positive k axis. The Jost function $f_l^{(-)}(k)$ is given by

$$f_l^{(-)}(k)=(f_l^{(+)}(k))^*, \tag{88}$$

and the S matrix by

$$\begin{aligned}S_l(k)&=f_l^{(+)}(k)/f_l^{(-)}(k)\\&=e^{2i\delta_l(k)}.\end{aligned} \tag{89}$$

We now obtain the approximate Jost function and S matrix by replacing $\tilde{g}_\mu^{(+)}$ by our approximate expressions $\tilde{g}_{j_0,\mu}^{(+)}$; thus,

$$f_{j_0,l}^{(+)}(k)=\prod_{\mu=0}^{\infty}\left(1-\frac{g}{g_{j_0,\mu}^{(+)}(l,G_0,E)}\right). \tag{90}$$

Since our goal is to obtain approximations which can be computed in finite terms we must next find a meaningful method of truncating the infinite product (90). In fact we shall next show that the approximation of (90) by

$$f_{j_0,l}^{(+)\prime}(k)\equiv\prod_{\mu=0}^{2j_0}\left(1-\frac{g}{g_{j_0,\mu}^{(+)}(l,G_0,E)}\right) \tag{91}$$

has an interesting physical interpretation.

We have noted previously that the algebra of $\mathfrak{L}_{j(j+1)}$ of quantum-mechanical operators develops an ideal when $j=\frac{1}{2}, 1, \frac{3}{2}, \cdots$. We have also noted that the structure constants of $\mathfrak{L}_{j(j+1)}$ are polynomials in $j(j+1)$ and are therefore invariant under $j\to -j-1$. Thus, $\mathfrak{L}_{j(j+1)}$ also has an ideal for $j=-\frac{3}{2}, -2, \cdots$. If one writes explicit infinite matrices for the operators $T_j(k,m)$ which generate $\mathfrak{L}_{j(j+1)}$ and then analytically continues in j to $j=-\frac{3}{2}, -2, \cdots$, one finds that these ideals manifest themselves explicitly. To see this we observe that the quantum-mechanical operators $T_j(k,m)$ are determined by (9), (8b), (8c'), the Hermiticity requirement $J_0=J_0^+$, $J_+=-J_-^+$, and the requirement that J_0 have only positive eigenvalues. These requirements restrict j to $j=0, \frac{1}{2}, 1, \cdots$ and lead in a familiar way to the matrix elements.

$$\begin{aligned}&(J_0)_{m'm}=m\delta_{m'm},\\&(J_\pm)_{m'm}=i\delta_{m',m\pm1}[m(m\pm1)-j(j+1)]^{1/2},\\&(e^{\alpha J+})_{m'm}=(i\alpha)^{m'm}\\&\quad\times\left\{\frac{\Gamma(m'-j)\Gamma(j+m'+1)}{\Gamma(m-j)\Gamma(j+m+1)}\right\}^{1/2}\frac{1}{(m'-m)!},\end{aligned} \tag{92}$$

with

$$m', m=j+1, j+2, \cdots.$$

If we define new indices n', n by $m=j+1+n$, $m'=j+1+n'$; $n, n'=0, 1, 2, \cdots$, we find, e.g.,

$$(e^{\alpha J+})_{n'n}=(i\alpha)^{n'-n}\left\{\frac{n'!}{n!}\frac{\Gamma(2j+2+n')}{\Gamma(2j+2+n)}\right\}^{1/2}\frac{1}{(n'-n)!},$$

and if we then analytically continue these matrix elements in j to $-j-1$ we obtain the expression

$$(i\alpha)^{n'-n}\left\{\frac{n'!}{n!}\frac{\Gamma(-2j+n')}{\Gamma(-2j+n)}\right\}^{1/2}\frac{1}{(n'-n)!}, \tag{93}$$

which has the property that it vanishes unless either:

(i) $n'\leq 2j$ and $n\leq 2j$,

or

(ii) $n'>2j$ and $n>2j$.

That is to say, the matrix $e^{\alpha J+}$ fragments into two blocks:

$$e^{\alpha J+}=\begin{pmatrix}A & 0\\0 & B\end{pmatrix}, \tag{94}$$

the A block being a $(2j+1)$-dimensional matrix which coincides with the representation of $e^{\alpha J+}$ in the spin-j

representation of J_+, J_-, J_0. If the infinite block B one introduces indices s, s' defined by $n=2j+1+s$ and $n'=2j+1+s'$, one observes that this matrix is identical with (93). Clearly the same thing happens for $e^{\alpha J_-}$ and hence for all $T_j(k,m)$. One may also see that the A blocks of the matrices $T_j(k,m)$ for $k=2j+1, 2j+2, \cdots$ vanish identically. Thus the A blocks of the $T_j(k,m)$ represent the algebra A_{2j} which as we have seen is the homomorphic image of $\mathfrak{L}_{j(j+1)}$ when $j=\pm\frac{1}{2}, \pm 1, \cdots$. Since the A blocks of the quantum mechanical operators $T_j(k,m)$ at $j=-\frac{3}{2}, -2, \cdots$ coincide with the matrices which we inserted for them in the calculation of the $2(-j-1)$th-order normal form we see that the approximate energy levels $E_\mu(j)$, $\mu=0, 1, 2, \cdots$, of H_j which we computed by our method are *exact* eigenvalues of H_j for $j=-(n+3)/2$, $n=0, 1, 2, \cdots$, and $\mu=0, 1, \cdots, n+1$.

The above observations may now be translated into Jost function properties by virtue of the reciprocal relationship between the Green's function and the Hamiltonian, namely, one finds that at $j=-j_0-1$ the infinite determinant (86) fragments into the product of two factors: a finite determinant which we denote ξ_{j_0} corresponding to the part of H_j which goes into the A block and an infinite determinant corresponding to the B block which in fact coincides with $f_{j_0}^{(+)}$. The finite factor ξ_{j_0} is the same for $f^{(+)}$ and $f^{(-)}$ so that

$$f_{-j_0-1}^{(\pm)}=\xi_{j_0}f_{j_0}^{(\pm)}. \tag{95}$$

(This is just the well-known Mandelstam symmetry of the S matrix: $S_{-j_0-1}=S_{j_0}$ for $j_0=0, \pm\frac{1}{2}, \pm 1, \cdots$.) When $f_{-j_0-1}^{(+)}(k)$ vanishes because of a zero of ξ_{j_0} then $f_{-j_0-1}^{(-)}(k)$ also vanishes and hence the S matrix develops no pole, becoming formally 0/0. Such zeros are therefore called "indeterminacy points" (cf. Ref. 21). Now it is easy to see that the truncated expression (91) coincides exactly with ξ_{j_0} when $l=-j_0-1$. Consequently the replacement of (90) by (91) gives an approximate Jost function with the property that it vanishes when $l=-\frac{3}{2}, -2, \cdots, -j_0-1$ at precisely the values of $k=\sqrt{E}$ at which the true Jost function develops a zero corresponding to an indeterminacy point.

Even though the indeterminancy points are not true bound states they do define points on the Regge trajectories which are defined as the zeros of $f_l(k)$. If one attempts to write dispersion relations for the Regge parameters (cf. Ref. 21), in which the unitarity condition is approximated by including only a finite number of trajectories, one finds that a finite set of indeterminacy points may be taken as the subtraction points and thus constitute the dynamical input. The approximate Jost function (91) and the approximate S matrix corresponding thereto contains and is fixed by precisely the same information. Thus we conclude that the approximate S matrices obtained by the method above described are the exact solutions of these approximate Regge dispersion relations.

We shall now take (91) as the final expression for our $2j_0$th-order approximation for obtaining the phase shifts and examine some properties of the simple Yukawa-potential S matrix as deduced from the second order ($j_0=1$) approximation. For this discussion it is convenient to take $\hbar=1$. First of all we may inquire as to the energy range for which we expect our method to yield an accurate expression for the phase shifts. Since our second-order approximation replaces (77) by

$$\frac{g}{r}\left(1-\frac{r}{r_0}+\frac{1}{2}\frac{r^2}{r_0}\right) \tag{96}$$

in the sense that higher order terms lead to zero operators in the computation of the normal form, we expect an accurate answer only if the energy $E=k^2$ is sufficiently large that $V(r)$ is negligible compared to E in the range of r values for which (96) deviates appreciably from $V(r)$. This leads to a condition

$$k^2\gg|g/r_0|. \tag{97}$$

How much greater depends on the accuracy desired but it is important to observe that (97) becomes weaker and weaker as we go to higher and higher approximations wherein more terms of $V(r)$ are included. Hence, any given energy can be discussed to any given degree of accuracy by considering a sufficiently high-order approximation. If we also restrict ourselves to "nuclear-type" potentials for which

$$|gr_0|\geq 1, \tag{98}$$

then (97) and (98) yield the condition

$$kr_0\gg 1. \tag{99}$$

In this range the inversion of (76) to obtain the quantities $\tilde{g}_{j_0,\mu}$ becomes quite simple, namely, one may assume that η is small, solve for $g_{1,\mu}$ and then show that the assumption is justified by (99) for $\mu=0, 1, 2$. The result for S waves is

$$\tilde{g}_{1,\mu}^{(+)}=-\frac{2(\mu+1)}{r_0}+2ik(\mu+1)\left(1+\frac{(\mu+1)^2}{2k^2r_0}\right)^{1/2}. \tag{100}$$

In the spirit of (99) one should expand the radical to first order. We have included the radical itself to indicate the manner in which the familiar left-hand energy cuts appear in our approximation scheme. The phase shifts may now be calculated in the $2j_0$th approximation from (91) and are given in general by

$$\delta_{j_0,l}(k)=\sum_{\mu=0}^{2j_0}\delta_{j_0,l}^{(\mu)}(k), \tag{101}$$

where

$$\delta_{j_0,l}^{(\mu)}(k)=\arg\left\{1-\frac{g}{g_{j_0,\mu}^{(+)}(l,G_0,E)}\right\}. \tag{102}$$

[21] S. C. Frautschi, P. Kaus, and F. Zachariasen, Phys. Rev. 133, B1607 (1964).

One then obtains as the second-order approximation for the S-wave phase shift for a simple Yukawa potential in the domain for which (98) and (99) are valid:

$$\delta_{1,0}(k)=\frac{11}{12}\frac{(gr_0)}{(kr_0)}\left\{1-\frac{1}{(kr_0)^2}\left[\frac{45}{11}+\tfrac{1}{2}(gr_0)+\frac{251}{4752}(gr_0)^2\right]\right\}\times o\left(\frac{1}{(kr_0)^3}\right). \quad (103)$$

In general, it will be necessary to carry out the above calculations with a computer but no computation involves anything more difficult than the solution of an algebraic or trancendental equation.

The methods described above also yield approximate expressions for the Regge parameters $\alpha(E)$ and $\beta(E)$. For example, $\alpha(E)$ is computed by solving

$$\tilde{g}_{j_0,\mu}(l,G_0,E)=g$$

for l as a function of g and E. In fact it is not necessary to first compute $\tilde{g}_{j_0,\mu}$ but suffices to solve

$$g=g_{j_0,\mu}(l,G,E)$$

for l and choose the root which goes into the known Coulomb value at $G_0=0$. Thus, for the $j_0=1$ approximation it is only necessary to solve (76) with $g_{1,\mu}$ set equal to g (which is a cubic in l) and choose the root $l=\alpha_{1,\mu}(E)$ which becomes

$$-\mu-1-\frac{g}{\sqrt{(-E)}}, \quad \mu=0, 1, \cdots,$$

when $G=0$. To find the residue parameter β it is necessary to calculate the approximate S matrix as described above and compute its residue at $l=\alpha_{1,\mu}$. Numerical results for the Yukawa case will be presented in a subsequent paper.

Thus we have obtained a closed form $2j_0$th approximation to the solution of the scattering problem which may be expected to converge as $j_0\to\infty$ to the exact solution. As a byproduct we have also discovered the group theoretic significance of the Mandelstam symmetry and indeterminacy points.[22]

It is important to point out that our $2j_0$th-order approximation really ends with (90) rather than (91). The approximation of (90) by (91) is outside of the group-theoretical framework and may only be expected to yield a convergent system of approximations under the circumstances (not yet fully understood) wherein Regge dispersion relations with the indeterminacy points used as subtractions converge to the correct answer when all indeterminacy points are included. This undoubtedly is the case if the so-called background integral can be moved to the infinitely far left-hand l plane and then thrown away. It may well be that this is always so when the potential admits a normal form to infinite order, but we have not as yet been able to establish this.

[22] In this connection, it is interesting to note a group-theoretic property of indeterminacy points which can be obtained directly from the radial Schrödinger equation: The indicial equation has the roots l, $-l-1$ for a Fuchsian singularity at the origin. It is well known that when the two roots of the indicial equation differ by an integer one solution will in general have a logarithmic singularity at the origin. That is, the group of the differential equation is infinite, i.e., we cannot find a linearly independent pair of solutions which return to themselves after a finite number of circuits about the origin of the complex r plane. When the parameters of the equation have a certain "fortuitous" values it may happen that the coefficient of the logarithm vanishes. In this case the group of the differential equation reduces to the identity. I have been able to prove that the vanishing of the coefficient of the logarithm is precisely the condition for an indeterminacy point.

XII. CONCLUSIONS

While we have succeeded in generating a purely group-theoretic technique for the approximate solution of two-body problems only, the method is suggestive of an approach to n-body problems by building the dynamical algebra out of that of an n-body system of harmonic oscillators. This will no doubt lead to structures based in higher order symplectic algebras, the occurrence of the 0_3 algebra in the two-body problem being only a manifestation of the familiar degeneracy among the low-rank algebras. The formalism will be more complicated but we see no intrinsic reason for believing that such an extension would not be possible for interactions without singular points at the origin. For singular interactions involving more than two particles it is not at all evident that one can get rid of more than one singularity by going to a conjugate problem via a canonical transformation as we did in the two-body problem and we do not at the moment see how this difficulty can be surmounted.

Finally we should like to point out that while there are strong indications of rapid convergence of the method developed (cf., the well-depth calculation for the Yukawa case) it is of utmost importance to make the conditions precise. From a mathematical point of view this entails putting some reasonable topology on the interpretation of an infinite Lie algebra as the limit of a sequence of finite Lie algebras. The theory of infinite Lie algebras is as yet in its infancy and it may well be that physics may suggest the most effective way of treating such problems.

ACKNOWLEDGMENTS

I wish to express my appreciation to Professor H. A. Bethe, Professor S. Frautschi, and Professor P. Kaus for stimulating an interest in "indeterminacy points" from which this investigation developed. I wish further to thank Professor L. Durand, III, and Professor U. Fano for some invaluable assistance with tensor techniques, and Professor A. Currier and Professor A. Kleppner for clarifying several matters concerning the structure of semisimple Lie algebras. Finally I wish to express my gratitude to the Aspen Institute for Humanistic Studies, Physics Division, whose hospitality made most of the above assistance possible.

APPENDIX A

An exact expression for the $C_{k_1m_1k_2m_2}{}^{km}(j(j+1))$ is here obtained. From the tensor property for any integer or half-integer j,

$$T_j(k_1m_2)\cdot T_j(k_2m_2)=\delta_{m,m_1+m_2}\sum_k(k_1m_1k_2m_2|\,k_1k_2km)\times[T_j(k_1m_1)\otimes T_j(k_2m_2)]^{(k)}{}_m,$$

where the tensor product is given by (cf. A. R. Edmonds, Ref. 12)

$$[T_j(k_1m_1)\times T_j(k_2m_2)]^{(k)}{}_m=\frac{(j\|[T_j(k_1)\otimes T_j(k_2)]^{(k)}\|j)}{(j\|T_j(k)\|j)}T_j(k,m)\delta_{m,m_1+m_2},$$

with the reduced matrix elements

$$(j\|[T_j(k_1)\times T_j(k_2)]^{(k)}\|j)=(-1)^k(2k+1)^{1/2}\times(j\|T_j(k_1)\|j)(j\|T_j(k_2)\|j)\begin{Bmatrix}k_1&k_2&k\\j&j&j\end{Bmatrix},$$

$$(-1)^k(j\|T_j(k)\|j)=\left[\frac{k!k!(2j+k+1)!}{(2k)!(2j-k)!}\right]^{1/2}\equiv(2k+1)^{1/2}\rho_{jk}.$$

Interchanging k_1 and k_2 one obtains the same expression except for a factor $(-1)^{k_1+k_2-k}$. Hence, from the commutator:

$$C_{k_1m_1k_2m_2}{}^{km}(j(j+1))=\delta_{m,m_1+m_2}[1-(1)^{k_1+k_2-k}]\times\left(\frac{\rho_{jk_1}\rho_{jk_2}}{\rho_{jk}}\right)((2k_1+1)(2k_2+1))^{1/2}\times(k_1m_1k_2m_2|\,k_1k_2km)\times(-1)^{k_1+k_2}\begin{Bmatrix}k_1&k_2&k\\j&j&j\end{Bmatrix}.$$

From the familiar expression for the $6-j$ symbol one may readily see that this is indeed a polynomial in $j(j+1)$. Hence, the above expression is valid for all complex j.

APPENDIX B

We may rewrite (20) as

$$\hbar\{2(2+\alpha_1)J_0+\alpha_1(J_--J_+)\}. \tag{B1}$$

For any γ we have the identity

$$e^{\gamma(J_++J_-)}J_0e^{-\gamma(J_++J_-)}=J_0\cosh(2\gamma)+\tfrac{1}{2}(J_--J_+)\sinh(2\gamma). \tag{B2}$$

Let

$$\tanh(2\gamma)=\alpha_1/(2+\alpha_1), \tag{B3}$$

so that

$$e^{-\gamma(J_++J_-)}\hbar\{2(2+\alpha_1)J_0+\alpha_1(J_--J_+)\}e^{\gamma(J_++J_-)}=2\hbar(1+\alpha_1)^{1/2}J_0=\tfrac{1}{2}\hbar(1+\alpha_1)^{1/2}H_0. \tag{B4}$$

Thus if $\alpha_1\neq-1$, a finite γ exists satisfying (B3) such that (B4) is a canonical transformation of (B1) into oscillator form.

APPENDIX C

Let

$$(n,s,k,m)\equiv\hbar^{n+k+2s-1}(j(j+1))^sT_j(k,m),\quad k=1,2,\cdots;\ n,\ s=0,1,2,\cdots. \tag{C1}$$

Then $[(n_1,s_1,k_1,m_1),(n_2,s_2,k_2,m_2)]$ is a linear combination of (n,s,k,m)'s with coefficients independent of $\hbar$ and j and with

$$n+k+2s-1=(n_1+k_1+2s_1-1)+(n_2+k_2+2s_2-1). \tag{C2}$$

Moreover,

$$[(n,s,k,m,T_j(1,0)]=\beta_{km}(n,s,k,m),$$

where β_{km} is independent of $\hbar$ and j and $\beta_{km}\neq0$ unless $m=0$. Hence, for $m\neq0$,

$$\exp\left\{-\frac{\alpha}{\beta_{km}}(n,s,k,m)\right\}T_j(1,0)\exp\left\{\frac{\alpha}{\beta_{km}}(n,s,k,m)\right\}=T_j(1,0)-\alpha(n,s,k,m). \tag{C3}$$

Now any regular H_j can (cf. Appendix A) be brought into the form

$$\hbar T_j(1,0)+\sum_{k=2}^{\infty}\sum_{m=-k}^{k}\alpha_{km}\hbar^kT_j(k,m)$$

by a preliminary canonical transformation. Hence by successive applications of inner automorphisms of the form

$$\exp\left\{-\frac{\alpha_{km}}{\beta_{km}}(n,s,k,m)\right\}$$

to H_j we can in virtue of (C2) and (C3) bring H_j into the form

$$\sum_{n,k,s}\gamma_{nks}\hbar^{n+k+2s}[j(j+1)]^sT_j(k,0)+o(\hbar^N)$$

for any given $N=1,2,\cdots$ where γ_{nks} are coefficients independent of $\hbar$ and j.