High-Energy Deuteron Cross Sections*

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A theoretical analysis of the collisions of particles with deuterons is carried out in the high-energy approximation. This approximation, which corresponds to a generalized form of diffraction theory, takes explicit account of double collision processes as well as single ones. It is used to express the amplitudes for elastic and inelastic scattering by deuterons in terms of the elastic-scattering amplitudes of the neutron and proton and the deuteron wave functions. The resulting expressions are used to evaluate the differential cross section for elastic scattering, and the angular distribution of inelastic scattering (i.e., the differential cross section for deuteron breakup integrated over final energies of the incident particle). The contributions to these cross sections of the various single and double scattering processes and the terms which represent their interference are exhibited individually. Expressions are derived for the total cross section of the deuteron and for its elastic and inelastic total scattering and absorption cross sections. The difference between the various types of deuteron cross sections and the sum of the corresponding cross sections for the free neutron and proton is explained in some detail. Spin-dependent interactions are treated, and for incident particles of spin $\frac{1}{2}$ an expression is given for the deuteron total cross section in terms of the general spin-dependent scattering amplitudes of the neutron and proton. The theory is applied to antiproton-deuteron collisions in the energy range from 0.13 to 17.1 BeV. The results for the total and absorption cross sections which are calculated for a variety of models of the deuteron wave function are found to be in good agreement with the measurements. The magnitudes of such effects as double scattering and the interference of single- and double-scattering amplitudes are seen to be appreciable.

I. INTRODUCTION

EXPERIMENTAL studies have been made in recent years of the interactions of a wide variety of high-energy particle beams with deuterium targets. Several of these studies have been directed toward determining the internal structure of the deuteron. More frequently, however, deuterons have been used as collision targets in the hope of estimating, by a simple subtraction procedure, the cross sections of stationary neutron targets. This method has been based on the presumption that at sufficiently high energies, i.e., when the incident-particle wavelength is much smaller than the deuteron radius, any deuteron cross section should be approximately the sum of the corresponding free-neutron and free-proton cross sections. If that is so, then once the deuteron and proton cross sections are measured, the neutron cross section is given by their difference.

It has become evident, however, through many experiments that the deuteron cross section may differ quite substantially, even at the highest available energies from the sum of the free-neutron and proton cross sections. An elementary discussion of the origin of this lack of additivity has been given by Glauber[1,2] who made theoretical estimates of its magnitude for the case of incident nucleons and π mesons. In subsequent experimental work much the same effect has been observed with incident beams of antinucleons[3–5] and K mesons.[5,6] In particular the deviation from additivity of the cross sections for the case of incident antinucleons has been found in some measurements[3,4] to amount to 20 to 40% of the individual antinucleon-nucleon cross sections. Since a detailed understanding of this correction is evidently of some importance in establishing fundamental cross sections we shall try to improve the earlier theoretical analysis in a number of respects and to extend its domain of applicability. At the same time we shall take the occasion to study in detail the various types of collision processes in which the deuteron may participate.[7] In particular we shall separate the differential cross sections for elastic and inelastic scattering into contributions coming from single and from double scattering and their various interference terms. We then illustrate the magnitudes and angular dependences of these cross sections by calculating them explicitly for the case of antiproton-deuteron collisions.

In each of the cases which has been studied experi-

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mentally at sufficiently high energy, the sum of the free neutron and proton cross sections has been found to exceed the deuteron cross section. If we call this difference the cross-section defect, then our first task is to explain why the cross-section defect is so consistently of positive sign at high energies. Part of the answer is immediately evident when the incident particle is capable of producing other particles on colliding with either the neutron or proton. Particle-production processes may be considered as absorption processes from the standpoint of their effect on the incident wave. Thus, when particle production occurs the neutron and proton in the deuteron cast individual shadows. When either particle lies in the shadow cast by the other it absorbs less effectively than when outside it. Hence the absorption cross section of the deuteron is smaller on the average than the sum of the absorption cross sections of the free neutron and proton.

The shadowing effect, however, as was noted in Refs. 1 and 2, is only one of several contributions to the cross-section defect. Other contributions of comparable magnitude come from double-interaction effects such as double scattering[8] and scattering by one nucleon followed by absorption by the other. Also present in the cross-section defect are the effects of interference of the amplitudes for the various single- and double-scattering processes which can occur. Although the contributing terms are many in number, their sum is easily shown by means of the optical theorem to be expressible in a fairly compact and simple formula for the cross-section defect. This expression was used as the basis of the analysis in Ref. 1 which mentioned the various contributing effects but did not treat them individually in any detail. One of the purposes of the present paper, in addition to presenting a more general analysis of the total cross section of the deuteron, is to discuss the relative magnitudes of these individual contributions to it.[7]

A good deal of insight into the behavior of the cross-section defect may be obtained, according to the analysis of Ref. 1, if we assume that the ranges of interactions of the incident particle with the neutron and the proton are considerably smaller than the radius of the deuteron. In that case the optical theorem shows that the cross-section defect is proportional to the real part of the product $-f_n(0)f_p(0)$ where $f_n(0)$ and $f_p(0)$ are the amplitudes for forward scattering of the incident particle by the neutron and proton, respectively. Since the imaginary parts of the forward-scattering amplitudes are positive and tend to exceed the magnitudes of the real parts at high energies we see immediately that the cross-section defect will tend consistently to be positive. That the cross-section defect is not due solely to the shadowing effect mentioned earlier may be seen from its presence at energies low enough to lie beneath the threshold for absorption, i.e., for particle production. At sufficiently low energies, however, there is no reason to expect the contribution of the imaginary parts of the scattering amplitudes to be dominant, and the cross-section defect may have either sign.

While the assumption that the interaction ranges are small compared with the size of the deuteron leads to several useful insights, it was noted in Ref. 1 that more accurate approximations must be used for numerical purposes. The double-interaction effects which contribute to the cross-section defect are all most intense when the neutron and proton are close together in the deuteron. The heavy weighting which is thus given to the smaller separations means that it is quite necessary, for quantitative purposes, to treat the interaction ranges as comparable in size to the average distance between the neutron and proton. This was done in Ref. 1 by assuming the regions of interaction surrounding the neutron and proton to be purely absorbing spheres with radii adjusted to fit known total cross sections. Since we often possess, at present, more detailed information based on measurements of differential cross sections, it is desirable to reformulate the analysis in terms which may be applied more accurately and more generally. In doing this we shall forego attempts to describe the interactions themselves in any detail. Instead we shall express the various contributions to deuteron cross sections directly in terms of the amplitudes for scattering by the neutron and proton and certain integrals of their products. The means by which we do this are provided by an approximate form of high-energy collision theory[2] which is related to optical-diffraction theory. Several aspects of the scattering of high-energy particles by deuterons have been discussed by means of this approximation by Harrington.[9] The theory of deuteron stripping at high energies may be developed in much the same terms. Several of the cross sections for stripping reactions may be found from the results of the present paper by simply transforming them to the rest system of the incident particle.[7,10]

The analysis is begun by presenting some of the necessary elements of the high-energy approximation in Sec. II. The various cross sections of the deuteron are then calculated in Sec. III under the simplifying assumption that spin-dependent interactions may be neglected. The evaluation of these cross sections is then discussed in the next section and numerical results are presented for the case of antiproton-deuteron scattering. Finally, the effects of spin dependence of the interactions are discussed in Sec. V.

[8] Since high-energy scattering takes place predominantly at extremely small angles, double scattering may occur with appreciably intensity within the deuteron. Triple and higher order multiple-scattering processes, however, must take place via one or more backward scatterings and hence tend to have negligibly small amplitudes.

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II. THE HIGH-ENERGY APPROXIMATION

We shall assume, as our definition of the high-energy domain, that the wavelength of the incident particle is much smaller than the ranges of its interaction with nucleons. When that condition holds, the scattering phase shifts corresponding to relatively large values of the orbital angular momentum take on values different from zero. The many partial waves which are thus scattered typically lead, through their interference, to angular distributions which are sharply peaked near the forward direction. We shall take advantage of these features of high-energy collisions in the work that follows by using an approximate form of collision theory which is asymptotically correct for high-energy scattering at small angles. It is the use of this approximation which makes it possible to calculate the collision cross sections for a two-particle target such as the deuteron in a reasonably simple and accurate way.

To illustrate the approximations used, let us assume that the interaction between the incident particle and single nucleon is spin-independent and spherically symmetric. Then the elastic-scattering amplitude $f(\theta)$ for this two-particle system may be written in the high-energy approximation[11] as

$$f(\theta)=ik\int_0^\infty J_0(2kb\sin\tfrac{1}{2}\theta)[1-e^{i\chi(b)}]b\,db, \quad (2.1)$$

where θ is the angle of scattering, k is the wave number of the incident particle and the variable of integration b corresponds to the impact parameter of the collision. The integration over the latter variable is simply an asymptotic representation of the sum over the angular momenta $l=kb-\frac{1}{2}$ of the partial waves which contribute. The Bessel function J_0 is an approximate form for the Legendre polynomial appropriate to small scattering angles. The function $\chi(b)$ which occurs in Eq. (2.1) is a complex phase shift which is related to the more familiar phase shifts δ_l through the definition

$$\chi[k^{-1}(l+\tfrac{1}{2})]=2\delta_l.$$

When a particle is incident upon a nucleus containing more than one nucleon the interaction which it encounters does not in general have spherical symmetry. The instantaneous configuration of the nucleons in the deuteron will not even have azimuthal symmetry in general about the direction of the incident momentum. For that reason we must use a more general form for the elastic-scattering amplitude than is given by Eq. (2.1). We require an expression for the amplitude of the scattering produced by an interaction of arbitrary shape. It is shown in Appendix A that a general expression for this amplitude, asymptotically correct for high-energy scattering near the forward direction, is

$$f(\mathbf{k}',\mathbf{k})=\frac{ik}{2\pi}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}] \times\{1-\exp[i\chi(\mathbf{b})]\}d^{(2)}\mathbf{b} \quad (2.2)$$

which we shall often abbreviate as

$$f(\mathbf{k}',\mathbf{k})=\frac{ik}{2\pi}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\Gamma(\mathbf{b})d^{(2)}\mathbf{b}, \quad (2.3)$$

where

$$\Gamma(\mathbf{b})=1-\exp[i\chi(\mathbf{b})]. \quad (2.4)$$

In these expressions $\chi(\mathbf{b})$ is a phase shift associated with a particular impact-parameter vector $\mathbf{b}$ which is perpendicular to the direction of the incident beam. The two-dimensional integration is performed over the plane of impact-parameter vectors.

The expression (2.2) is obtained by first writing the elastic-scattering amplitude in a series of spherical harmonics and then expressing the associated Legendre functions for small angles in terms of Bessel functions. The resulting summation over angular momenta is then transformed to an integration over impact parameters. It is easy to show that this general result reduces to Eq. (2.1) if the interaction has azimuthal symmetry.

The formulas we have written for the scattering amplitude are of the correct form for describing the collision of the incident and target particles in their center-of-mass system. In order to compare deuteron cross sections with those of free nucleons by calculating scattering amplitudes in center-of-mass systems, however, we should have to make use of at least two such systems. It is considerably simpler instead to refer all calculations of the scattering amplitudes to the laboratory system. We demonstrate in Appendix B that the expressions (2.2) and (2.3) for the scattering amplitude in fact undergo very little change of form when they are transformed to the laboratory system. Because of the simple geometry and small recoil effects which are associated with nearly forward scattering, the scattering amplitudes in the laboratory system may be found from the expressions (2.2) and (2.3) simply by substituting in them the laboratory values of $\mathbf{k}$ and $\mathbf{k}'$, or of the incident momentum and scattering angle.

It is characteristic of collisions as seen in the laboratory system that the incident particle transfers a certain recoil energy to the target particle and that the magnitude of $\mathbf{k}'$ is thus smaller than that of $\mathbf{k}$. This energy transfer, which is proportional to the square of the momentum transfer, is typically quite small for scattering near the forward direction. No appreciable error is introduced, therefore, by neglecting the energy transfer altogether and evaluating the scattering amplitudes for $k'=k$.

It is worth noting that the derivation of the formulas (2.1) and (2.2) for the scattering amplitude does not assume the existence of a potential function to describe

[11] See Ref. 2, p. 345.

the interaction of the incident and target particles. It is possible to show, however, that a complex potential[2] may always be found which describes high-energy diffraction scattering. The relationship between a potential function V and the phase-shift function χ is given in the high-energy approximation by

$$\chi(\mathbf{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V(\mathbf{b}+\mathbf{z})dz\,, \tag{2.5}$$

where v is the relative velocity of the incident and target particles and $\mathbf{z}$ is a vector parallel to the incident momentum.

We next consider the more realistic problem of scattering by a system which possesses internal degrees of freedom such as the deuteron (or more generally, any A-particle nucleus). The detailed justification of the use of the high-energy approximation in treating problems of this type has been given by Glauber.[12] For our present purposes we shall use a simplified form of this approximation which takes advantage of the fact that the motion of a nucleon which is part of a scattering nucleus is characteristically rather slow in comparison to that of a high-energy incident particle. The approximation is made therefore that the scattering nucleons are frozen in their instantaneous positions during the passage of the incident particle through the nucleus. The detailed analysis shows that this approximation amounts to neglecting the energy communicated to the target nucleons by the incident particle. Since we are discussing only collision processes in which the momentum transfer is small, the energy taken up by the nucleus will be small in comparison to the incident particle energy, and neglecting it leads quantitatively to little error.

Let us imagine the coordinates of the target nucleons for the moment to have the fixed values $\mathbf{r}_1, \cdots, \mathbf{r}_A$. Then the wave which represents the incident particle, when it passes through this system, will accumulate a total phase shift which depends on the coordinates $\mathbf{r}_1, \cdots, \mathbf{r}_A$ as well as on the impact parameter vector $\mathbf{b}$. If we write this phase-shift function as $\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ and introduce the function

$$\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A) = 1-\exp[i\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)]\,, \tag{2.6}$$

then we see from Eq. (2.3) that the scattering amplitude for the fixed configuration of the nucleons would be

$$\frac{ik}{2\pi}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)d^{(2)}\mathbf{b}\,. \tag{2.7}$$

We may take account of the fact that the nucleons are not rigidly fixed in the positions $\mathbf{r}_1, \cdots, \mathbf{r}_A$ by noting that $\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ can be regarded as an operator which induces changes in the state of the nucleus through its dependence on the nucleon coordinates, much as it changes the momentum state of the incident particle through its dependence on the coordinate $\mathbf{b}$. The scattering amplitude then for a particular nuclear transition is simply the matrix element of the expression (2.7) taken between the appropriate nuclear states. The amplitude for a collision process in which the incident particle suffers a deflection from momentum $\hbar\mathbf{k}$ to $\hbar\mathbf{k}'$ while the target nucleus makes a transition from the initial state $|i\rangle$ to the final state $|f\rangle$ may therefore be written as

$$F_{fi}(\mathbf{k}',\mathbf{k}) = \frac{ik}{2\pi}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}] \times\langle f|\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)|i\rangle d^{(2)}\mathbf{b}\,. \tag{2.8}$$

The initial state $|i\rangle$ will ordinarily be a nuclear ground state, but the final state $|f\rangle$ may be the ground state for the case of elastic scattering or any excited state including those with unbound nucleons when the scattering is inelastic.

We have noted earlier that the recoil of the target nucleus leads to negligible energy transfer for nearly forward scattering. We therefore ignore the recoil by assuming that the center of mass of the target nucleus remains fixed at the origin. Then if we introduce the configuration space wave functions ψ_i and ψ_f for the initial and final nuclear states, we may write the scattering amplitude (2.8) as

$$F_{fi}(\mathbf{k}',\mathbf{k}) = \frac{ik}{2\pi}\int \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]d^{(2)}\mathbf{b} \times\int \psi_f^*(\mathbf{r}_1,\cdots,\mathbf{r}_A)\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A) \times\psi_i(\mathbf{r}_1,\cdots,\mathbf{r}_A)\prod_j d\mathbf{r}_j\,. \tag{2.9}$$

The $\mathbf{r}_j$ integrations are carried out over $A-1$ nucleon positions which may be chosen independently relative to the nuclear center of mass. The states ψ_i and ψ_f may depend on the nucleon spins and the function Γ_{tot} may be spin-dependent as well. Until spin dependences are examined at a later point, however, we shall not introduce any explicit notation for them.

If the incident particle interacts with the nucleons through two-body interactions then its total phase shift $\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)$ will be the sum of the phase-shift functions produced by each of the nucleons considered individually. If the components of the coordinates $\mathbf{r}_1, \cdots, \mathbf{r}_A$ perpendicular to the direction of incidence (i.e., parallel to the plane containing the impact parameter vectors $\mathbf{b}$) are $\mathbf{s}_1, \cdots, \mathbf{s}_A$, then we may write the total phase shift as a sum of the form

$$\chi_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A) = \sum_{j=1}^{A}\chi_j(\mathbf{b}-\mathbf{s}_j)\,. \tag{2.10}$$

[12] See Ref. 2, p. 372.

The function Γ_{tot} then becomes

$$\Gamma_{\text{tot}}(\mathbf{b},\mathbf{r}_1,\cdots,\mathbf{r}_A)=1-\exp[i\sum_{j=1}^{A}\chi_j(\mathbf{b}-\mathbf{s}_j)]. \quad (2.11)$$

When the scattering amplitude obtained by substituting this expression into Eq. (2.9) is analyzed in detail it may be shown to take implicit account of all the significant ways in which the incident particle can be multiply scattered by the target nucleons. (The correct treatment of multiple scattering is a consequence of the fact that phase shifts rather than scattering amplitudes are summed.) In treating collisions with deuteron targets we shall resolve the expression (2.9) into a sum of terms which exhibit single- and double-scattering processes explicitly.

III. SCATTERING BY DEUTERONS

We shall now apply the expressions obtained in the preceding section to the case of collisions of arbitrary particles with deuterons. To afford the most direct insight we shall simplify the calculations for the present by neglecting any spin dependence of the interaction between the incident particle and the target nucleons. Cases in which the interactions are spin-dependent are treated in Sec. V.

We let $\mathbf{r}_n$ and $\mathbf{r}_p$ be the coordinates of the neutron and proton in the deuteron so that the internal coordinate is $\mathbf{r}=\mathbf{r}_p-\mathbf{r}_n$. We take $\phi_i(\mathbf{r})$ to be the internal ground-state wave function of the deuteron and $\phi_f(\mathbf{r})$ its internal final-state wave function. The function $\phi_f(\mathbf{r})$ may represent either an excited state (i.e., an unbound two-particle state) or, in the case of elastic scattering, the ground state once again. If we use the approximation described in the preceding section, we may write the amplitude for the process in which the deuteron is left in a final state $|f\rangle$ and the incident particle transfers momentum $\hbar\mathbf{q}=\hbar(\mathbf{k}-\mathbf{k}')$ to the deuteron as

$$F_{fi}(\mathbf{q})=\frac{ik}{2\pi}\int \exp[i\mathbf{q}\cdot\mathbf{b}]\, d^{(2)}\mathbf{b} \times\int \phi_f^*(\mathbf{r})\{1-\exp[i\chi_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s})+i\chi_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s})]\} \times\phi_i(\mathbf{r})\, d\mathbf{r}. \quad (3.1)$$

In this relation $\mathbf{s}$ is the projection of $\mathbf{r}$ on the plane perpendicular to the direction of the incident beam and $\chi_n(\mathbf{b}-\frac{1}{2}\mathbf{s})$ and $\chi_p(\mathbf{b}+\frac{1}{2}\mathbf{s})$ are the phase shifts produced by the neutron and proton in their instantaneous positions. Since k, the magnitude of the propagation vector, remains constant in the collision, we have omitted explicit reference to it in writing the scattering amplitude as $F_{fi}(\mathbf{q})$.

As a first step in separating the contributions of the individual nucleons to the scattering, we introduce the functions Γ_n and Γ_p for the neutron and proton, respectively. These are defined in terms of χ_n and χ_p by means of Eq. (2.4) and therefore obey the identity

$$1-\exp[i\chi_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s})+i\chi_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s})] =\Gamma_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s})+\Gamma_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s})-\Gamma_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s})\Gamma_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s}). \quad (3.2)$$

Since the amplitudes for scattering of the incident particle by the nucleons can be determined from experiment more readily than can the functions Γ_n and Γ_p, there will be considerable advantage in expressing the latter functions in terms of the scattering amplitudes. This step, as we shall see, will make it possible, within the approximations stated, to express all of the cross sections of the deuteron in terms of the neutron and proton elastic-scattering amplitudes f_n and f_p.

In this connection we note from Eq. (2.3) that the nucleon elastic-scattering amplitude is a Fourier transform of Γ. An approximate inversion of the transform may be achieved by multiplying Eq. (2.3) by $\exp(-i\mathbf{q}\cdot\mathbf{b}')$ and integrating the variable $\mathbf{q}$ over a plane perpendicular to the direction of incidence. In this way we find

$$\Gamma(\mathbf{b})=\frac{1}{2\pi ik}\int \exp(-i\mathbf{q}\cdot\mathbf{b})f(\mathbf{q})\, d^{(2)}\mathbf{q}. \quad (3.3)$$

The accuracy of this expression for $\Gamma(\mathbf{b})$ will of course depend on the extent to which our underlying assumptions are fulfilled. If the scattering is indeed highly concentrated near the forward direction, the contributions to the integral (3.3) will come from momentum transfers $\hbar\mathbf{q}$ which are quite small. In that case the difference between integrating $\mathbf{q}$ over a plane and integrating it over the sphere, which more correctly represents the locus of momentum transfers $\hbar\mathbf{q}$ for fixed energy, becomes negligibly small.

We shall denote the matrix element of any operator, $G(\mathbf{s})$, with respect to the internal coordinate by

$$\langle f|G(\mathbf{s})|i\rangle\equiv\int\phi_f^*(\mathbf{r})G(\mathbf{s})\phi_i(\mathbf{r})d\mathbf{r}.$$

If we now substitute the identity (3.2) into the integral (3.1) and shift the origin in the $\mathbf{b}$ plane in the first two resulting integrals we obtain

$$F_{fi}(\mathbf{q})=\langle f|\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_n(\mathbf{b})d^{(2)}\mathbf{b} +\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_p(\mathbf{b})d^{(2)}\mathbf{b} -\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s})\Gamma_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s})d^{(2)}\mathbf{b}|i\rangle. \quad (3.4)$$

The first two integrals may be easily expressed in terms of the neutron and proton elastic-scattering amplitudes by means of Eq. (2.3). By expressing Γ_n and Γ_p

in the third integral in terms of f_n and f_p and making use of the Fourier integral representation for the two-dimensional delta function, we find a result which may be written as

$$F_{fi}(\mathbf{q})=\langle f|F(\mathbf{q},\mathbf{s})|i\rangle, \tag{3.5}$$

where

$$F(\mathbf{q},\mathbf{s})=\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})f_n(\mathbf{q})+\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})f_p(\mathbf{q})+\frac{i}{2\pi k}\int\exp(i\mathbf{q}'\cdot\mathbf{s})f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\times f_p(-\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\,d^{(2)}\mathbf{q}'. \tag{3.6}$$

The effects of single and double scattering have been separated in this expression. The first two terms in Eq. (3.6) are the single-scattering amplitudes contributed by the neutron and proton, respectively. The third term is an approximate form of the double-scattering amplitude, i.e., it represents processes in which the incident particle collides with both the neutron and proton. The absence of triple and higher order multiple scattering terms in the expansion is due to our implicit assumption that large-angle scattering is negligible. When scattering is confined to small angles the geometry of a two-particle scattering system only allows single and double scattering. Double scattering, in particular, tends to take place only when the position vector of either scattering particle relative to the other is nearly parallel to the incident propagation vector **k**.

To evaluate the elastic-scattering amplitude we must make use of integrals having the form

$$\int\exp(i\mathbf{q}\cdot\mathbf{s})|\phi(\mathbf{r})|^2\,d\mathbf{r}.$$

Since **s** is the component of the coordinate **r** lying parallel to the plane which contains the momentum transfers **q**, this integral is equivalent to the expression

$$S(\mathbf{q})=\int\exp(i\mathbf{q}\cdot\mathbf{r})|\phi(\mathbf{r})|^2\,d\mathbf{r} \tag{3.7}$$

which we recognize to be the form factor of the deuteron ground state. If we make use of this expression for the form factor the elastic-scattering amplitude may be written as

$$F_{ii}(\mathbf{q})=S(\tfrac{1}{2}\mathbf{q})f_n(\mathbf{q})+S(-\tfrac{1}{2}\mathbf{q})f_p(\mathbf{q})+\frac{i}{2\pi k}\int S(\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\,d^{(2)}\mathbf{q}'. \tag{3.8}$$

The optical theorem, which relates the total cross section to the imaginary part of the amplitude for elastic scattering in the forward direction, allows us to write the deuteron total cross section as

$$\sigma_d=\frac{4\pi}{k}\,\mathrm{Im}F_{ii}(0). \tag{3.9}$$

Similarly if we let the subscripts n and p refer to the neutron and proton, respectively, then the free-nucleon cross sections may be written as

$$\sigma_j=\frac{4\pi}{k}\,\mathrm{Im}f_j(0),\quad j=n,p. \tag{3.10}$$

If we note that the form factor of the deuteron is unity for the forward direction $\mathbf{q}=0$, and make use of the imaginary part of Eq. (3.8) with $\mathbf{q}=0$, we see that the total cross section of the deuteron may be written in the form

$$\sigma_d=\sigma_n+\sigma_p-\delta\sigma \tag{3.11}$$

with the cross-section defect $\delta\sigma$ given by

$$\delta\sigma=-\frac{2}{k^2}\int S(\mathbf{q})\,\mathrm{Re}[f_n(\mathbf{q})f_p(-\mathbf{q})]d^{(2)}\mathbf{q}. \tag{3.12}$$

The deuteron cross section is thus expressed as the sum of the free-nucleon cross sections plus a correction term dependent upon the nucleon elastic-scattering amplitudes and the deuteron form factor. Included in this sum are the contributions of the full variety of effects which are due to the simultaneous presence of the two target particles. We shall discuss some of these effects shortly.

The cross-section defect may also be written as

$$\delta\sigma=\frac{2}{k^2}\int S(\mathbf{q})[\mathrm{Im}f_n(\mathbf{q})\,\mathrm{Im}f_p(-\mathbf{q})-\mathrm{Re}f_n(\mathbf{q})\,\mathrm{Re}f_p(-\mathbf{q})]d^{(2)}\mathbf{q}. \tag{3.13}$$

In this form we see that it is positive when the scattering amplitudes are predominantly imaginary near the forward direction, as indeed they appear to be quite generally at high energies. However, we may expect that at lower energies, where the real parts of the forward amplitudes can be relatively large, $\delta\sigma$ can become negative, i.e., the deuteron cross section can exceed the sum of the free-particle cross sections.

It is of interest to investigate the effects of the simultaneous presence of two target nucleons on the angular distributions which may be found for the scattered particles. Let us consider first the total scattered intensity, $(d\sigma/d\Omega)_{\mathrm{sc}}$, which is obtained by summing the squared modulus of the amplitude in Eq. (3.5) over a complete set of final deuteron states, $|f\rangle$. We then have a differential cross section for the sum of elastic plus inelastic scattering which we may write as

$$(d\sigma/d\Omega)_{\mathrm{sc}}=\sum_f|F_{fi}(\mathbf{q})|^2. \tag{3.14}$$

Since we have neglected the energy differences of the various final states of the deuteron we may use the completeness relation

$$\sum_f\phi_f^*(\mathbf{r})\phi_f(\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$$

in summing over all final states in Eq. (3.14). We may then make use of the amplitude $F(\mathbf{q},\mathbf{s})$ defined by Eq. (3.6) to write the summed differential cross section as

$$(d\sigma/d\Omega)_{sc}=\langle i|\,|F(\mathbf{q},\mathbf{s})|^2|i\rangle \tag{3.15}$$

$$=|f_n(\mathbf{q})|^2+|f_p(\mathbf{q})|^2+2S(\mathbf{q})\,\mathrm{Re}[f_n(\mathbf{q})f_p^*(\mathbf{q})]-\frac{1}{\pi k}\mathrm{Im}f_n^*(\mathbf{q})\int S(\mathbf{q}'-\tfrac{1}{2}\mathbf{q})f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}'$$
$$-\frac{1}{\pi k}\mathrm{Im}f_p^*(\mathbf{q})\int S(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')d^{(2)}\mathbf{q}'$$
$$+\frac{1}{(2\pi k)^2}\int d\mathbf{r}|\phi(\mathbf{r})|^2\left|\int d^{(2)}\mathbf{q}'e^{i\mathbf{q}'\cdot\mathbf{s}}f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\right|^2. \tag{3.16}$$

Each term in this expression has a simple physical interpretation. The first and second terms are just the intensities for scattering by a free neutron and a free proton, respectively, and are assumed to be peaked in the forward direction. The third term results from the interference of the two wave amplitudes scattered by the neutron and by the proton. We may expect it to be even more sharply peaked than the first two terms because the deuteron form factor $S(\mathbf{q})$ is itself sharply peaked at $q=0$. If double-scattering effects were neglected (as they have usually been in applications of the impulse approximation), the total scattered intensity would consist of only these three terms. The fourth term corresponds to the interference between the double-scattering amplitude and the neutron single-scattering amplitude, and the fifth term has a similar interpretation with the neutron single-scattering amplitude replaced by that of the proton. Since the form factor is sharply peaked in the forward direction, the major contribution to the integral in the fourth term will occur for $\mathbf{q}'$ close to $\frac{1}{2}\mathbf{q}$. This indicates that the neutron single-scattering amplitude interferes appreciably with the double-scattering amplitude only if in the double-scattering process the scattering by the neutron occurs with a momentum transfer very close to $\mathbf{q}$ and the additional scattering by the proton occurs with nearly zero momentum transfer. A similar argument holds for the fifth term. The last term is the intensity for pure double scattering and is simply the average, taken over orientations of the deuteron, of the squared modulus of that part of $F(\mathbf{q},\mathbf{s})$ due to double scattering.

Another angular distribution of interest is the elastically scattered intensity, $(d\sigma/d\Omega)_{el}$, which is obtained by squaring the modulus of the diagonal element of $F(\mathbf{q},\mathbf{s})$ in the deuteron ground state. This matrix element is given in Eq. (3.8) and yields

$$(d\sigma/d\Omega)_{el}=|F_{ii}(\mathbf{q})|^2 \tag{3.17}$$

$$=S^2(\tfrac{1}{2}\mathbf{q})\{|f_n(\mathbf{q})|^2+|f_p(\mathbf{q})|^2+2\,\mathrm{Re}[f_n(\mathbf{q})f_p^*(\mathbf{q})]\}$$
$$-\frac{1}{\pi k}S(\tfrac{1}{2}\mathbf{q})\,\mathrm{Im}\left\{[f_n^*(\mathbf{q})+f_p^*(\mathbf{q})]\int S(\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\,d^{(2)}\mathbf{q}'\right\}$$
$$+\frac{1}{(2\pi k)^2}\left|\int S(\mathbf{q}')f_n(\tfrac{1}{2}\mathbf{q}+\mathbf{q}')f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\,d^{(2)}\mathbf{q}'\right|^2. \tag{3.18}$$

The inelastic angular distribution, $(d\sigma/d\Omega)_{inel}$, is simply the difference between the total and elastically scattered intensities

$$(d\sigma/d\Omega)_{inel}=(d\sigma/d\Omega)_{sc}-(d\sigma/d\Omega)_{el}. \tag{3.19}$$

The scattering processes which are regarded as inelastic in the present context are those which excite the deuteron and thereby dissociate it into two free nucleons.

The integrated scattering cross section, σ_{sc}, is defined as

$$\sigma_{sc}=\int\left(\frac{d\sigma}{d\Omega}\right)_{sc}d\Omega, \tag{3.20}$$

where the solid-angle integration is carried out over all directions of $\mathbf{k}'$. Since the scattering is predominantly close to the forward direction, the integration over the surface of the sphere $|\mathbf{k}'|=k$ may be approximated by an integration over the plane, in momentum space, which is tangent to the sphere at the forward direction, $\mathbf{k}'=\mathbf{k}$. The solid angle $d\Omega$ may therefore be represented approximately by $d^{(2)}\mathbf{k}'/k^2$ where $d^{(2)}\mathbf{k}'$ lies in the tangent plane mentioned. Since the angular distribution is a function of $\mathbf{q}=\mathbf{k}-\mathbf{k}'$ and $\mathbf{k}$ is fixed we may equivalently replace $d\Omega$ by $d^{(2)}\mathbf{q}/k^2$. With this approximation we carry out the integration in Eq. (3.20) and find that the total scattering cross section is

$$\sigma_{\text{sc}}=\int\left(\frac{d\sigma}{d\Omega}\right)_{\text{sc}}\frac{d^{(2)}\mathbf{q}}{k^2} \tag{3.21}$$

$$=\sigma_{n\,\text{sc}}+\sigma_{p\,\text{sc}}+\frac{2}{k^2}\int S(\mathbf{q})\,\text{Re}[f_n(\mathbf{q})f_p^*(\mathbf{q})]\,d^{(2)}\mathbf{q}$$

$$-\frac{1}{\pi k^3}\int[S(\mathbf{q}'-\tfrac{1}{2}\mathbf{q})\,\text{Im}f_n^*(\mathbf{q})f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')+S(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\,\text{Im}f_p^*(\mathbf{q})f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')]\,d^{(2)}\mathbf{q}\,d^{(2)}\mathbf{q}'$$

$$+\frac{1}{(2\pi k^2)^2}\int S(\mathbf{q}'-\mathbf{q}'')f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')f_n^*(\mathbf{q}''+\tfrac{1}{2}\mathbf{q})f_p^*(\tfrac{1}{2}\mathbf{q}-\mathbf{q}'')\,d^{(2)}\mathbf{q}\,d^{(2)}\mathbf{q}'\,d^{(2)}\mathbf{q}'', \tag{3.22}$$

where

$$\sigma_{j\,\text{sc}}=\frac{1}{k^2}\int|f_j(\mathbf{q})|^2\,d^{(2)}\mathbf{q},\quad j=n,p \tag{3.23}$$

are the elastic-scattering cross sections of the free nucleons.

The elastic-scattering cross section of the deuteron is easily obtained by integrating the elastic-scattered intensity, given by Eq. (3.18) in the same manner. The result is

$$\sigma_{\text{el}}=\int(d\sigma/d\Omega)_{\text{el}}\,d^{(2)}\mathbf{q}/k^2 \tag{3.24}$$

$$=\frac{1}{k^2}\int S^2(\tfrac{1}{2}\mathbf{q})\{|f_n(\mathbf{q})|^2+|f_p(\mathbf{q})|^2+2\,\text{Re}[f_n(\mathbf{q})f_p^*(\mathbf{q})]\}\,d^{(2)}\mathbf{q}$$

$$-\frac{1}{\pi k^3}\int S(\tfrac{1}{2}\mathbf{q})S(\mathbf{q}')\,\text{Im}[f_n^*(\mathbf{q})+f_p^*(\mathbf{q})]f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\,d^{(2)}\mathbf{q}\,d^{(2)}\mathbf{q}'$$

$$+\frac{1}{(2\pi k^2)^2}\int\left|\int d^{(2)}\mathbf{q}'\,S(\mathbf{q}')f_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q})f_p(\tfrac{1}{2}\mathbf{q}-\mathbf{q}')\right|^2 d^{(2)}\mathbf{q}. \tag{3.25}$$

The deuteron form factor occurs in each of the terms of this cross section since the final state of the deuteron remains the ground state.

The inelastic-scattering cross section is simply the difference between the total-scattering and elastic-scattering cross sections

$$\sigma_{\text{inel}}=\sigma_{\text{sc}}-\sigma_{\text{el}}. \tag{3.26}$$

By examining each of the various terms which occur in the angular distributions and cross sections (3.17–3.26) we may easily identify them as contributions of single or double scattering or their interference terms. The terms of σ_{sc} given by Eq. (3.22) and σ_{el} given by Eq. (3.25) for example are in one-to-one correspondence with those described earlier for the total-scattered intensity, Eq. (3.16).

An additional cross section we may obtain is the "absorption" cross section, which corresponds to processes in which the incident particle fails to reappear after the collision or reappears together with one or more particles such as mesons which are produced in the collision. The "absorption" cross section σ_{abs} is given by the difference between the total cross section and the integrated scattering cross section,

$$\sigma_{\text{abs}}=\sigma_d-\sigma_{\text{sc}}. \tag{3.27}$$

The latter cross sections are given by Eqs. (3.11) and (3.22).

It is interesting to observe the form the integrated cross sections take when expressed in terms of the total phase shift function $\chi_{\text{tot}}(\mathbf{b},\mathbf{s})$. For example, the deuteron total cross section as obtained from Eq. (3.1) and the optical theorem may be written as

$$\sigma_d=2\int\{1-\langle\text{Re}\exp[i\chi_{\text{tot}}(\mathbf{b},\mathbf{s})]\rangle\}d^{(2)}\mathbf{b}, \tag{3.28}$$

where $\langle\mathcal{O}\rangle$ means the expectation value of the operator $\mathcal{O}$ in the deuteron ground state. The total scattering cross section is obtained by squaring the modulus of $F_{fi}(\mathbf{q})$ as given in Eq. (3.1), summing over all final states $|f\rangle$, and integrating over all angles of $\mathbf{k}'$. If we use the closure approximation and express the angular integration as an integration over a plane of momentum transfers, we may write the total scattering cross section as

$$\sigma_{\rm sc}=\int\langle|F(\mathbf{q},\mathbf{s})|^2\rangle\frac{d^{(2)}\mathbf{q}}{k^2}.$$

$$=\left\langle\frac{1}{4\pi^2}\int d^{(2)}\mathbf{b}\,d^{(2)}\mathbf{b}'\exp[i\mathbf{q}\cdot(\mathbf{b}-\mathbf{b}')]\times\{1-\exp[i\chi_{\rm tot}(\mathbf{b},\mathbf{s})]\}\times\{1-\exp[-i\chi_{\rm tot}^*(\mathbf{b}',\mathbf{s})]\}d^{(2)}\mathbf{q}\right\rangle.$$

If we now recognize in this equation the Fourier integral representation of the two-dimensional δ function of argument $\mathbf{b}-\mathbf{b}'$ and perform the $\mathbf{b}'$ integration, we find

$$\sigma_{\rm sc}=\int\langle|1-\exp[i\chi_{\rm tot}(\mathbf{b},\mathbf{s})]|^2\rangle\,d^{(2)}\mathbf{b}. \quad (3.29)$$

The absorption cross section defined by Eq. (3.27) is then

$$\sigma_{\rm abs}=\int\{1-\langle|\exp[i\chi_{\rm tot}(\mathbf{b},\mathbf{s})]|^2\rangle\}\,d^{(2)}\mathbf{b}, \quad (3.30)$$

which is of precisely the form one would find by calculating the average decrease in the intensity of the incident particle wave brought about by its passage through the deuteron.[2]

If the function $\exp[i\chi_{\rm tot}(\mathbf{b},\mathbf{s})]$ in Eqs. (3.28), (3.29), and (3.30) is expressed in terms of the neutron and proton scattering amplitudes, the equivalent expressions for the cross sections given by Eqs. (3.11), (3.12), (3.22), and (3.27) may easily be reproduced.

IV. EVALUATION OF CROSS SECTIONS: SPIN-INDEPENDENT INTERACTIONS

To illustrate a typical application of the results of Sec. III we shall consider the case of high-energy antinucleons incident upon a deuteron target nucleus.[7] Since the available data on the basic antinucleon-nucleon-scattering amplitudes are incomplete, we shall make a number of simple assumptions to complete their specification. Scattering problems involving other types of incident particles may require somewhat different expressions for the basic scattering amplitudes, but the calculations in which these amplitudes are used will, in general, closely resemble those presented here.

We shall begin the treatment of antinucleon-deuteron collisions by assuming that the amplitudes for elastic scattering of the antinucleon by the neutron and proton are equal. Theoretical arguments suggesting that these amplitudes should be equal in the high-energy limit have been given by Pomeranchuk.[13] (Furthermore, we may note that some of the results of the present calculation show that this assumption is not inconsistent with experimental values for the $\bar{p}$-p and $\bar{p}$-d cross sections.) We therefore write

$$f_n(\mathbf{q})=f_p(\mathbf{q})=f(\mathbf{q}). \quad (4.1)$$

If the antinucleon-nucleon total cross section approaches a constant limiting value as the incident energy becomes infinite then, according to the optical theorem, the imaginary part of the forward-scattering amplitude increases linearly with k. Since furthermore the predominance of absorption (i.e., particle production processes) tends to lead in the high-energy approximation to purely imaginary scattering amplitudes, we may expect that at sufficiently high energies $\mathrm{Im}f(0)$ greatly exceeds $|\mathrm{Re}f(0)|$. If we assume that the real part of the forward-scattering amplitude is negligibly small in comparison to the imaginary part then the optical theorem relates the elastic-scattering intensity in the forward direction, $d\sigma_N(0)/d\Omega$, to the antinucleon-nucleon total cross section σ_N by the equation

$$d\sigma_N(0)/d\Omega=(k\sigma_N/4\pi)^2, \quad (\mathrm{Re}f(0)=0). \quad (4.2)$$

The assumption that $|\mathrm{Re}f(0)|$ is negligible compared to $\mathrm{Im}f(0)$ may therefore be tested by comparing the measured forward-scattering intensity to the value obtained from the measured total cross section by means of Eq. (4.2). Recent experiments[14] with 4-BeV/c antiprotons scattered by protons find the ratio of the extrapolated elastic-scattering intensity at zero degrees to the value given by Eq. (4.2) to be 0.98 ± 0.07. In the following calculations we shall assume that near the forward direction, where nearly all the elastic scattering takes place, the scattering amplitude is purely imaginary.[15,16] In that case the antinucleon-nucleon elastic scattering amplitude is determined by the measured angular distribution of elastic scattering to be

$$f(q)=i[d\sigma_N(q)/d\Omega]^{1/2}. \quad (4.3)$$

According to Eq. (3.12) then, the cross-section defect may be expressed in terms of the deuteron form factor and the differential cross section $d\sigma_N/d\Omega$ by

$$\delta\sigma=\frac{4\pi}{k^2}\int_0^\infty S(q)(d\sigma_N(q)/d\Omega)q\,dq. \quad (4.4)$$

[13] I. Ia. Pomeranchuk, Zh. Eksperim. i Teor. Fiz. **30**, 423 (1956) [English transl.: Soviet Phys.—JETP **3**, 306 (1956)].

[14] O. Czyzewski, B. Escoubès, Y. Goldschmidt-Clermont, M. Guinea-Moorhead, D. R. O. Morrison, and S. De Unamuno-Escoubès, Phys. Letters **15**, 188 (1965).

[15] If, however, the scattering amplitude has a small real part, as recent experiments on p-p and π-p scattering, described in Ref. 16, have indicated for those amplitudes, the resulting errors incurred in the cross-section defect and the various deuteron cross sections and angular distributions would be small. For example, if $\mathrm{Re}f/\mathrm{Im}f$ were as large as 0.25 for the $\bar{p}$-p elastic amplitudes, (i.e., as large as experiments indicate for the corresponding p-p and π-p amplitudes), then neglecting $\mathrm{Re}f$ would result in an error of less than 2% in the deuteron total cross section for the energy range we have considered.

[16] K. J. Foley, R. S. Gilmore, R. S. Jones, S. J. Lindenbaum, W. A. Love, S. Ozaki, E. H. Willen, R. Yamada, and L. C. L. Yuan, Phys. Rev. Letters **14**, 74 (1965); **14**, 862 (1965).

It will be convenient for numerical purposes to adopt an analytical representation for the angular distribution $d\sigma_N/d\Omega$ which is consistent with small-angle scattering measurements in high-energy experiments. It is simplest, of course, to choose one that allows an analytical evaluation of at least some of the multiple integrals we shall encounter. Such an angular distribution is the Gaussian

$$d\sigma_N/d\Omega = Ae^{-\alpha^2 q^2}. \tag{4.5}$$

(It is worth noting that some theoretical arguments have been put forward to suggest that scattering distributions are indeed Gaussian in shape at high energies and small momentum transfers.[17]) The parameter A may be determined from the measured value of the total cross section by means of the optical theorem as expressed by Eq. (4.2). That value is given by

$$A = (k\sigma_N/4\pi)^2. \tag{4.6}$$

The parameter α^2 may be determined, for example, from the measured values of the total cross section and the elastic-scattering cross section. The elastic-scattering cross section $\sigma_{N\,\mathrm{sc}}$ is given by the integral

$$\sigma_{N\,\mathrm{sc}} = \int \frac{d\sigma_N}{d\Omega} d\Omega \tag{4.7}$$

which is carried out over the direction of $\mathbf{k}'$. This integral may be evaluated in the approximate way we have discussed in the preceding section by regarding it as an integral over $d^{(2)}\mathbf{q}$ and letting the momentum transfers range over a plane perpendicular to the direction of incidence. In this way we obtain the relation

$$\alpha^2 = \sigma_N^2/16\pi\sigma_{N\,\mathrm{sc}}. \tag{4.8}$$

When the differential cross section for scattering by a nucleon is represented by means of Eqs. (4.5) and (4.6) the total-cross-section defect given by Eq. (4.4) becomes

$$\delta\sigma = \frac{\sigma_N^2}{4\pi}\int_0^\infty S(q)e^{-\alpha^2 q^2} q\,dq. \tag{4.9}$$

The same assumptions regarding the form of the antinucleon-nucleon scattering amplitude may be used to evaluate the various types of angular distributions and integrated cross sections for scattering by deuterons which were discussed in Sec. III. For this purpose we use Eqs. (4.3), (4.5) and (4.6) to write the scattering amplitude $f(q)$ as

$$f(q) = i(k\sigma_N/4\pi)e^{-\frac{1}{2}\alpha^2 q^2}. \tag{4.10}$$

If we substitute this expression, for example, for the amplitudes f_n and f_p in Eq. (3.16) we may evaluate the differential cross section summed over elastic and inelastic processes. Several of the integrations indicated in Eq. (3.16) may be carried out analytically before the form factor of the deuteron, $S(q)$, is specified. When these are done we find

$$(d\sigma/d\Omega)_{\mathrm{sc}} = (k\sigma_N/4\pi)^2 2[1+S(q)]e^{-\alpha^2 q^2} - \frac{4}{k}\left(\frac{k\sigma_N}{4\pi}\right)^3 \left[\int_0^\infty S(q')e^{-\alpha^2 q'^2} I_0(\alpha^2 qq')q'\,dq'\right]e^{-\alpha^2 q^2} + \left(\frac{k\sigma_N}{4\pi}\right)^4 \frac{1}{(2\alpha k)^2}\left[\int_0^\infty S(q')e^{-\frac{1}{2}\alpha^2 q'^2} q'\,dq'\right]e^{-\frac{1}{2}\alpha^2 q^2}, \tag{4.11}$$

where $I_0(\alpha^2 qq') = J_0(i\alpha^2 qq')$ is the Bessel function of zeroth order with purely imaginary argument. In obtaining this expression we have used an integral representation for the Bessel function $I_0(\alpha^2 qq')$ given by

$$I_0(\alpha^2 qq') = \frac{1}{2\pi}\int_0^{2\pi} e^{-\alpha^2 qq' \cos\varphi} d\varphi \tag{4.12}$$

and have utilized the result

$$4\alpha^2 \int_0^\infty e^{-2\alpha^2 q'^2} I_0(2\alpha^2 q'q'')q'dq' = e^{\frac{1}{2}\alpha^2 q''^2}. \tag{4.13}$$

The angular distribution of elastically scattered particles is obtained from Eq. (3.18) and may be written in the simple form

$$(d\sigma/d\Omega)_{\mathrm{el}} = [2(k\sigma_N/4\pi)S(\tfrac{1}{2}q)e^{-\frac{1}{2}\alpha^2 q^2} - (k\delta\sigma/4\pi)e^{-\frac{1}{4}\alpha^2 q^2}]^2. \tag{4.14}$$

The intensity of inelastically scattered particles may be found by subtracting the elastic differential cross section from the summed differential cross section as given by Eq. (4.11).

The scattering cross section integrated over solid angles (for elastic plus inelastic scattering) is obtained from Eq. (3.22) and may be written as

$$\sigma_{\mathrm{sc}} = 2\sigma_{N\,\mathrm{sc}} + \delta\sigma - \frac{1}{4\pi}\sigma_{N\,\mathrm{sc}}\sigma_N \int_0^\infty S(q)e^{-\frac{3}{4}\alpha^2 q^2} q\,dq + \frac{1}{2\pi}\sigma_{N\,\mathrm{sc}}^2 \int_0^\infty S(q)e^{-\frac{1}{2}\alpha^2 q^2} q\,dq \tag{4.15}$$

$$= 2\sigma_{N\,\mathrm{sc}} - \delta\sigma_{\mathrm{sc}}, \tag{4.16}$$

[17] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).

where Eq. (4.16) defines the defect, $\delta\sigma_{sc}$, of the scattering cross section. Similarly Eq. (3.25) yields the integrated elastic-scattering cross section

$$\sigma_{el}=\frac{1}{2\pi}\sigma_N^2\int_0^\infty S^2(\tfrac{1}{2}q)e^{-\alpha^2q^2}qdq$$
$$-\frac{1}{2\pi}\sigma_N\delta\sigma\int_0^\infty S(\tfrac{1}{2}q)e^{-\frac{3}{4}\alpha^2q^2}qdq+2\sigma_{N\,sc}\left(\frac{\delta\sigma}{\sigma_N}\right)^2. \quad (4.17)$$

The absorption cross section is the difference between the total cross section of the deuteron and the integrated scattering cross section

$$\sigma_{abs}=\sigma_d-\sigma_{sc} \quad (4.18)$$
$$=2(\sigma_N-\sigma_{N\,sc})-\delta\sigma_{abs}. \quad (4.19)$$

The latter equation defines the absorption cross-section defect $\delta\sigma_{abs}$ of the deuteron.

To evaluate the expressions we have given for the various deuteron cross sections we must have some knowledge of the form factor $S(q)$ for the ground state of the deuteron. Since the ground-state wave function of the deuteron is not known very accurately, particularly for small neutron-proton separations, we can only use theoretical models to evaluate $S(q)$. The values of the cross-section defects have been shown[1] on the other hand to depend more sensitively on the form taken by the deuteron wave function at small distances, than on its form at large distances. It will therefore be interesting to see how model-dependent the deuteron cross sections are. To investigate this sensitivity we have considered seven forms of the deuteron wave function in calculating the deuteron total cross section. The choice of these wave functions, we must emphasize, has been guided by the need for analytical simplicity as well as variety. Six of the wave functions we have chosen represent simply plausible functions which are available in manageable analytic form. The seventh is a particular function available only in numerical form. Our use of these wave functions as illustrations is not intended, of course, to constitute endorsement of these products.

The first of these wave functions, and one of the simplest, is the one which corresponds to the zero-range approximation for the neutron-proton force,

$$\phi_1(\mathbf{r})=(a/2\pi)^{1/2}e^{-ar}/r, \quad (4.20)$$

where the parameter a may be regarded as the reciprocal of the radius of the deuteron. It is given in terms of the deuteron binding energy E_d by $a=(2\mu\hbar^{-2}E_d)^{1/2}$, where μ is the reduced mass of the proton and neutron, and has the numerical value $a=0.232$ fermi^{-1}.

The second of the wave functions is given by

$$\phi_2(\mathbf{r})=(\text{const})(e^{-ar}-e^{-br})/r, \quad (4.21)$$

where the parameter a is the same as in Eq. (4.20). The value used for b is that given by Moravcsik[18] in his fit of the Gartenhaus wave function, $b=1.202$ fermi^{-1}.

The third type of wave function is an improved fit to the Gartenhaus wave function obtained by Moravcsik.[18] It is given by

$$\phi_3(\mathbf{r})=(\text{const})(e^{-ar}-e^{-cr})(1-e^{-cr})/r, \quad (4.22)$$

where $c=1.59$ fermi^{-1}.

The fourth wave function is a still more accurate fit to the Gartenhaus wave function and is given by[18]

$$\phi_4(\mathbf{r})=(\text{const})(e^{-ar}-e^{-dr})(1-e^{-cr})(1-e^{-gr}), \quad (4.23)$$

where $d=1.90$ fermi^{-1} and $g=2.5$ fermi^{-1}.

Although the wave functions (4.20–23) clearly have the correct asymptotic form for large neutron-proton separations, their exponential character leads to expressions for the integrals in Eqs. (4.9), (4.15), and (4.17), for example, which can only be evaluated by numerical integration. Since, as we have noted earlier, the double interaction effects arise largely from small neutron-proton separations, there is no need to restrict the approximate wave functions to choices having the correct behavior for large separations. It becomes possible to evaluate the integrals in Eqs. (4.9), (4.15), and (4.17) analytically, for example, if we approximate the deuteron wave function by means of the simple Gaussian

$$\phi_5(\mathbf{r})=(2\beta^2/\pi)^{3/4}e^{-\beta^2r^2}. \quad (4.24)$$

For this wave function we take $\beta^2=0.0961$ fermi^{-2}, the value obtained by Verde[19] by minimizing the energy of the trial function for the deuteron ground state by means of a variational method.

A possibly more accurate wave function for which the calculations may still be carried out analytically is given by the sum of three Gaussians

$$\phi_6(\mathbf{r})=0.02133e^{-0.03r^2}+0.08582e^{-0.16r^2}+0.18115e^{-0.76r^2}, \quad (4.25)$$

which was suggested by Christian and Gammel.[20] This expression represents a fit to a wave function they obtained for a specific Gaussian form of interaction.

None of the analytical representations of the deuteron ground-state wave function, $\phi_1, \cdots, \phi_6$, allows for the presence of a hard core in the neutron-proton force. To illustrate the effect of a hard core on the cross-section defect we consider as a seventh type of wave function, ϕ_7, the one obtained by Hamada and Johnston[21] which has a hard-core radius of 0.486 fermi. This wave function, which has been presented in the form of numerical tables, includes a representation of the small D-state admixture in the deuteron ground state.

[18] M. J. Moravcsik, Nucl. Phys. **7**, 113 (1958).
[19] M. Verde, Helv. Phys. Acta **22**, 339 (1949).
[20] R. S. Christian and J. L. Gammel, Phys. Rev. **91**, 100 (1958).
[21] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).

The form factor $S_1(\mathbf{q})$ for the wave function $\phi_1(\mathbf{r})$ is

$$S_1(\mathbf{q}) = (2a/q)\tan^{-1}(q/2a), \qquad (4.26)$$

and the form factor $S_5(\mathbf{q})$ for the wave function $\phi_5(\mathbf{r})$ is

$$S_5(\mathbf{q}) = e^{-q^2/8\beta^2}. \qquad (4.27)$$

The form factors for the wave functions ϕ_2, ϕ_3, and ϕ_4 are just linear combinations of arctangent functions multiplied by q^{-1}, and for ϕ_6 the form factor is just a linear combination of Gaussians. The form factor for the wave function ϕ_7 has been obtained by numerical integration.

We have summarized the experimental input data and the results of one of our calculations in Table I. Columns 2, 3, and 4 of that table show the experimental values of the elastic and total cross sections of the proton[4,5,14,22–25] and total cross sections of the deuteron[4,5] for instances in which at least two of these cross sections have been measured at the same energy. The measurements of the elastic-scattering cross section of the proton which have been carried out at energies above 4 BeV have, with a single exception, been made at energies other than those listed in the table. We find that those measurements are consistent with an energy-independent value of the parameter α^2, defined by Eq. (4.5), of ~0.50 fermi². Since this value is consistent with direct measurements of the high-energy antiproton-proton scattering angular distribution, we have used it in place of the values of the elastic cross section at the energies above 4 BeV in the table. The fifth column of Table I shows the values of the deuteron total cross section which are calculated according to Eq. (4.9) by making use of the wave function ϕ_4 which presumably is the most accurate of the analytically presented wave functions we have considered. The over-all agreement of the calculated total cross sections of the deuteron with the measurements is seen to be quite good. It is perhaps indicative of some systematic error, either theoretical or experimental, that at the energies above 4 BeV the calculated values of the cross section exceed the measured values, though only by amounts lying within or near the quoted probable errors.

TABLE I. Deuteron total cross sections. Measured antiproton-proton and antiproton-deuteron cross sections are shown in columns 2, 3, and 4. The antiproton-deuteron total cross sections calculated using the wave function ϕ_4 are shown in the last column. All cross sections are in millibarns.

T_{lab} (BeV)	σ_N (expt)	$\sigma_{N\,\mathrm{sc}}$ (expt)	σ_d (expt)	σ_d (calc)
0.133	166 ±8	72 ±10		285
0.197	152 ±7	64 ±8		263
0.265	124 ±7	50 ±7		219
0.333	114 ±4	49 ±6		202
0.534	118 ±6	42 ±5	210 ±5	211
0.700	116 ±5	42 ±4	189 ±5	207
0.816	108 ±5	38 ±4	196 ±6	194
0.948	96 ±3	33 ±3	178 ±5	174
1.00	100 ±3	32 ±2		181
1.068	96 ±3	30 ±2	184 ±3	175
1.25	89 ±4	28 ±2		163
2.00	80 ±6	25 ±4		147
2.47	75.4±2.0	21.9±1.1		139.6
3.17	71 ±1	19.8±0.7		132
5.1	59.3±1.1		106.9±1.3	111.6
7.1	56.4±0.8		102.7±1.3	106.4
11.1	51.7±0.8	11.6±0.4	96.1±1.3	98.1
13.1	50.7±0.9		95.0±1.4	96.3
15.1	49.2±0.8		93.2±1.6	93.6
17.1	50.3±3.6		87.2±6.1	95.5

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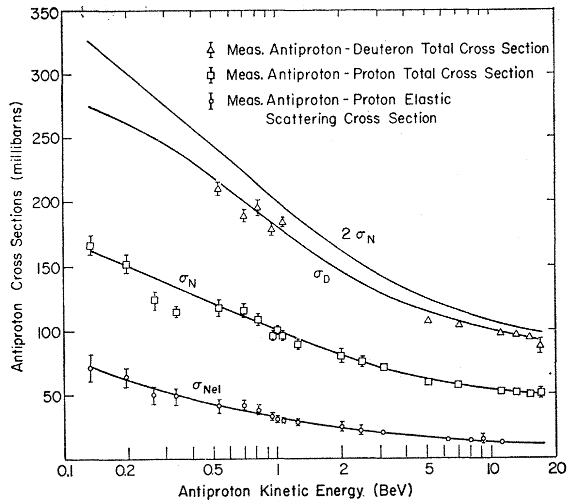

FIG. 1. The theoretical total cross sections for antiproton-deuteron collisions as a function of the incident antiproton laboratory kinetic energy. The deuteron wave function used is ϕ_3, Eq. (4.22). The two lower curves are smoothed representations of the indicated measurements of the total and elastic antiproton-proton cross sections. The deuteron total cross sections which are calculated from the smoothed data are given by the curve labeled σ_d. The uppermost curve represents twice the antiproton-proton cross section. The total cross-section defect is the difference of the ordinates of the two uppermost curves.

We show in Fig. 1 the way in which the actual cross section of the deuteron differs, as a function of energy, from the sum of the cross sections of the two nucleons. As input data in the calculations for this graph we have used the curves indicated there which have been drawn to fit the measured values of the total and elastic proton cross sections. The deuteron wave function used was ϕ_3. We observe that the cross section defect decreases quite slowly with increasing energy. The graph shows quite clearly the importance of $\delta\sigma$ in calculating the cross section of the deuteron.

The dependence of the total cross-section defect upon the deuteron model used is indicated in Table II for the energy range 0.13 to 3.2 BeV. The columns headed $\delta\sigma_j$ represent the values calculated for the total

TABLE II. The calculated total-cross-section defect. The deuteron wave functions used in calculating $\delta\sigma_j$ ($j=1, \cdots, 7$) are $\phi_1,\cdots,\phi_7$, respectively. All cross-section defects are in millibarns.

$T_{\rm lab}$ (BeV)	$\delta\sigma_1$	$\delta\sigma_2$	$\delta\sigma_3$	$\delta\sigma_4$	$\delta\sigma_5$	$\delta\sigma_6$	$\delta\sigma_7$
0.133	83	50.5	47.6	47.3	53	72	48.6
0.197	72	43.5	40.9	40.6	46	66	41.7
0.265	54	31.1	29.0	28.7	32	45	29.5
0.333	49	28.0	25.9	25.5	28	41	26.3
0.534	46	27.3	25.5	25.3	28	39	26.0
0.700	46	26.7	25.0	24.7	28	39	25.5
0.816	41	23.6	22.0	21.8	24	34	22.4
0.948	34	19.4	18.0	17.8	20	29	18.3
1.00	35	20.4	19.0	18.8	21	30	19.3
1.068	32	18.7	17.4	17.2	19	27	17.7
1.25	29	16.6	15.4	15.2	17	24	15.7
2.00	25	14.0	12.9	12.7	14	21	13.1
2.47	22	12.4	11.4	11.2	13	18	11.6
3.17	20	11.1	10.1	9.9	11	16	10.3

cross-section defect by using the deuteron form factor appropriate to the wave function ϕ_j. We observe that the cross section defects predicted on the basis of the wave functions ϕ_2, ϕ_3, ϕ_4, ϕ_5, and ϕ_7 cluster fairly closely in value at corresponding energies. They all appear to furnish total cross sections which agree quite well with the experimental values. The cross-section defects which correspond to the zero-range wave function, ϕ_1, and the sum of Gaussians, ϕ_6, on the other hand are rather too large. The reason for the disagreement is clear in the case of the zero-range wave function, ϕ_1, since this wave function becomes infinite for small neutron-proton separations. It clearly describes too compact a structure for the deuteron. It is interesting to note that the presence of the hard core in the Hamada-Johnston wave function, ϕ_7, does not lead to any significant decrease of the cross-section defect even though its effect is to keep the neutron and proton from overlapping one another.

As we have noted in Eq. (4.19), the absorption cross section of the deuteron also shows a cross-section defect,

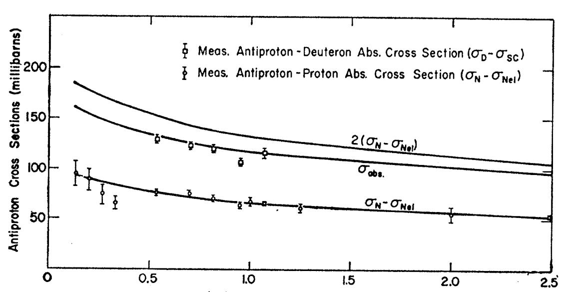

FIG. 2. The theoretical absorption cross section for antiproton-deuteron collisions as a function of the incident antiproton laboratory kinetic energy. The deuteron wave function used is ϕ_3, Eq. (4.22). The lowest curve is an approximate fit to the values of the antiproton-proton absorption cross section, i.e., the difference between the measured values of the total and elastic-scattering cross sections. The calculated deuteron absorption cross section is given by the curve labeled $\sigma_{\rm abs}$. The uppermost curve represents twice the antiproton-proton absorption cross section. The cross-section defect for absorption is given by the difference of the ordinates of the two uppermost curves.

$\delta\sigma_{\rm abs}$. Calculations of the absorption cross section of the deuteron have been carried out for the energy range 0.13 to 2.5 BeV using the wave function ϕ_3. The influence of double interactions on the absorption cross section is shown as a function of energy in Fig. 2. We observe that the absorption cross section has a positive defect which, like that of the total cross section, decreases quite slowly with increasing energy. The absorption cross sections which are predicted by taking double interactions into account clearly agree much more closely with the experimental values of the absorption cross section of the deuteron than those which neglect them.

It is interesting to compare our numerical results with those which follow from the approximate formula for the cross-section defect which holds when the average neutron-proton distance greatly exceeds the force range. If the size of the deuteron is quite large in comparison to the range of antinucleon-nucleon forces, we would expect the form factor of the deuteron to decrease to zero much more rapidly than the scattering amplitudes $f(q)$ as a function of q, the momentum transferred. In that case the integral (3.12) may be approximated by

$$\delta\sigma\sim-\frac{2}{k^2}\,{\rm Re}[f_n(0)f_p^{\dagger}(0)]\int S(\mathbf{q})\,d^{(2)}\mathbf{q}. \quad (4.28)$$

But the integral of the form factor is related to the average inverse square of the neutron-proton distance in the deuteron ground state, $\langle r^{-2}\rangle_d$, by

$$\int S(\mathbf{q})\,d^{(2)}\mathbf{q}=2\pi\left\langle\frac{1}{r^2}\right\rangle_d. \quad (4.29)$$

When this integral is substituted in Eq. (4.28) we obtain the result

$$\delta\sigma\approx-(4\pi/k^2)\,{\rm Re}[f_n(0)f_p(0)]\langle r^{-2}\rangle_d. \quad (4.30)$$

Since we have assumed that the elastic scattering amplitudes f are purely imaginary, we may use the optical theorem (3.10) to reduce the result to the simple form

$$\delta\sigma\approx(1/4\pi)\sigma_n\sigma_p\langle r^{-2}\rangle_d. \quad (4.31)$$

This formula makes it particularly evident that the central regions of the deuteron wave function contribute most strongly to the determination of the cross-section defect. The approximation on which it is based, however, is inaccurate for small values of the neutron-proton separation, and no radial dependence as singular as r^{-2} is present in the more accurate treatments which we have described earlier. The effect of the $1/r^2$ dependence in the formula (4.31) is to exaggerate the magnitude of the cross-section defect when the parameters are given their proper physical values. This tendency may be illustrated by noting that for the wave function ϕ_2 the value of $\langle r^{-2}\rangle_d$ is 0.61 fermi^{-2}. The values of the cross section defects given by Eq. (4.31) at energies of 0.133, 0.534, 1.00, and 2.47 BeV are then 134, 68, 49, and 28 mb, respectively. These estimates are more than a factor of two larger than the values listed for the same energies in the column headed $\delta\sigma_2$ in Table II. The simple formula (4.31) is evidently not very reliable in accuracy.

The presence of a hard-core interaction, which tends to keep the neutron and proton apart, can improve the accuracy of the approximate expressions given by Eqs. (4.30) and (4.31) since it gives zero weight to the region in which r^{-2} is singular. To test this possibility we note[26] that for the wave function ϕ_7, the average value of r^{-2} is 0.273 fermi^{-2}. This value, used in Eq. (4.31), leads to estimates of the cross-section defect of 30.2, 21.7, and 12.3 mb at energies of 0.534, 1.00, and 2.47 BeV. These values are 6 to 16% larger than the values of the cross-section defect for the same energies given in the column headed $\delta\sigma_7$ of Table II.

To test the sensitivity of the cross sections to the assumption that the angular distributions of scattering are Gaussian in form we have also carried out a calculation in which the angular distribution is taken to be that due to diffraction by a grey disk. For this interaction the function $\Gamma(\mathbf{b})$ takes the form

$$\Gamma(\mathbf{b})=1-\zeta\,, \qquad b<\rho$$
$$=0\,, \qquad b>\rho\,, \tag{4.32}$$

where ρ is the radius of the disk and ζ is a positive parameter less than unity. The elastic scattering amplitude is then

$$f(\mathbf{q})=i(1-\zeta)k\rho q^{-1}J_1(\rho q)\,. \tag{4.33}$$

The parameters ζ and ρ are determined from Eqs. (4.2) and (4.7) to be

$$1-\zeta=2\sigma_{N\ \mathrm{sc}}/\sigma_N\,, \tag{4.34}$$

and

$$\rho=\tfrac{1}{2}\sigma_N(\pi\sigma_{N\ \mathrm{sc}})^{-1/2}\,. \tag{4.35}$$

The angular distribution for elastic scattering is then

$$d\sigma_N/d\Omega=(k\sigma_N/4\pi)^2[2J_1(\rho q)/\rho q]^2\,. \tag{4.36}$$

The wave function used for this calculation is the form ϕ_2 given by Eq. (4.21).

Typical values for the cross section defects are then found to be 48, 26, 20, and 12 mb at energies 0.133, 0.534, 1.00, and 2.47 BeV, respectively. These values are seen to differ little from the ones listed for the same energies in the column headed $\delta\sigma_2$ in Table II. The latter values were calculated with the same deuteron wave function ϕ_2, but with the Gaussian form of angular distribution for scattering. This insensitivity of $\delta\sigma$ to the form used for the scattering amplitudes is in fact suggested by the approximate expression (4.30) which depends only on the forward-scattering amplitudes. The more correct expression, Eq. (3.12), depends on the scattering amplitudes for small but nonvanishing momentum transfers. The power series expansions of the two scattering angular distributions we have used are in fact the same until the q^4 term is reached.

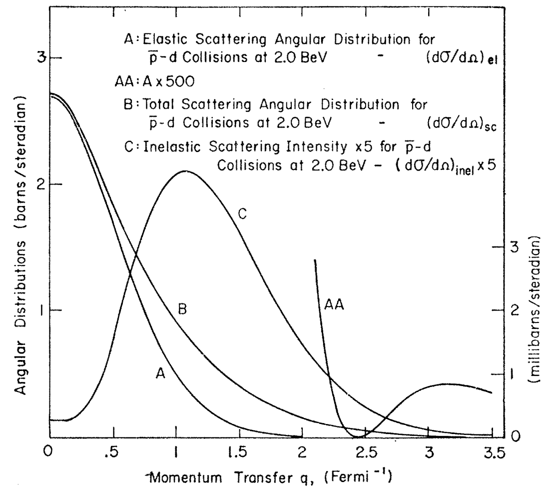

FIG. 3. The theoretical angular distributions for elastic scattering, inelastic scattering, and their sum for antiproton-deuteron collisions as a function of momentum transfer q. The incident antiproton laboratory kinetic energy is 2.0 BeV and the deuteron wave function which is used is ϕ_2, Eq. (4.21). The angular distribution for elastic scattering for $q>2.1$ fermi^{-1} is given by curve AA with the scale on the right side. The intensity for inelastic scattering (multiplied by a factor of 5) is shown in curve C.

As we have noted in Sec. III, the high-energy approximation may also be used to investigate the angular distribution for elastic scattering and that for the sum of elastic plus inelastic scattering. We have calculated these angular distributions[7] for an incident antiproton energy of 2.0 BeV by means of Eqs. (4.11), (4.14) and the wave function ϕ_2. As input data we have used the antiproton-proton cross-section measurements shown in Table I. The results are shown in Fig. 3. We see that elastic scattering dominates near the forward direction, as might be expected. We also observe that the difference shown in curve C between the angular distribution for elastic plus inelastic scattering and that for purely elastic scattering does not vanish for zero momentum transfer. It is of course not surprising that some of the forward scattering should be inelastic in character, but the statement that this scattering is associated with zero momentum transfer requires some clarification. For this purpose we must recall that the energy change of the incident particle upon scattering has been assumed negligibly small. For small momentum transfers, therefore, $\hbar\mathbf{q}$ represent only the part of the momentum transfer which is transverse to the direction of incidence. This component naturally vanishes when the angle of scattering vanishes. Double-scattering processes in which the transverse momenta communicated to the two nucleons are equal and opposite and sufficiently strong to dissociate the deuteron make up the inelastic contributions to forward scattering.

[26] R. A. J. Riddle, A. Langsford, P. H. Bowen, and G. C. Cox, Nucl. Phys. 61, 457 (1965).

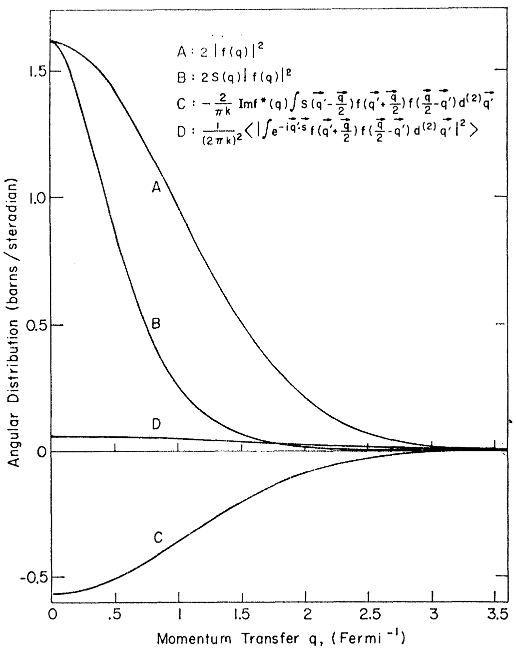

FIG. 4. The theoretical components of the total angular distribution for scattering (inelastic plus elastic) in antiproton-deuteron collisions as a function of momentum transfer q. The incident antiproton laboratory kinetic energy is 2.0 BeV and the deuteron wave function used is ϕ_2, Eq. (4.21). The terms of Eq. (3.16) to which the four curves correspond are indicated in the figure. Curve A represents the sum of the single-scattering cross sections and B is the contribution of the interference of the two single-scattering processes. Curve C is the contribution of the interference of the single- and double-scattering amplitude (which is negative) and D is the angular distribution for double scattering.

We note in Fig. 3 that there is a zero in the elastic scattering angular distribution at $q \approx 2.5$ fermi^{-1}. This occurs as a result of a cancellation of the elastic single scattering amplitudes which predominate at small angles with the elastic double-scattering amplitude which predominates at large angles. It is also evident, however, that the elastic scattering is quite weak in intensity for $q \gtrsim 2.5$ fermi^{-1}.

We have shown in the preceding section that the angular distributions of scattering may, by means of the high-energy approximation, be resolved into their various single and double interaction components. In Figs. 4, 5, and 6 the various components of the angular distributions are shown for antiproton-deuteron collisions at 2.0 BeV. These calculations[7] are based on the wave function ϕ_2.

Plotted in Fig. 4 are the single and double scattering and interference terms which contribute to the angular distribution for elastic plus inelastic scattering. By referring to Eq. (3.16) and recalling that the neutron- and proton-scattering amplitudes have been assumed equal, we see that curve A in Fig. 4 represents the contribution to the total scattered intensity from the two nucleons considered individually. It furnishes the major contribution to the scattered intensity at all angles for which the intensity is appreciable, except for the forward direction. There it shares this distinction with the simple interference term for the nucleon-scattering amplitudes, which is represented by curve B and is highly peaked near the forward direction. Curve C corresponds to the contribution from the interference between the single- and double-scattering amplitudes and is seen to be negative, and curve D represents the direct effect of double scattering. The total scattered intensity shown in Fig. 3 is obtained by adding all the curves.

Figure 5 shows a similar analysis of the contributions of the various collision processes to the angular distribution for purely elastic scattering. In this distribution the interference term contributed by single scattering takes the same form as curve A which represents the sum of the elastic single-scattering terms contributed individually by the two nucleons. Curves B and C represent the interference between single and double scattering and the contribution of double scattering, respectively.

The various contributions to the intensity of inelastic scattering, or scattering accompanied by deuteron break-up, are given in Fig. 6. Curve D in this figure shows quite clearly that the inelastic scattering which

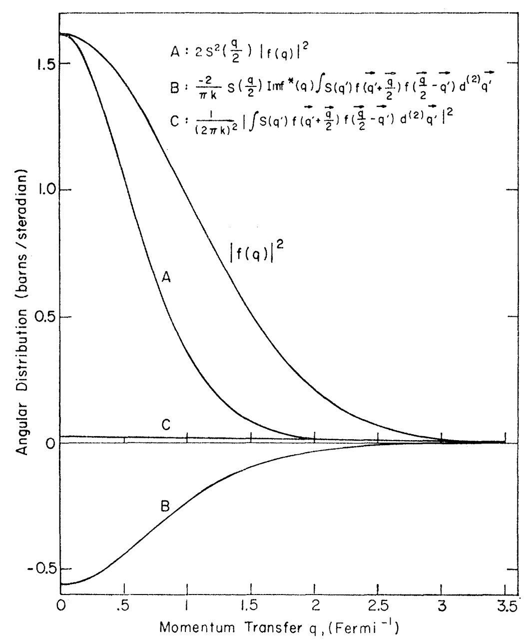

FIG. 5. The theoretical components of the elastic-scattering angular distribution for antiproton-deuteron collisions as a function of momentum transfer q. The incident-antiproton laboratory kinetic energy is 2.0 BeV, and the deuteron wave function used is ϕ_2, Eq. (4.21). The terms of Eq. (3.18) to which the curves A, B, and C correspond are indicated in the figure. Curve A represents both the sum of the single-scattering contributions and their interference term. Curve B represents the contribution of the interference of the single- and double-scattering processes, and C is the angular distribution for elastic-double scattering. The curve labeled $|f(q)|^2$ is the antinucleon-nucleon elastic-scattering angular distribution and is shown for reference.

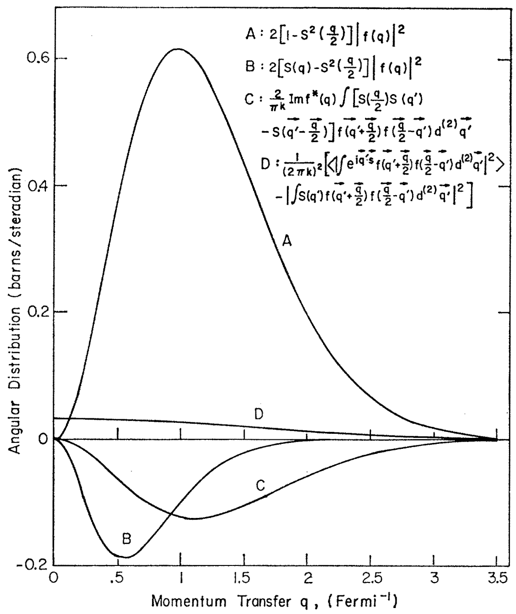

FIG. 6. The theoretical components of the intensity of inelastic scattering in antiproton-deuteron collisions as a function of momentum transfer q. The incident antiproton laboratory kinetic energy is 2.0 BeV and the deuteron wave function used is ϕ_2, Eq. (4.21). The expressions which the four curves represent are derived from Eqs. (3.16), (3.18), and (3.19) and are indicated in the figure. The physical processes which give rise to these contributions are the same as for the correspondingly labeled curves of Fig. 4.

is observed in the forward direction is contributed by double-scattering processes. The inelastic angular distribution shown in curve C of Fig. 3 is the sum of the four curves in Fig. 6.

V. SPIN-DEPENDENT EFFECTS

In this section we consider the influence of the spin dependence of nuclear forces on the deuteron total cross section.[7] We do this by expressing the spin-dependent deuteron scattering matrix in terms of the nucleon scattering matrices and the deuteron wave function. We begin by discussing an expression for the nucleon scattering matrix in the high-energy approximation.

Since the interaction between an incident spin-$\frac{1}{2}$ particle and a neutron or a proton is spin-dependent, the nucleon scattering matrices may be regarded as operators in spin space. These operators may be expressed in terms of the Pauli spin operators $\boldsymbol{\sigma}$, for the incident particle, and $\boldsymbol{\sigma}_n$ and $\boldsymbol{\sigma}_p$, for the neutron and the proton, respectively. In the high-energy approximation the nucleon scattering amplitude operator is given by the generalization of Eq. (2.3),

$$M_j(\mathbf{q},k,\boldsymbol{\sigma},\boldsymbol{\sigma}_j)=\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_j(\mathbf{b},k,\boldsymbol{\sigma},\boldsymbol{\sigma}_j)\,d^{(2)}\mathbf{b},\qquad j=n,\,p. \tag{5.1}$$

The function Γ_j is to be regarded as an operator in spin space, as is the scattering amplitude M_j. The total cross section for unpolarized nucleons is given, according to the optical theorem, by

$$\sigma_j=\frac{4\pi}{k}\,\mathrm{Im}\tfrac{1}{4}\,\mathrm{Tr}M_j(0,k),\quad j=n,\,p, \tag{5.2}$$

where the symbol Tr stands for the trace taken over the composite spin space of the incident particle and the target nucleon. The trace occurs in this expression together with the factor $\frac{1}{4}$ because the particles are initially unpolarized and we must average the cross section over their possible spin orientations.

We observe that because of the spin-dependence, the interactions of the incident particle with the neutron and proton are not in general commuting operators, a property which is reflected by the fact that the functions Γ_n and Γ_p, for the neutron and the proton, will in general not commute. To give as simple an insight as possible into the spin-dependence of the double scattering, we shall use an approximate form for the spin-dependent scattering amplitude of the deuteron, which we construct as a generalization of Eq. (3.4). We take this to be

$$\mathfrak{M}_{fi}(\mathbf{q},\boldsymbol{\sigma},\boldsymbol{\sigma}_n,\boldsymbol{\sigma}_p)=\langle f|\exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})\frac{ik}{2\pi}\int\exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_n(\mathbf{b},\boldsymbol{\sigma},\boldsymbol{\sigma}_n)d^{(2)}\mathbf{b}+\exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s})\frac{ik}{2\pi}\int\exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_p(\mathbf{b},\boldsymbol{\sigma},\boldsymbol{\sigma}_p)\,d^{(2)}\mathbf{b}$$
$$-\frac{ik}{2\pi}\int\exp(i\mathbf{q}\cdot\mathbf{b})\tfrac{1}{2}\{\Gamma_n(\mathbf{b}-\tfrac{1}{2}\mathbf{s},\,\boldsymbol{\sigma},\boldsymbol{\sigma}_n),\Gamma_p(\mathbf{b}+\tfrac{1}{2}\mathbf{s},\,\boldsymbol{\sigma},\boldsymbol{\sigma}_p)\}\,d^{(2)}\mathbf{b}\,|i\rangle, \tag{5.3}$$

where $\{\Gamma_n,\Gamma_p\}$ is the anticommutator of Γ_n and Γ_p, defined by

$$\{\Gamma_n,\Gamma_p\}=\Gamma_n\Gamma_p+\Gamma_p\Gamma_n.$$

The use of this anticommutator in the double-scattering term of Eq. (5.3) is indicated by the fact that the individual collisions with the neutron and proton may take place in either of two orders.

This expression is only an approximate rendering of the scattering amplitude in the high-energy approximation. The further approximation which has been implicitly made lies in neglecting the commutator of the spin-dependent parts of the interaction of the incident particle with the neutron and proton when the neutron and proton are so close together that the interaction regions surrounding them overlap appreciably. The approximation may be a good one to the extent that the deuteron is loosely bound or that the spin-dependence is weak.

To obtain the deuteron scattering amplitude, $\mathfrak{M}_{fi}$, in terms of the neutron and proton amplitudes, we invert, Eq. (5.1) to express Γ_n and Γ_p in terms of M_n and M_p, and substitute the result in Eq. (5.3). We then obtain

$$\mathfrak{M}_{fi}(q) = \langle f| M_n(\mathbf{q}) \exp(\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s}) + M_p(\mathbf{q}) \exp(-\tfrac{1}{2}i\mathbf{q}\cdot\mathbf{s}) + \frac{i}{2\pi k}\int \exp(i\mathbf{q}'\cdot\mathbf{s})\tfrac{1}{2}\{M_n(\mathbf{q}'+\tfrac{1}{2}\mathbf{q}), M_p(-\mathbf{q}'+\tfrac{1}{2}\mathbf{q})\} \times d^{(2)}\mathbf{q}'|i\rangle. \quad (5.4)$$

To find the deuteron total cross section σ_d we use the optical theorem,

$$\sigma_d = \frac{4\pi}{k}\,\mathrm{Im}\tfrac{1}{6}\langle \mathrm{Tr}_3\mathfrak{M}(0)\rangle, \quad (5.5)$$

where the brackets $\langle\ \rangle$ mean the average in coordinate space, taken with respect to the deuteron ground state. Tr_3 means the trace over the spin space of the incident particle and the triplet states of the deuteron, and may be obtained by using the triplet-state projection operator, P_3, given by

$$P_3 = \tfrac{1}{4}(3+\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p), \quad (5.6)$$

in connection with the trace over the composite spin space of the neutron, proton, and incident particle. In this way we obtain the expression for the cross-section defect

$$\delta\sigma = -\frac{2}{k^2}\int S(\mathbf{q})\,\mathrm{Re}\{\tfrac{1}{6}\,\mathrm{Tr}[P_3 M_n(\mathbf{q}) M_p(-\mathbf{q})]\}\,d^{(2)}\mathbf{q}. \quad (5.7)$$

To furnish a concrete illustration of the formalism let us assume that a complex spin-dependent potential operator exists. We may assume, for example, that the interaction potential between two spin-$\frac{1}{2}$ particles (e.g., the incident particle and a nucleon) has the specific form

$$V(\mathbf{r}) = V_c(r) + V_s(\mathbf{r})(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{L} + V_\sigma(\mathbf{r})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2, \quad (5.8)$$

where $\mathbf{L}$ is the orbital angular momentum operator for the incident particle in units of $\hbar$,

$$\mathbf{L} = \hbar^{-1}\mathbf{r}\times\mathbf{p},$$

and $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the Pauli spin operators for the particles.

We begin by investigating elementary collisions between two spin-$\frac{1}{2}$ particles in order to obtain the form of the scattering matrix in the high-energy approximation. This analysis will show how the various spin amplitudes and the corresponding phase-shift functions are derived from the interaction potentials. When the interaction takes the general form in Eq. (5.8), the techniques of Ref. 2 may be used to show that the operator Γ assumes the form

$$\Gamma = 1 - \exp\{i[\chi_c(\mathbf{b}) + \chi_s(\mathbf{b})(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{b}\times\mathbf{k}) + \chi_\sigma(\mathbf{b})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2]\}. \quad (5.9)$$

The phase-shift functions in this expression are given by the integrals

$$\chi_i(\mathbf{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V_i(\mathbf{b}+\mathbf{z})dz, \quad i = c, s, \sigma, \quad (5.10)$$

which are taken along straight-line paths as in other applications of the high-energy approximation. If we define

$$u(\mathbf{b}) = kb\chi_s(\mathbf{b}), \quad (5.11)$$

and use the algebraic properties of the spin operators to express the exponential in Eq. (5.9) in terms that are linear and bilinear in the $\boldsymbol{\sigma}$'s, we obtain the relation[7]

$$M(\mathbf{q},k,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = \frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})[\Gamma_c(\mathbf{b}) + \Gamma_\sigma(\mathbf{b})\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + \Gamma_R(\mathbf{b})\boldsymbol{\sigma}_1\cdot(\mathbf{b}\times\mathbf{k})\boldsymbol{\sigma}_2\cdot(\mathbf{b}\times\mathbf{k}) + \Gamma_s(\mathbf{b})(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{b}\times\mathbf{k})]d^{(2)}\mathbf{b}, \quad (5.12)$$

where

$$\Gamma_c(\mathbf{b}) = 1 - \tfrac{1}{4}\exp(i\chi_c(\mathbf{b})) \times (\{\exp[-3i\chi_\sigma(\mathbf{b})] + 3\exp[i\chi_\sigma(\mathbf{b})]\}\cos^2 u(\mathbf{b}) + \{\exp[-3i\chi_\sigma(\mathbf{b})] - \exp[i\chi_\sigma(\mathbf{b})]\}\sin^2 u(\mathbf{b})), \quad (5.13)$$

$$\Gamma_\sigma(\mathbf{b}) = \tfrac{1}{4}\exp[i\chi_c(\mathbf{b})] \times \{\exp[-3i\chi_\sigma(\mathbf{b})] - \exp[i\chi_\sigma(\mathbf{b})]\}, \quad (5.14)$$

$$\Gamma_R(\mathbf{b}) = \exp\{i[\chi_c(\mathbf{b}) + \chi_\sigma(\mathbf{b})]\}\sin^2 u(\mathbf{b}), \quad (5.15)$$

$$\Gamma_s(\mathbf{b}) = -i\exp\{i[\chi_c(\mathbf{b}) + \chi_\sigma(\mathbf{b})]\} \times \sin u(\mathbf{b})\cos u(\mathbf{b}). \quad (5.16)$$

If we recall that these formulas describe small-angle scattering and specialize to the case of potentials with axial symmetry, we find that the scattering matrix may be expressed in the general form

$$M(\mathbf{q},k,\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2) = A(\mathbf{q},k) + B(\mathbf{q},k)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + C(\mathbf{q},k)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{n} + D(\mathbf{q},k)\boldsymbol{\sigma}_1\cdot\hat{n}\boldsymbol{\sigma}_2\cdot\hat{n} + E(\mathbf{q},k)\boldsymbol{\sigma}_1\cdot\hat{q}\boldsymbol{\sigma}_2\cdot\hat{q}, \quad (5.17)$$

where the unit vectors are defined by

$$\hat{q} = (\mathbf{k}-\mathbf{k}')/|\mathbf{k}-\mathbf{k}'|, \quad (5.18)$$

and

$$\hat{n} = \hat{q}\times\hat{K}, \quad (5.19)$$

in which

$$\hat{K} = (\mathbf{k}+\mathbf{k}')/|\mathbf{k}+\mathbf{k}'|. \quad (5.20)$$

The algebraic form of Eq. (5.17) is in fact the most general one the scattering matrix may take when restricted by the assumption of charge symmetry and invariance under rotation, reflection and time reversal.

For the case of the combined spin-orbit and spin-spin interaction of Eq. (5.8) the explicit form taken by the five complex scattering amplitudes is easily shown to be

$$A(\mathbf{q},k)=\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_c(\mathbf{b})\,d^{(2)}\mathbf{b}\,, \tag{5.21}$$

$$B(\mathbf{q},k)=\frac{ik}{2\pi}\int \exp(i\mathbf{q}\cdot\mathbf{b})\Gamma_\sigma(\mathbf{b})\,d^{(2)}\mathbf{b}\,, \tag{5.22}$$

$$C(\mathbf{q},k)=-k\int_0^\infty J_1(qb)\Gamma_s(b)b\,db\,, \tag{5.23}$$

$$D(\mathbf{q},k)=\frac{ik}{2}\int_0^\infty [J_0(qb)-J_2(qb)]\Gamma_R(b)b\,db\,, \tag{5.24}$$

$$E(\mathbf{q},k)=\frac{ik}{2}\int_0^\infty [J_0(qb)+J_2(qb)]\Gamma_R(b)b\,db\,. \tag{5.25}$$

The nucleon total cross section may be seen from the optical theorem, Eq. (5.2), to be determined by the amplitude $A(0,k)$ alone, i.e., we have

$$\sigma_N=(4\pi/k)\,\mathrm{Im}A(0,k)\,. \tag{5.26}$$

We may obtain an expression for the total cross-section defect in terms of the ten complex neutron and proton amplitudes by substituting the expression (5.17) for the nucleon amplitudes into Eq. (5.7) and taking the trace of $P_3M_nM_p$. The result may then be expressed as[7]

$$\begin{aligned}\delta\sigma=-\frac{2}{k^2}\int S(\mathbf{q})\,\mathrm{Re}\{&A_n(\mathbf{q})A_p(-\mathbf{q})+B_n(\mathbf{q})B_p(-\mathbf{q})\\&-\tfrac{4}{3}C_n(\mathbf{q})C_p(-\mathbf{q})+\tfrac{1}{3}B_n(\mathbf{q})[D_p(-\mathbf{q})+E_p(-\mathbf{q})]\\&+\tfrac{1}{3}B_p(\mathbf{q})[D_n(-\mathbf{q})+E_n(-\mathbf{q})]\\&+\tfrac{1}{3}[D_n(\mathbf{q})D_p(-\mathbf{q})+E_n(\mathbf{q})E_p(-\mathbf{q})]\}d^{(2)}\mathbf{q}\,.\end{aligned} \tag{5.27}$$

We note that if the range of the interaction of the incident particle with the neutron and the proton is much smaller than the average neutron-proton separation of the deuteron, Eq. (5.27) reduces to the approximate expression

$$\begin{aligned}\delta\sigma\approx-(4\pi/k^2)\langle 1/r^2\rangle_d\,\mathrm{Re}\{&A_n(0)A_p(0)+B_n(0)B_p(0)\\&+\tfrac{2}{3}[B_n(0)D_p(0)+D_n(0)B_p(0)+D_n(0)D_p(0)]\}\end{aligned} \tag{5.28}$$

which is the spin-dependent counterpart of Eq. (4.30). In deriving this expression we have made use of the relations

$$C(0)=0$$

and

$$D(0)=E(0)\,,$$

which are evident from Eqs. (5.23)–(5.25).

Since there are insufficient high-energy scattering data at present to determine the five complex amplitudes in the nucleon scattering matrix, some further simplifying assumptions will be made in order to obtain an indication of the effect of spin dependence on the deuteron cross section. We shall assume that at high energies the amplitudes B, D, and E may be neglected in comparison with A and C, and further, that

$$A_n(\mathbf{q})=A_p(\mathbf{q})=A(\mathbf{q})\,, \tag{5.29}$$

and

$$C_n(\mathbf{q})=C_p(\mathbf{q})=C(\mathbf{q})\,. \tag{5.30}$$

The cross-section defect then becomes

$$\delta\sigma=-\frac{2}{k^2}\int S(\mathbf{q})\,\mathrm{Re}(A^2-\tfrac{4}{3}C^2)d^{(2)}\mathbf{q}\,. \tag{5.31}$$

The corresponding angular distribution for elastic scattering by nucleons is

$$d\sigma_N(\mathbf{q})/d\Omega=|A(\mathbf{q})|^2+2|C(\mathbf{q})|^2\,, \tag{5.32}$$

and the polarization, P, of the scattered particle is given by

$$(d\sigma_N/d\Omega)P=2\,\mathrm{Re}(AC^*)\,. \tag{5.33}$$

If we assume furthermore that at high energies the real part of A is negligibly small and write the amplitude C as

$$C=|C|e^{i\beta}\,, \tag{5.34}$$

then the cross-section defect may be written as

$$\delta\sigma=\frac{2}{k^2}\int S(\mathbf{q})\frac{d\sigma_N}{d\Omega}[1+Y(\mathbf{q})]d^{(2)}\mathbf{q}\,, \tag{5.35}$$

where

$$Y(\mathbf{q})=-\tfrac{1}{6}(3-2\cos 2\beta)[1-(1-2P^2\csc^2\beta)^{1/2}]\,. \tag{5.36}$$

We note that $-\frac{5}{6}\leq Y(\mathbf{q})\leq 0$, so that the effect of spin dependence, with the assumptions we have employed, is to decrease the magnitude of the cross-section defect. Since the polarization is linear in q for small values of momentum transfer, the contribution of $Y(\mathbf{q})$ to the integral (5.35) is quite small.

APPENDIX A: HIGH-ENERGY SCATTERING AMPLITUDE FOR ASYMMETRIC INTERACTIONS

The wave function of the incident particle at large distances from the scatterer is assumed to be the sum of the incident plane wave and an outgoing spherical wave with elastic-scattering amplitude $f(\theta,\varphi)$. It has the form

$$\exp(i\mathbf{k}\cdot\mathbf{r})+(e^{ikr}/r)f(\theta,\varphi)\,,$$

where $\mathbf{k}$ is the propagation vector of the incident plane wave and $\mathbf{r}$ is the position vector measured from a point in the scatterer. The angle θ is the scattering angle measured with respect to the direction of the incident beam, which we take to be that of the positive z axis, and φ is the azimuthal scattering angle measured with respect to the positive x axis. The elastic-scattering amplitude may be expanded in a series of spherical harmonics which takes the form

$$f(\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_{lm}(\theta,\varphi)\,, \tag{A1}$$

where

$$f_{lm}(\theta,\varphi)=\frac{1}{2ik}(2l+1)(S_{lm}-\delta_{m0})e^{im\varphi}P_l^m(\cos\theta) \tag{A2}$$

and the coefficients S_{lm} are determined by the scattering interaction. In particular, for azimuthally symmetric potentials S_{lm} may be written in the form $\delta_{m0}\exp(2i\delta_l)$ where δ_l is the phase shift for the lth partial wave.

Near the forward direction the associated Legendre function is related to the Bessel function by the asymptotic relation

$$P_l^m(\cos\theta)\sim(-)^m(l+\tfrac{1}{2})^m J_m[(2l+1)\sin\tfrac{1}{2}\theta], \tag{A3}$$

which is due to Macdonald.[27] If we insert Eq. (A3) into Eq. (A2) and replace the Bessel function by its integral representation

$$J_m[(2l+1)\sin\tfrac{1}{2}\theta]=\frac{i^{-m}}{2\pi}\int_0^{2\pi}e^{i(2l+1)\sin\frac{1}{2}\theta\cos\alpha+im\alpha}d\alpha \tag{A4}$$

we obtain the expression

$$f(\theta,\varphi)=\frac{1}{2\pi ik}\int_0^{2\pi}\sum_{l=0}^{\infty}(l+\tfrac{1}{2})e^{2i(l+\frac{1}{2})\sin\frac{1}{2}\theta\cos\alpha}\times[\sum_{m=-l}^{l}i^{-m}(l+\tfrac{1}{2})^m S_{lm}e^{im(\alpha+\varphi-\pi)}-1]d\alpha \tag{A5}$$

for the scattering amplitude. By defining

$$\psi\equiv\alpha+\varphi-\pi \tag{A6}$$

and

$$A(l,\psi)=\sum_{m=-l}^{l}i^{-m}(l+\tfrac{1}{2})^m S_{lm}e^{im\psi} \tag{A7}$$

we may write Eq. (A5) as

$$f(\theta,\varphi)=\frac{1}{2\pi ik}\int_0^{2\pi}\sum_{l=0}^{\infty}(l+\tfrac{1}{2})e^{2i(l+\frac{1}{2})\sin\frac{1}{2}\theta\cos\alpha}d\alpha[A(l,\psi)-1]. \tag{A8}$$

Because the wavelength of the incident particle is assumed to be much smaller than the range of the scattering interaction, the summation over l in the integrand of Eq. (A8) contains many terms and may be accurately approximated by an integral. If we define an impact distance b by the relation $kb=l+\frac{1}{2}$, the integral may be expressed as one carried out over the variable b. It is convenient for this purpose to define the complex phase shift $\chi(b,\psi)$ via the relation $\exp[i\chi(b,\psi)]=A(l,\psi)$. We may then rewrite Eq. (A8) as the approximate expression

$$f(\theta,\varphi)=\frac{ik}{2\pi}\int_0^{2\pi}\int_0^{\infty}e^{2ikb\sin\frac{1}{2}\theta\cos\alpha}[1-e^{i\chi(b,\psi)}]bdbd\alpha \tag{A9}$$

for the scattering amplitude.

By taking α to be the angle that the impact parameter vector, $\mathbf{b}$, makes with the positive x direction and using the small-angle expression

$$(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}=2kb\sin\tfrac{1}{2}\theta\cos(\alpha-\varphi+\pi)$$

we obtain

$$f(\theta,\varphi)=\frac{ik}{2\pi}\int\exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}]\{1-\exp[i\chi(\mathbf{b})]\}d^{(2)}\mathbf{b}, \tag{A10}$$

where the integration is over the plane of impact parameter vectors.

APPENDIX B

A number of applications have been made in the text of the fact that the formulas which describe small-angle scattering in the laboratory system are of essentially the same form as those which describe it in the center-of-mass system. To demonstrate this property we consider the transformation of the expression (2.3) for the scattering amplitude from the center-of-mass system to the laboratory system.

We let γ be the ratio of the incident-particle total energy in the laboratory system (i.e., kinetic energy plus rest energy) to its rest energy, and λ be the ratio of the mass of the incident particle to that of the target particle. We define relativistic factors γ_1 and γ_2 to be

$$\gamma_1=(1+\lambda\gamma)(1+2\lambda\gamma+\lambda^2)^{-1/2},$$

and

$$\gamma_2=(\lambda+\gamma)(1+2\lambda\gamma+\lambda^2)^{-1/2},$$

and use subscripts C and L to denote quantities associated with the center-of-mass and laboratory systems, respectively. The familiar Lorentz transformation from the center-of-mass system to the laboratory system yields[28]

$$k_L=(\gamma_1+\lambda\gamma_2)k_C$$

[27] H. M. Macdonald, Proc. London Math. Soc. **23**, Ser. 2, 220 (1914).

[28] See, for example, K. B. Mather and P. Swan, *Nuclear Scattering* (Cambridge University Press, Cambridge, England, 1958).

and

$$f_L(\mathbf{k}_L',\mathbf{k}_L)=[\sin^2\theta_C+(\gamma_C\cos\theta_C+\lambda\gamma_2)^2]^{3/4}\times(\gamma_1+\lambda\gamma_2\cos\theta_C)^{-1/2}f_C(\mathbf{k}_C',\mathbf{k}_C).$$

For small-angle scattering we may write the latter relation as

$$f_L(\mathbf{k}_L',\mathbf{k}_L)=(\gamma_1+\lambda\gamma_2)f_C(\mathbf{k}_C',\mathbf{k}_C).$$

We also note that since $\mathbf{b}$ is a vector perpendicular to the direction of the incident particle, and since the Lorentz transformation does not affect transverse components of the momentum we have

$$(\mathbf{k}_L'-\mathbf{k}_L)\cdot\mathbf{b}=(\mathbf{k}_C'-\mathbf{k}_C)\cdot\mathbf{b}.$$

Hence for small-angle scattering we may write the scattering amplitude (2.3) in the laboratory system as

$$f_L(\mathbf{k}_L',\mathbf{k}_L)=(\gamma_1+\lambda\gamma_2)\frac{ik_C}{2\pi}\int\exp[i(\mathbf{k}_C-\mathbf{k}_C')\cdot\mathbf{b}]\Gamma(\mathbf{b})\,d^{(2)}\mathbf{b}=\frac{ik_L}{2\pi}\int\exp[i(\mathbf{k}_L-\mathbf{k}_L')\cdot\mathbf{b}]\Gamma(\mathbf{b})\,d^{(2)}\mathbf{b}.$$

The phase shifts are not changed by the transformation to the laboratory system so that the function $\Gamma(\mathbf{b})$ takes the same form in the expression for f_C and f_L. We see, therefore, that Eq. (2.3) represents the scattering amplitude correctly in the laboratory system as soon as the laboratory values of $\mathbf{k}$ and $\mathbf{k}'$ are used to evaluate it.

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Nonleptonic Decays of the Intermediate Vector Meson*

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The nonleptonic decay modes of the intermediate vector meson are discussed using SU_3 symmetry. The decay rates into two pseudoscalar mesons and two baryons are calculated, estimating the effects of the strong interactions. It is found that for a W mass of 2.5 and 3.0 BeV the baryon decay rate is comparable to the leptonic decay rate whereas the meson decays are negligible.

INTRODUCTION

EXPERIMENTS to detect the intermediate vector meson using high-energy neutrino beams have been concerned mainly with the search for the leptonic decay products of the vector meson, after it has been produced by neutrino-nucleon collisions.[1]

The interpretation of the results of these experiments, which places the lower limit of the meson mass at about 2.0 BeV, depends on what one takes for the branching ratio into leptons.[1]

There have been several estimates of the decays into mesons, pseudoscalar and vector.[2–5] These results indicate that the total nonleptonic decay rate is comparable to the leptonic decay rate. However, in the two-body and three-body decays, the full effect of the strong interactions on the decay rates is not included. When the effects of the strong interactions are included in a simple model in the four-pion decay, the decay rate is found to be much smaller than the corresponding two- and three-body decays.[4] Therefore, it is important to attempt to include the strong interactions in all calculations of nonleptonic decay rates.

In this paper, we shall calculate the decay rate into two pseudoscalar mesons and into two baryons, assuming SU_3 symmetry, but breaking the symmetry where it is possible to do so. The effects of the strong interactions will be taken into account, either by the use of a simple model as in the pseudoscalar-meson decays, or by extrapolating experimental form factors, as in the baryon decays.

DECAY INTO TWO PSEUDOSCALAR MESONS

For the interaction Lagrangian density of the weak interactions we take

$$\mathcal{L}(x)=-g[\mathcal{J}_\alpha(x)W_\alpha(x)+\text{H.c.}], \qquad (2.1)$$

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