Consistent Theory of Weak Interactions*

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(Received 8 September 1965)

In this paper we present an S-operator theory of weak interactions which fulfills all requirements of consistency, including unitarity. In an expansion to first order in G the results are identical to those obtained in first-order perturbation theory from the standard Lagrangian formulation of weak interactions, but higher order corrections can be calculated. They are finite apart from a single infinite parameter which can be absorbed in a coupling-constant renormalization. It is shown that the renormalizations of the coupling constants in μ decay and n decay are different. Higher order corrections to e-ν_e scattering are discussed. The complete renormalization of the theory is carried out. One of the main differences from standard Lagrangian theory is that the theory is not crossing symmetric.

1. INTRODUCTION

IT is well known that the $V-A$ theory of weak interactions in its Lagrangian formulation is not renormalizable. Several attempts have been made to recast it into a true Lagrangian field theory yielding finite results for observable quantities to all orders in perturbation theory but the attempts have failed so far in this respect.[1] On the other hand, the answers obtained in first-order perturbation theory are all finite and show excellent agreement with experiments. This has led some authors[2] to define the weak interaction Lagrangian as an "effective Lagrangian" which should be treated only to first order in perturbation theory. This means, of course, that one uses the Lagrangian formalism as a guide line to derive the type of interaction but that one then discards it immediately. It is very tempting to go one step further and drop the Lagrangian formalism altogether. This can be done by formulating the theory of weak interactions as an effective S-operator theory. The effective S operator is then given by

$$S_{\mathrm{eff}}=I-\frac{iG}{\sqrt{2}}\int d^4x\,{:}j^\lambda(x)j_\lambda{}^\dagger(x){:}\,, \qquad (1)$$

where[3]

$$j^\lambda(x)=\bar{\psi}_e(x)\gamma^\lambda(1+\gamma_5)\psi_{\nu_e}(x)+\bar{\psi}_\mu(x)\gamma^\lambda(1+\gamma_5)\psi_{\nu_\mu}(x)\,, \qquad (2)$$

if we confine our attention to purely leptonic processes as we shall do for the most part in this paper. All the field operators obey the *free*-field equations if electromagnetic processes are neglected. Of course, the effective S operator (1) will reproduce all the results of the first-order Lagrangian theory exactly, but the troublesome higher orders are excluded in a trivial way.

The effective S operator of Eq. (1) can only be valid at low energies, of course, because it is not unitary. In this respect, it is plagued by the same difficulties as the first-order Lagrangian theory. For example, the cross section for elastic electron-neutrino scattering as calculated from (1) will increase quadratically with the neutrino energy and at about 300 BeV in the c.m. system it will intersect the upper limit set by unitarity.[2] It should be emphasized that unitarity is a ***purely formal*** requirement in order to have a consistent formalism (conservation of probability). It thus differs in an essential way from physical requirements such as time-reversal invariance or even microcausality which have to be checked experimentally.

We propose here an S-operator theory of weak interactions which, in an expansion to first order in G, coincides with the effective S operator (1), but which also allows for a calculation of higher order effects. The S operator is defined by the relation (3) below.

$$S=\exp\left\{-\frac{iG}{\sqrt{2}}\int d^4x\,{:}j^\lambda(x)j_\lambda{}^\dagger(x){:}\right\}. \qquad (3)$$

It is immediately seen that the only difference with regard to the Lagrangian formulation is the absence of the time-ordering operator in front of the exponential. It is precisely this fact that makes higher order contributions meaningful. As a matter of fact, higher order terms are all finite apart from a single infinite parameter which can be absorbed in a coupling-constant renormalization.

In the Lagrangian formulation, the time-ordering operator is essential to ensure causal propagators, that is, microcausality. We do not feel that its presence is necessary in an S-operator formulation as the concept of propagators does not enter at all. However, the problem of microcausality is an intricate one and will be taken up again in the concluding section. The theory as it is presented here fulfills all requirements of consistency, including unitarity. Its (finite) results can thus be checked experimentally and if they turn out correct, one will have to restrict the postulate of microcausality.

* Supported in part by the U. S. Atomic Energy Commission.
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[1] For example, G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963); **133**, B477 (1964).
[2] For example, T. D. Lee, in *Particle Physics*, CERN-Report 61–30, 1961 (unpublished), or T. D. Lee and C. N. Yang, Phys. Rev. Letters **4**, 307 (1960).
[3] We use the notation of S. S. Schweber, H. A. Bethe, and F. deHoffmann, *Mesons and Fields* (Row, Peterson and Company, Evanston, Illinois, 1955), except that our $\gamma_5=i\gamma_0\gamma_1\gamma_2\gamma_3$.

For a particular matrix element of (3), each term in an expansion with respect to G can be represented formally by a number of graphs. However, these graphs are *not* Feynman graphs as their lines do not correspond to particle propagators. A discussion of the various types of graphs which occur in such an expansion is given in Sec. 2. In Sec. 3 it will be shown that for decay processes higher order corrections only contribute to a renormalization of the coupling constant. Section 4 will be devoted to electron-neutrino scattering as a typical example for scattering processes and in Sec. 5, the general renormalization of the theory will be discussed.

2. THE STRUCTURE OF THE S OPERATOR

In order to study the type of contributions to a matrix element of S, (3) will be taken between two general states for which the total number of particles is four.

$$\langle\alpha|S|\beta\rangle=\langle\alpha|\exp\left\{\frac{G}{i\sqrt{2}}\int d^4x[:\bar{\psi}_\mu\gamma^\lambda(1+\gamma_5)\psi_{\nu_\mu}\times\bar{\psi}_{\nu_e}\gamma_\lambda(1+\gamma_5)\psi_e:+\text{H.c.}]\right\}|\beta\rangle+(\text{similar terms with }\mu\text{ or }e\text{ only}). \quad (4)$$

Actual calculations are much simplified if one applies the following generalized Fierz transformation (see Appendix A):

$$\bar{\psi}_a\gamma^\lambda(1+\gamma_5)\psi_b\bar{\psi}_c\gamma_\lambda(1+\gamma_5)\psi_d=-2\bar{\psi}_a(1-\gamma_5)\mathcal{C}\bar{\psi}_c{}^T\psi_d{}^T\mathcal{C}^{-1}(1+\gamma_5)\psi_b. \quad (5)$$

Here, "transpose" refers only to spinor space and $\mathcal{C}$ is the familiar unitary (4×4) matrix with the following properties:

$$\mathcal{C}^{-1}\gamma_\mu\mathcal{C}=-\gamma_\mu{}^T,\quad \mathcal{C}^{-1}\gamma_5\mathcal{C}=\gamma_5{}^T, \quad (6)$$

$$\bar{u}\mathcal{C}^T=v^T,\quad \mathcal{C}^T=-\mathcal{C}, \quad (7)$$

and

$$\psi^C(x)=C\psi(x)C^{-1}=\mathcal{C}\bar{\psi}^T(x), \quad (8)$$

where $\psi^C(x)$ is the charge-conjugated spinor corresponding to $\psi(x)$.

It was explained in the Introduction that to first order in G the contribution S_1 to the matrix element (4) coincides with the conventional one obtained in the Lagrangian approach. Equation (4) can be rewritten in the following way:

$$\langle\alpha|S|\beta\rangle=S_1-2G^2\int d^4xd^4y\langle\alpha|:\bar{\psi}_a(x)(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_b}{}^T(x)\times\psi_b{}^T(x)\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_a}(x):Q:\bar{\psi}_c(y)(1-\gamma_5)\times\mathcal{C}\bar{\psi}_{\nu_d}{}^T(y)\psi_d{}^T(y)\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_c}(y):|\beta\rangle, \quad (9)$$

where a, b, c, and d stand for μ or e, depending on what kind of particles the states α and β represent. In general, several combinations will contribute and one has to sum over all of them. The operator Q is given by

$$Q=\frac{1}{2!}+\frac{i\Omega}{3!}+\frac{(i\Omega)^2}{5!}+\cdots=(i\Omega)^{-2}(S-I-i\Omega), \quad (10)$$

where S is the S operator, I is the identity operator, and Ω is defined by

$$\Omega=\sqrt{2}G\int d^4x:\{\bar{\psi}_\mu(x)(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_\mu}{}^T(x)\psi_\mu{}^T(x)\mathcal{C}^{-1}\times(1+\gamma_5)\psi_{\nu_\mu}(x)+\bar{\psi}_\mu(x)(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_e}{}^T(x)\psi_e{}^T(x)\times\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_\mu}(x)+\text{H.c.}+\bar{\psi}_e(x)(1-\gamma_5)\times\mathcal{C}\bar{\psi}_{\nu_e}{}^T(x)\psi_e{}^T(x)\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_e}(x)\}:. \quad (11)$$

To evaluate (9) it is important to notice that if "final-state interactions" are neglected[4] one may consider the particles of the initial states $|\beta\rangle$ annihilated at the point y and the final-state particles of $|\alpha\rangle$ created at the point x. The remaining field operators acting at these two points are then "contracted" into Q. Of course, the particular choice of the points x and y is arbitrary, but it can easily be shown that different choices amount to a relabelling of the space-time arguments of the field operators and that the result is independent of such a relabelling, as one would expect. This situation does not occur in conventional field theory because of the time-ordering there. The application of Wick's theorem then leads to a sum over all permutations of the space-time points at which interactions take place. This property of conventional field theory leads to crossing symmetry and it then follows that the theory presented here will *not be crossing symmetric.*

After the field operators which create and destroy the particles in the initial and final state are taken out, one is left with a vacuum expectation value of the following general structure:

$$\langle 0|:\bar{\psi}_{a_1}{}^{\alpha_1}(x)\cdots\psi_{a_n}{}^{\alpha_n}(x):Q:\bar{\psi}_{b_1}{}^{\beta_1}(y)\cdots\psi_{b_m}{}^{\beta_m}(y):|0\rangle,\quad n,m<4, \quad (12)$$

where the number and type of field operators occurring in (12) depends on the states $\langle\alpha|$ and $|\beta\rangle$. A vacuum expectation value of this form can always be represented as a sum of terms each of which is a product of S^+ functions (or S^- functions), for which the following

[4] By final-state interaction we refer to the fact that in μ decay the electron and the ν_e may rescatter. For other processes all initial particles are annihilated at one point and the same is true for the final-state particles.

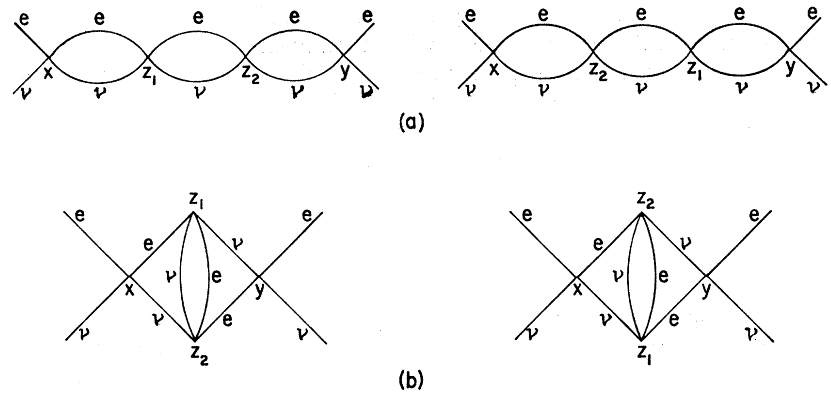


FIG. 1. Fourth order graphs for e-ν_e scattering.

relations hold:

$$
\begin{aligned}
S_{\alpha\beta}^{+}(x-y;m) & \\
&=-(i\gamma^{\mu}\partial_{\mu}+m)_{\alpha\beta}\Delta^{+}(x-y;m) \\
&=\frac{i}{(2\pi)^3}\int d^4k\, e^{-ik(x-y)}\Theta(k^0)\delta(k^2-m^2)(\boldsymbol{k}+m)_{\alpha\beta} \quad (13)\\
&=-i\langle 0|\psi_{\alpha}(x)\bar{\psi}_{\beta}(y)|0\rangle,
\end{aligned}
$$

and

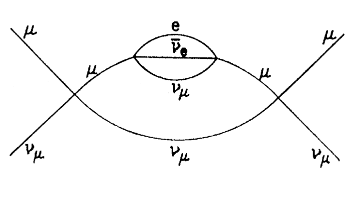
$$
\begin{aligned}
S_{\beta\alpha}^{-}(y-x;m)&=-i\langle 0|\bar{\psi}_{\alpha}(x)\psi_{\beta}(y)|0\rangle \\
&=-i(\gamma^{\mu}\partial_{\mu}+im)_{\beta\alpha}\Delta^{+}(x-y;m). \quad (14)
\end{aligned}
$$

A graphical description of the expansion (9) is then obtained by representing each function $iS^{\pm}(x-y)$ by a solid line from x to y. It should be emphasized again that these graphs are *not* Feynman graphs. For instance, all the lines correspond to particles *on* the mass shell and they do *not* describe the propagation of virtual (nor real) particles.

At each "vertex," 4 lines have to emerge. Figure 1 gives as an example all fourth-order graphs contributing to e-ν_e scattering. Since the 2 points at which initial and final particles emerge are fixed, each nth-order graph occurs exactly $(n-2)!$ times. Thus, the $n!$ from the exponential is not exactly cancelled. As a consequence, a summation of certain chains of diagrams will not lead to the familiar geometrical series but to a so-called Saalschützian series.

Owing to the fact that all lines correspond to particles on the mass shell, no vacuum graphs will occur. For the same reason graphs of the type shown in Fig. 2 do not occur in e-ν_e scattering.

FIG. 2. A graph contributing to μ-ν_μ scattering, but not to e-ν_e scattering.



For practical purposes, rules for calculations of diagrams (analogous to the Feynman rules in the usual theory) are collected in Table I. Only those factors are given which differ from standard Feynman rules. However, it will become clear in Appendices B and D that it is often more economical to start from the Fierz-transformed S operator as given in Eq. (9); for that form of the S operator graphical rules become cumbersome and a direct computation is preferable.

TABLE I. Diagrammatic rules for an nth-order graph.

from the expansion	$\frac{1}{n!}\left[\frac{-iG}{\sqrt{2}}\right]^n$
from the multiplicity	$(n-2)!$ if $n>2$
at each "vertex"	$[\gamma_\lambda(1+\gamma_5)]_{\alpha\beta}[\gamma^\lambda(1+\gamma_5)]_{\gamma\delta}$
for each particle line	$iS^{+}(x_1-x_k)$
for each antiparticle line	$iS^{-}(x_1-x_k)$

3. THE DECAY OF THE MUON

Following the general rules previously derived in Sec. 2 and neglecting e-ν_e scattering in the final state, the most general graph contributing to μ decay is easily seen to be given by Fig. 3. From the mere structure of this graph one can infer that one will encounter the same difficulties as in the calculation of wave-function renormalization in standard quantum electrodynamics[5] because both μ lines connected to the "black box" have the four-momentum of a free particle, i.e., $p^2=m^2$ and the internal line will thus contribute a

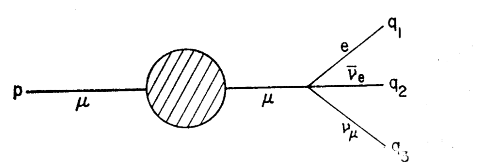


FIG. 3. The most general graph for μ decay.

[5] G. Lüders, Z. Naturforsch. **7a**, 206 (1952) or J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (North-Holland Publishing Company, Amsterdam, 1956), p. 221.

factor $\delta(0)$. In quantum electrodynamics this problem is of a formal nature[5] and can easily be circumvented, for example, by making explicit use of the adiabatic hypothesis. In an S-operator formulation this is not easily possible and an undetermined constant will remain, which is infinite in the limit of sharply defined masses.

Equation (9) specialized to μ decay reads

$$\langle q_1q_2q_3|S|p\rangle = \frac{-i\sqrt{2}G}{(2\pi)^6}\left[\frac{m_e m_\mu m_\nu^2}{E_e E_\mu E_{\nu_e} E_{\nu_\mu}}\right]^{1/2} \bar{u}_e(\bar{q}_1)(1-\gamma_5)v_{\nu_\mu}(\bar{q}_3) \times\sum_{\alpha\beta} u_\mu{}^\alpha(\bar{p})[\mathfrak{C}^{-1}(1+\gamma_5)v_{\nu_e}(\bar{q}_2)]^\beta \times\int d^4x d^4y\, e^{ix(q_1+q_2+q_3)-ipy} \times\{\delta_{\alpha\beta}\delta^{(4)}(x-y)+R_{\beta\alpha}(x-y)\}, \quad (15)$$

where the first term in the curly bracket gives the standard first order S-matrix element for μ decay and

$$R_{\beta\alpha}(x-y) = -i\sqrt{2}G\langle 0|\psi_\mu{}^\beta(x)Q'{:}\bar{\psi}_e(y)(1-\gamma_5)\mathfrak{C}\bar{\psi}_{\nu_\mu}{}^T(y) \times[\mathfrak{C}^{-1}(1+\gamma_5)\psi_{\nu_e}(y)]^\alpha{:}|0\rangle. \quad (16)$$

Since only odd orders of G contribute to μ decay, Q' is given by

$$Q' = \frac{i\Omega}{3!}+\frac{(i\Omega)^3}{5!}+\cdots = i\Omega^{-1}[1-\Omega^{-1}\sin\Omega]. \quad (17)$$

For normalization purposes, the neutrinos have been given a small mass which will be set equal to zero in all final results.

Defining the Fourier transform of $R_{\alpha\beta}(x-y)$ by

$$R_{\alpha\beta}(x-y) = \frac{1}{(2\pi)^4}\int d^4k\, e^{-ik(x-y)}\tilde{R}_{\alpha\beta}(k). \quad (18)$$

Equation (15) takes on the simple form

$$\langle q_1q_2q_3|S|p\rangle = -i\sqrt{2}G(2\pi)^{-2}\delta^{(4)}(q_1+q_2+q_3-p)\left[\frac{m_e m_\mu m_\nu^2}{E_e E_\mu E_{\nu_e} E_{\nu_\mu}}\right]^{1/2} \times\bar{u}_e(\bar{q}_1)(1-\gamma_5)v_{\nu_\mu}(\bar{q}_3)[\bar{u}_{\nu_e}(\bar{q}_2)(1+\gamma_5)]^\alpha u_\mu{}^\beta(\bar{p}) \times\{\delta_{\alpha\beta}+\tilde{R}_{\alpha\beta}(p)\}, \quad (19)$$

where Eqs. (6) and (7) have been used to get rid of the matrix $\mathfrak{C}$. (Summation over α and β is understood.) As $\tilde{R}_{\alpha\beta}(k)$ depends only on a single four-vector, its most general form will be

$$\tilde{R}_{\alpha\beta}(k) = [(a+b\mathbf{k})(1+c\gamma_5)]_{\alpha\beta}R, \quad (20)$$

where a, b, c, and R are certain constants. The γ algebra in Eq. (19) is now easily carried out with the following result:

$$[\bar{u}_{\nu_e}(\bar{q}_2)(1+\gamma_5)]^\alpha u_\mu{}^\beta(\bar{p})\tilde{R}_{\alpha\beta}(p) = \bar{u}_{\nu_e}(\bar{q}_2)(1+\gamma_5)u_\mu(\bar{p})R(a+m_\mu b+ac-m_\mu bc), \quad (21)$$

so that the total effect of higher order corrections to μ decay (neglecting final-state interactions) can be expressed through a renormalization of the coupling constant.

$$G_\mu = G\{1+[a(1+c)+m_\mu b(1-c)]R\}. \quad (22)$$

μ decay is the only example of a purely leptonic process. The essentials of the calculation above do not depend on what kind of particles are involved and, therefore, the results can be taken over to semileptonic decays of baryons with only trivial modifications.

A calculation of $\tilde{R}_{\alpha\beta}(k)$ to order G^2 is carried out in Appendix B with the result

$$\tilde{R}_{\alpha\beta}(k) = -[(m_\mu+\mathbf{k})(1+\gamma_5)]_{\alpha\beta}\frac{G^2m_\mu{}^5}{576\pi^2}\Lambda + O(m_e{}^2/m_\mu{}^2)+O(G^4), \quad (23)$$

where

$$\Lambda = \lim_{p^2\to m_\mu{}^2}\delta(p^2-m_\mu{}^2) \quad (24)$$

is the formally infinite constant, entering the theory.

The fact that Λ is infinite comes from the fact that the internal μ line in Fig. 3 has been given a sharp mass m_μ. Actually, the mass of the μ has a certain width of the order $1/\tau_\mu$ (τ_μ being the μ lifetime) because the muon is an unstable particle. It thus seems reasonable to replace the sharp δ function in (24) by a Lorentzian of width $1/\tau_\mu$:

$$\Lambda = \lim_{p^2\to m_\mu{}^2}\frac{1}{\pi}\frac{\tau_\mu{}^2}{1+\tau_\mu{}^4(p^2-m_\mu{}^2)^2} = \frac{\tau_\mu{}^2}{\pi}. \quad (25)$$

If this is the right value for Λ it leads to $m_\mu R = -1.2\times10^{17}$ in Eq. (22) so that a lowest order calculation is totally unreliable. But the summation of chain diagrams shown in Fig. 4 is easily achieved. In coordinate space, a graph with n bubbles in the chain is related to the one with $(n-1)$ bubbles through the simple iteration formula

$$R_{\alpha\beta}{}^{(n)}(x-y) = (-1)^{n+1}\frac{[(2n-1)!]^2\times3!}{(2n+1)!(2n-3)!}\int d^4z \times R_{\alpha\gamma}{}^{(n-1)}(x-z)R_{\gamma\beta}{}^{(1)}(z-y), \quad (26)$$

where $R_{\gamma\beta}{}^{(1)}(z-y)$ is, of course, the Fourier transform of (23). The factor in front of the integral stems from

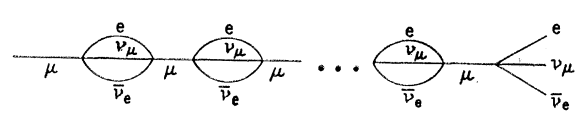


FIG. 4. Chain diagram for μ decay.

the fact, that the endpoints of each graph (of nth order, say) are fixed so that its multiplicity is $(n-2)!$. As pointed out in Sec. 2, this leads to a Saalschützian series. In this particular case Eq. (22) reads

$$G_\mu=G\left\{1-\sum_{\nu=1}^{\infty}\frac{(2\nu-1)!}{(2\nu+1)!}x^\nu\right\}, \tag{27}$$

where

$$x=\frac{G^2m_\mu{}^6}{48\pi^2}\Lambda. \tag{28}$$

The summation of the Saalschützian series in (27) is carried out in Appendix C, leading to

$$G_\mu=G\left\{\frac{1}{2\sqrt{x}}\ln\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right|+\tfrac{1}{2}\ln|1-x|\right\}; \tag{29}$$

with Λ given by (25) this gives $G_\mu=20.9G$. This is a much smaller renormalization than the one derived above from the lowest order calculation, but it is still very large. However, it will be shown in Sec. 4 that a simple summation of chain diagrams is still a very poor approximation (at least in e-ν_e scattering) in the sense that it still increases quadratically with ν momentum at high energies. Thus, one is tempted to conclude that the full expression for $\tilde{R}_{\alpha\beta}(k)$ inserted in (22) will give a substantially smaller renormalization. In any case, the important conclusion so far of a qualitative nature is that there exists a non-negligible renormalization of the coupling constant in μ decay due to higher order weak corrections.

It hardly needs to be said that the conclusion above has important implications for the conserved-vector-current theory and the persistent small difference between G_μ and G_β.[6] For that reason we next consider the decay of the free neutron neglecting effects of strong interactions. Of course the calculation is similar to that of μ decay but the operator Ω in Eq. (17) has to include the term [cf. Eq. (11)]

$$\Omega'=\sqrt{2}G\int d^4x:\bar{\psi}_p(x)(1-\gamma_5)$$
$$\times\mathcal{C}\bar{\psi}_e{}^T(x)\psi_{\nu_e}{}^T(x)\mathcal{C}^{-1}(1+\gamma_5)\psi_n(x):+\text{H.c.} \tag{30}$$

The lowest order contribution to $\tilde{R}_{\alpha\beta}'(k)$ for n decay is shown in Fig. 5. Details of the calculation are again found in Appendix B. The result is

$$\tilde{R}_{\alpha\beta}'(k)=-[(M_n+k)(1+\gamma_5)]_{\alpha\beta}\frac{G^2m_e{}^5}{6\pi^2}\lambda\Lambda', \tag{31}$$

with

$$\lambda=\xi\ln|\xi+(\xi^2-1)^{1/2}|$$
$$+(1/15)(2\xi^4-9\xi^2-8)(\xi^2-1)^{1/2}\approx7.8, \tag{32}$$

$$\xi=(M_n-M_p)/m_e\approx2.6. \tag{32'}$$

[6] C. S. Wu, Rev. Mod. Phys. 36, 618 (1964).

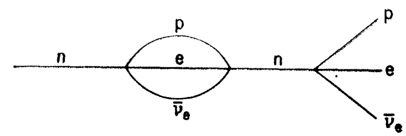


FIG. 5. Lowest order weak correction to n decay.

Λ' is defined similarly to Λ, Eq. (24), with $m_\mu{}^2$ replaced by $M_n{}^2$. Because of similar arguments to those presented after Eq. (24), the most natural value for the parameter Λ' is

$$\Lambda'=\tau_n{}^2/\pi, \tag{33}$$

where τ_n is the neutron lifetime. The summation of chains now leads to [cf. Eq. (29)]

$$G_\beta=G\left\{\frac{1}{2\sqrt{x'}}\ln\left|\frac{1+\sqrt{x'}}{1-\sqrt{x'}}\right|+\tfrac{1}{2}\ln|1-x'|\right\}, \tag{34}$$

where

$$x'=(2G^2/\pi^2)\lambda M_nm_e{}^5\Lambda'. \tag{35}$$

The quantitative information of Eq. (34) is of no particular interest for the same reason as before when μ decay was discussed. The qualitative conclusion, however, that there is a *different* weak renormalization of the coupling constants in μ decay and n decay is of the utmost importance. This difference in G_μ and G_β may very well be as large as a few percent and may thus account for the persistent experimental difference.

4. ELECTRON-NEUTRINO SCATTERING

There are several scattering processes involving only leptons. The e-ν_e scattering provides a typical example for all of them and it will be the only scattering process studied here. It was pointed out in the Introduction that the most interesting feature of this process is the behavior of the elastic scattering cross section at high energies. For simplicity only the last term in Ω [Eq. (11)] will be considered here; that is, muons and their neutrinos will be totally neglected. This does not affect the results in any essential way.

For scattering of an electron of four-momentum p_1 on a neutrino of four-momentum q_1 into an electron neutrino pair with four-momenta p_2 and q_2, the S-matrix element is given by [cf. Eq. (9)]

$$\langle p_2q_2|S|p_1q_1\rangle$$
$$=\delta_{fi}-\frac{i\sqrt{2}G}{(2\pi)^6}\frac{m_em_\nu}{[p_1{}^0p_2{}^0q_1{}^0q_2{}^0]^{1/2}}\bar{u}_e(\bar{p}_2)(1-\gamma_5)$$
$$\times v_\nu(\bar{q}_2)\bar{v}_e(\bar{p}_1)(1+\gamma_5)u_\nu(\bar{q}_1)$$
$$\times\int d^4xd^4y\,e^{ix(p_2+q_2)-iy(p_1+q_1)}$$
$$\times\{\delta^{(4)}(x-y)+K(x-y)\}, \tag{36}$$

where

$$K(x-y)=i\sqrt{2}G\langle 0|\psi_e^T(x)\mathcal{C}^{-1}(1+\gamma_5)\times\psi_\nu(x)Q\bar{\psi}_e(y)(1-\gamma_5)\mathcal{C}\bar{\psi}_\nu^T(y)|0\rangle \quad (37)$$

and Q is defined in Eq. (10). Again, the term with the δ function in the curly bracket represents the first-order contribution, being identical to that of standard Lagrangian field theory.

Introducing the Fourier transform of $K(x-y)$,

$$K(x-y)=\frac{1}{(2\pi)^4}\int d^4k\, e^{-ik(x-y)}\tilde{K}(k)\,, \quad (38)$$

reduces Eq. (36) to

$$\langle p_2q_2|S|p_1q_1\rangle =\delta_{fi}-\frac{i\sqrt{2}G}{(2\pi)^2}\frac{m_em_\nu}{[p_1^0p_2^0q_1^0q_2^0]^{1/2}}\delta^{(4)}(p_2+q_2-p_1-q_1) \times\bar{u}_e(\bar{p}_2)(1-\gamma_5)v_\nu(\bar{q}_2)\bar{v}_e(\bar{p}_1)(1+\gamma_5)u_\nu(\bar{q}_1) \times\{1+\tilde{K}(p_1+q_1)\}\,, \quad (39)$$

from which the spin-averaged differential elastic cross section is readily obtained:

$$\frac{d\sigma}{d\Omega}=\frac{G^2}{2\pi^2}\frac{(E+q)q^2}{E}|1+\tilde{K}(s)|^2. \quad (40)$$

Here E and q are electron and neutrino energies in the center-of-momentum system. Furthermore,

$$s=(p_1+q_1)^2 \quad (41)$$

is the total energy in this system. (For invariance reasons, K depends only on s.)

Up to third order in G, only bubble diagrams of the type shown in Fig. 1(a) contribute. The calculation is carried out in Appendix D with the following result:

$$\tilde{K}(k)=-i\frac{\sqrt{2}G}{4\pi}\frac{(k^2-m_e^2)^2}{k^2}-\frac{G^2}{12\pi^2}\frac{(k^2-m_e^2)^4}{k^4}+O(G^3), \quad (42)$$

This leads to the following cross section:

$$\frac{d\sigma}{d\Omega}=\frac{G^2q^2}{\pi^2}-\frac{2}{3}\frac{G^4q^6}{\pi^4}+O(m_e^2/q^2)+O(G^6). \quad (43)$$

Because that term of Eq. (42) which is linear in G is purely imaginary, there is no interference term of order G^3 in the cross section (43). The cross section as given by Eq. (43) is plotted in Fig. 6. It is seen that there is practically no deviation from the first-order result until the unitarity limit is approached. At that point the cross section has a maximum and then it decreases sharply. Unfortunately it is in this energy region that all the higher order terms become important. In fact, the cross section (43) even becomes negative for sufficiently large q values. There is thus a need to include higher order terms in the expansion of $\tilde{K}(k)$. The easiest thing to do is again to sum the chains of bubble diagrams. This leads to the following Saalschützian series (see Appendix C):

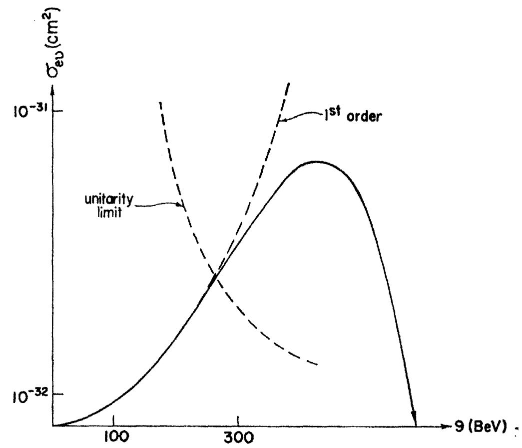


FIG. 6. The e-ν_e scattering cross section to order G^4.

$$1+\tilde{K}\approx\sum_{\nu=0}^{\infty}\frac{(\nu-1)!}{(\nu+1)!}x^\nu=2+\frac{1-x}{x}\ln(1-x)\,, \quad (44)$$

with

$$x=-i2\sqrt{2}Gq^2/\pi\equiv-i\xi\,. \quad (45)$$

Equation (44) inserted in Eq. (40) yields

$$\frac{d\sigma}{d\Omega}=\frac{G^2}{\pi^2}q^2\left\{4+\frac{\xi^2+1}{\xi^2}[\tfrac{1}{4}\ln^2(1+\xi^2)+\arctan^2\xi] -2\ln(1+\xi^2)-\frac{4}{\xi}\arctan\xi\right\}, \quad (46)$$

with ξ defined by (45). For large q, (46) diverges as $q^2\ln^2(Gq^2)$ and hence worse than the lowest order term. This proves nothing but the crudeness of the chain approximation.

In an attempt to go one step further, one may calculate the fourth-order graphs of Fig. 1(b). This is done in Appendix D. One can then easily sum up diagrams of the type shown in Fig. 7 with the following result:

$$\tilde{K}\approx\sum_{\nu=1}^{\infty}\frac{(3\nu-1)!}{(3\nu+1)!}z^\nu, \quad (47)$$

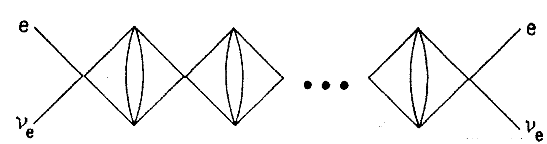


FIG. 7. Diagrams summed for e-ν_e scattering.

with

$$z=-i\frac{\sqrt{2}G^3}{3}\frac{(2q)^6}{(2\pi)^3}\equiv(i\alpha)^3. \tag{48}$$

The summation is carried out in Appendix C, leading to

$$\begin{aligned}K\approx 1+\frac{1}{3\alpha}\left[\frac{\sqrt{3}}{2}\ln\frac{[1-\alpha^2+\alpha^4]^{1/2}}{1+\alpha\sqrt{3}+\alpha^2}-\tfrac{1}{2}\arctan\frac{3\alpha(1-\alpha)}{1-2\alpha-\alpha^2}\right]\\ -\tfrac{1}{6}\ln(1+\alpha^6)-\frac{i}{3\alpha}\left[\tfrac{1}{2}\ln\frac{1+\alpha^2}{[1-\alpha^2+\alpha^4]^{1/2}}\right.\\ \left.+\sqrt{3}\frac{\pi}{6}-\frac{\sqrt{3}}{2}\arctan\frac{\sqrt{3}}{1-2\alpha^2}\right]-\tfrac{1}{3}i\arctan\alpha^3. \end{aligned}\tag{49}$$

$\tilde{K}$ as given by (49) diverges at large q as badly as (44). The conclusion one draws from this fact is that the interference terms between different classes of diagrams will be important to depress the cross section below the unitarity limit. Therefore, in order to obtain the cross section accurately enough for comparison with future ultra-high-energy neutrino experiments, one will have to endeavor the task of computing all diagrams of each order step by step up to a sufficiently high order of Gq^2.

5. RENORMALIZATION

In Sec. 3 it has been shown that higher order corrections to decay processes lead to a renormalization of the coupling constant. This renormalization depends upon the masses of the particles involved. It has been discussed in detail in Sec. 3 and we have no further comment on that.

There is, however, another type of renormalization connected with internal lines of unstable particles. A typical example is shown in Fig. 2. It is also recalled that all particles are on the mass shell so that this type of correction to an internal line can only occur if it represents an unstable particle. The three particles in the "bubble" must form a possible decay system of the "mother particle."

The correction to the internal μ line as shown in Fig. 2 looks like a self-energy correction. It will be shown, however, that it does not lead to a mass renormalization but rather to a new kind of coupling constant renormalization. (As the lines do not correspond to particle propagators, an interpretation in terms of a mass renormalization seems impossible.)

An internal μ line, say, is represented by a function $S_{\alpha\beta}^+(x-y, m_\mu)$. Including only the lowest order correction shown in Fig. 2, the "renormalized line" will be represented by

$$S_{\alpha\beta}'^+(x-y, m_\mu)=S_{\alpha\beta}^+(x-y, m_\mu)+\tfrac{2}{3}G\, I_{\alpha\beta}(x-y)\,, \tag{50}$$

where

$$\begin{aligned}I_{\alpha\beta}(x-y)=-\int d^4z_1d^4z_2\left[\left(\gamma\frac{\partial}{\partial x}-im_\mu\right)(1-\gamma_5)\gamma^\lambda(1+\gamma_5)\left(\gamma\frac{\partial}{\partial y}-im_\mu\right)\right]_{\alpha\beta}\\ \times\left\{\mathrm{Tr}\left[\gamma^\sigma(1+\gamma_5)\left(\gamma\frac{\partial}{\partial z_1}-im_e\right)(1-\gamma_5)\right]\Delta^+(z_1-z_2, m_e)\right\}\\ \times\Delta^+(x-z_1, m_\mu)\Delta^+(z_2-y, m_\mu)\Delta_{;\lambda}^+(z_1-z_2, 0)\Delta_{;\sigma}^+(z_1-z_2, 0)\end{aligned} \tag{51}$$

and

$$\Delta_{;\mu}^+(x,m)=\frac{\partial}{\partial x^\mu}\Delta^+(x,m) \tag{52}$$

The computation of the integral (51) follows the standard pattern outlined in Appendix B. Inserting the result in Eq. (50), one obtains

$$\begin{aligned}S_{\alpha\beta}'^+(x-y, m_\mu)\\ =S_{\alpha\beta}^+(x-y, m_\mu)\left\{1+\frac{G^2m_\mu^6}{144\pi^2}\Lambda+O(m_e^2/m_\mu^2)\right\},\end{aligned} \tag{53}$$

where Λ is defined in (24). As the particular structure of the bubble does not enter the calculations in an essential way, one arrives at the very important conclusion that self-energy-like graphs in internal lines change the function representing the line S^+ by a multiplicative factor. It has been noted above that an interpretation of that factor as mass renormalization is unphysical. It is not even possible formally, since (see Appendix E)

$$\begin{aligned}\Delta^+(x, m^2+\delta m^2)\\ =\Delta^+(x,m^2)\left\{1+\frac{\delta m^2}{m^2}\right\}-\frac{\delta m^2}{m^2}\frac{x^2}{4}\int_0^{m^2}ds\,\Delta^+(x,s)\end{aligned} \tag{54}$$

and the last term in Eq. (54) is missing in Eq. (53). But an interpretation as coupling-constant renormalization is straightforward. Each internal line is connected to 2 vertices and the square root of the renormalizing factor in Eq. (53) can be absorbed into each of the 2 coupling constants, thereby leading to a coupling-constant renormalization. The new feature is that the vertices connected to external lines will have a different renormalization because in scattering processes, external lines are not renormalized. (Remember that all initial lines emerge from the same point and similarly all the final lines emerge from a single point.) In order to be able to renormalize the theory from the very beginning, one has to compensate for this fact by renormalizing the state vectors as well.

Define a renormalization constant Y for each particle through

$$S_{\alpha\beta}'^{+}(x,m_a)=Y_a S_{\alpha\beta}^{+}(x,m_a) \tag{55}$$

so that, for example,

$$Y_\mu=1+\frac{G^2m_\mu^6}{144\pi^2}\Lambda+O(m_e^2/m_\mu^2)+O(G^4). \tag{56}$$

Y_e and Y_ν are, of course, unity because these particles are stable. The whole theory can now be formulated in a renormalized way with (11) replaced by

$$\begin{aligned}\Omega_r=\sqrt{2}\int d^4x:\{&(Y_\mu G)\bar{\psi}_\mu(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_\mu}{}^T\psi_\mu{}^T\mathcal{C}^{-1}(1+\gamma_5)\psi_\nu\\&+(Y_\mu{}^{1/2}G)\bar{\psi}_\mu(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_e}{}^T\psi_e{}^T\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_\mu}+\text{H.c.}\\&+G\bar{\psi}_e(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_e}{}^T\psi_e{}^T\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_e}\}:,\end{aligned} \tag{57}$$

and the following renormalization of state vectors

$$|a\rangle\rightarrow|a\rangle_r=Y_a{}^{-1/2}|a\rangle. \tag{58}$$

If calculations are made with (57) and (58), all graphs with internal line renormalization parts (self-energy-like parts on internal lines) have to be dropped.

It should be pointed out that e-ν_e scattering provides a means for measuring the ("bare") coupling constant G. This is so because at energies far below the unitarity limit higher order corrections can be neglected, thus leaving the coupling constant unchanged. Hence all renormalization constants Y_a are measurable quantities. If the interpretation (25) is adopted, they all are finite.

6. CONCLUSION

In the preceding sections, an S-operator theory of weak interactions has been presented; it yields finite answers and fulfills all mathematical requirements, in particular unitarity of the S operator. The question is how this theory may be tested experimentally. Of course, the most direct test would be e-ν_e scattering (or possibly $\nu_\mu+n\rightarrow\mu+p$) at ultra-high energies. Since no appreciable deviations from the first-order term are predicted by the theory until one almost reaches the unitarity limit at about 300 BeV in the c.m. system, this check has to wait until ultra-high energies are accessible.

Fortunately, there are other predictions of the theory, which can be tested at low energies. These are the differences in coupling constants. It would be premature to regard the difference of G_μ and G_β as a success of the theory but if a refined measurement of the coupling constant in e-ν_e scattering or even ordinary μ capture[7] shows a definite deviation from both G_μ and G_β then that may be taken as evidence in favor of the theory.

There are many other processes in which the theory can be checked. These are mainly processes where strongly interacting particles participate. The most obvious check of the theory in this field is of course the K_1^0-K_2^0 mass difference, the only measured quantity proportional to G^2. Other processes forbidden to lowest order such as e-ν_μ scattering or ν_μ-p scattering can eventually provide means to check the theory.

All this can be summarized by saying that this is a theory that allows one to calculate every possible process. Such a theory is bound to lead either to success or to failure.

A final word about causality is in order. It has been pointed out in the Introduction that the fact that internal lines are not represented by causal propagators is not disturbing. The graphical analysis of S-matrix elements is purely formal and does not represent any virtual process developing in time. On the other hand, it is often stated that analytic properties of the S-matrix elements are consequences of microcausality. A check on the lowest order diagrams shows that they have fewer singularities than the analogous ones in standard Lagrangian field theory. (Note that the G^2 diagram in e-ν_e scattering happens to be the imaginary part of the corresponding Feynman diagram in the conventional theory.) Initially, all matrix elements are only defined in their physical region as they carry a Θ function which restricts them to this region [see Appendix D, Eq. (D2)]. But an analytic continuation is trivially achieved. Though it seems that fewer singularities occur, the theory is based on a unitary S operator and all singularities required by unitarity are necessarily present. There is of course the possibility that complicated diagrams introduce singularities incompatible with microcausality. (Further elaboration on this point might eventually be desirable.) If this is not so, the matrix elements of this theory satisfy ordinary dispersion relations. However, it should be repeated that the theory is not crossing symmetric.

APPENDIX A: FIERZ TRANSFORMATIONS

In this Appendix we derive the generalized Fierz transformation of Eq. (5) in the text. To this end it is convenient to start from the well-known Fierz identities[8]

$$\delta_{\alpha\sigma}\delta_{\rho\beta}=\tfrac{1}{4}\sum_{i=SVTAP}\Gamma_{\rho\sigma}{}^i\Gamma_{\alpha\beta}{}^i, \tag{A1}$$

[7] Note that any first-order process with 2 incoming and 2 outgoing particles is proportional to the unrenormalized coupling constant G.

[8] H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956), p. 119.

where

$$\begin{aligned}\Gamma^i&=I &&\text{if } i=S\\ &=\gamma^\mu &&\text{if } i=V\\ &=\frac{1}{\sqrt{2}}\sigma^{\mu\nu} &&\text{if } i=T\\ &=i\gamma^\mu\gamma_5 &&\text{if } i=A\\ &=\gamma_5 &&\text{if } i=P.\end{aligned} \tag{A2}$$

If Eq. (A1) is inserted in an expression like

$$\bar{\psi}_aO\psi_b\bar{\psi}_cO\psi_d=(\bar{\psi}_aO)^\alpha\delta_{\alpha\sigma}\psi_b{}^\sigma(\bar{\psi}_cO)^\rho\delta_{\rho\beta}\psi_d{}^\beta, \tag{A3}$$

one obtains the well-known transformation of (A3) into

$$\bar{\psi}_aO'\psi_d\bar{\psi}_cO'\psi_b, \tag{A4}$$

where O' is in general a linear combination of Dirac operators, related to O via the Fierz matrix.[8] To be specific, this kind of Fierz transformation shall be called a Fierz transformation of the first kind.

To prove Eq. (5), a "Fierz transformation of the second kind," it is best to start with the right-hand side of Eq. (5) and insert Eq. (A1) in the following way:

$$\begin{aligned}&\bar{\psi}_a(1-\gamma_5)\mathcal{C}\bar{\psi}_c{}^T\psi_d{}^T\mathcal{C}^{-1}(1+\gamma_5)\psi_b\\ &\quad=[\bar{\psi}_a(1-\gamma_5)]^\alpha\delta_{\alpha\sigma}[\mathcal{C}\bar{\psi}_c{}^T]^\sigma[\psi_d{}^T\mathcal{C}^{-1}]^\rho\\ &\qquad\qquad\times\delta_{\rho\beta}[(1+\gamma_5)\psi_b]^\beta\\ &\quad=-\tfrac{1}{4}\textstyle\sum_i\bar{\psi}_a(1-\gamma_5)\Gamma^i(1+\gamma_5)\psi_b\psi_d{}^T\mathcal{C}^{-1}\Gamma^i\mathcal{C}\bar{\psi}_c{}^T,\end{aligned} \tag{A5}$$

where the minus stems from the anticommutativity of the field operators. The combination $(1-\gamma_5)\Gamma^i(1+\gamma_5)$ assures that only V and A survive in the sum over i and if one uses the properties of $\mathcal{C}$, given in Eq. (6) of the text, one arrives directly at Eq. (5) of the text.

APPENDIX B: COMPUTATION OF $\tilde{R}_{\alpha\beta}(k)$

The quantity one has to compute for the renormalization of the μ-decay coupling constant is $\tilde{R}_{\alpha\beta}(k)$ whose Fourier transform is defined in Eq. (16) of the text. To order G^2, it is given by

$$\begin{aligned}&R_{\alpha\beta}(x-y)\\ &=-\frac{(i\sqrt{2}G)^2}{3!}\int d^4z\langle 0|\psi_\mu{}^\alpha(x):\bar{\psi}_{\nu_e}(z)(1-\gamma_5)\mathcal{C}\bar{\psi}_\mu{}^T(z)\\ &\quad\times\psi_{\nu_\mu}{}^T(z)\mathcal{C}^{-1}(1+\gamma_5)\psi_e(z)::\bar{\psi}_e(y)(1-\gamma_5)\mathcal{C}\bar{\psi}_{\nu_\mu}{}^T(y)\\ &\qquad\times[\mathcal{C}^{-1}(1+\gamma_5)\psi_{\nu_e}(y)]^\beta:|0\rangle.\end{aligned} \tag{B1}$$

Recalling Eqs. (13) and (14), this can be recast into the form [cf. Eq. (52)]

$$\begin{aligned}R_{\alpha\beta}(x-y)=&\frac{-G^2}{3}\int d^4z[(1-\gamma_5)\mathcal{C}]^{\alpha_1\beta_1}[\mathcal{C}^{-1}(1+\gamma_5)]^{\alpha_2\beta_2}[(1-\gamma_5)\mathcal{C}]^{\alpha_3\beta_3}[\mathcal{C}^{-1}(1+\gamma_5)]^{\alpha\beta_4}\\ &\times S_{\beta\beta_1}{}^+(x-z,m_\mu)S_{\beta_4\alpha_1}{}^-(y-z,0)S_{\alpha_2\beta_3}{}^+(z-y,0)S_{\beta_2\alpha_3}{}^+(z-y,m_e)\\ =&-\frac{G^2}{3}\int d^4z\left[\left(\gamma\frac{\partial}{\partial x}-im_\mu\right)\mathcal{C}^T(1-\gamma_5)^T\gamma_\lambda{}^T(1+\gamma_5)^T\mathcal{C}^{-1T}\right]_{\beta\alpha}\Delta^+(x-z,m_\mu)\Delta_{;\lambda}{}^+(z-y,0)\\ &\times\left\{\mathrm{Tr}\left[\mathcal{C}^{-1}(1+\gamma_5)\left(\gamma\frac{\partial}{\partial z}-im_e\right)(1-\gamma_5)\mathcal{C}\gamma_\sigma{}^T\right]\Delta^+(z-y,m_e)\right\}\Delta_{;\sigma}{}^+(z-y,0).\end{aligned} \tag{B2}$$

One can combine two of the Δ^+ functions with the same space-time dependence into a spectral integral[9]

$$\begin{aligned}&\Delta_{;\mu}{}^+(x,m^2)\Delta_{;\nu}{}^+(x,0)\\ &=\frac{-i}{48\pi^2}\int_{m^2}^\infty\frac{ds}{s^2}\left\{(s-m^2)^3\frac{g_{\mu\nu}}{4}-(s-m^2)(s+2m^2)\frac{\partial_\mu\partial_\nu}{2s}\right\}\\ &\qquad\times\Delta^+(x,s).\end{aligned} \tag{B3}$$

Contraction with $g^{\mu\nu}$ and insertion of the definition of Δ^+, Eq. (13), yields the form required in (B2)

$$\begin{aligned}g^{\mu\nu}\Delta_{;\mu}{}^+(x,m^2)\Delta_{;\nu}{}^+(x,0)=&\frac{1}{16\pi}\frac{1}{(2\pi)^4}\int d^4k\;\Theta(k^0)\\ &\times\Theta(k^2-m^2)e^{-ikx}\frac{(k^2-m^2)^2}{k^2}.\end{aligned} \tag{B4}$$

A collection of formulas of the type (B3) can be found in Ref. 10. Insertion of (B4) and the definition of Δ^+ in Eq. (B2) leads directly to the Fourier transform $\tilde{R}_{\alpha\beta}(k)$

$$\tilde{R}_{\alpha\beta}(k)=-\frac{G^2}{12\pi^3}\Theta(k)\delta(k^2-m^2)J_{\alpha\beta}(k), \tag{B5}$$

where

$$\begin{aligned}&J_{\alpha\beta}(k)\\ &=\int d^4k'\Theta(k')\Theta(k-k')\Theta(k'^2-m_e{}^2)\delta(m_\mu{}^2-2kk'+k'^2)\\ &\qquad\times[(\mathbf{k}+m_\mu)(\mathbf{k}-\mathbf{k}')(1+\gamma_5)]_{\alpha\beta}\frac{(k^2-m_e{}^2)^2}{k^2}.\end{aligned} \tag{B6}$$

Using

$$\int d^4k'k_\mu f(k,k')=\frac{k_\mu}{k^2}\int d^4k'(kk')f(k,k'), \tag{B7}$$

[9] W. Thirring, *Principles of Quantum Electrodynamics* (Academic Press Inc., New York, 1958), Appendix II.

[10] H. V. R. Pietschmann, Phys. Rev. 139, B446 (1965).

one can write the integral (B6) in the following form:

$$J_{\alpha\beta}(k)=[(m_\mu+\mathbf{k})(1+\gamma_5)]_{\alpha\beta}J_1(k), \tag{B8}$$

where

$$J_1(k)=\int d^4k'\Theta(k')\Theta(k-k')\Theta(k'^2-m_e^2)\delta(k'^2-2kk'+m^2)\times\frac{(2kk'-m_\mu^2-m_e^2)^2}{2kk'-m_\mu^2}\frac{m_\mu^2-kk'}{m_\mu}. \tag{B9}$$

The integral (B9) is most easily evaluated in the rest frame of k. The result inserted in Eqs. (B8) and (B5) yields directly Eq. (23) of the text.

The computation of $\tilde{R}_{\alpha\beta}'(k)$ as given in Eq. (31) of the text follows an analogous pattern with trivial changes in the mass values. Instead of (B4) one has to use

$$g^{\mu\nu}\Delta_{;\mu}^+(x,M)\Delta_{;\nu}^+(x,m)=\frac{1}{16\pi}\frac{1}{(2\pi)^4}\int d^4k\ \Theta(k)\times\Theta[k^2-(m+M)^2]e^{-ikx}\frac{k^2-(M^2+m^2)}{k^2}\times[M^4+m^4+k^4-2(M^2m^2+M^2k^2+m^2k^2)]^{1/2}, \tag{B10}$$

so that (B9) will change into

$$J_1'(k)=\int d^4k'\Theta(k')\Theta(k-k')\Theta[k^2-(m_e+M_p)^2]\times\delta(k'^2-2kk'+M_n^2)\frac{k'^2-(M_p^2+m_e^2)}{k'^2}\frac{M_n^2-kk'}{M_n}\times[M_p^4+m_e^4+k'^4-2(M_p^2m_e^2+M_p^2k'^2+m_e^2k'^2)]^{1/2}. \tag{B11}$$

A computation of (B11) in the rest frame leads to Eqs. (31) and (32) of the text.

APPENDIX C: THE SUMMATION OF CERTAIN SAALSCHÜTZIAN SERIES

The summation of the following three infinite series is required [cf. Eqs. (27), (44), and (47)]

$$S_1=\sum_{\nu=1}^{\infty}\frac{(2\nu-1)!}{(2\nu+1)!}x^\nu, \tag{C1}$$

$$S_2=\sum_{\nu=0}^{\infty}\frac{(\nu-1)!}{(\nu+1)!}x^\nu, \tag{C2}$$

$$S_3=\sum_{\nu=1}^{\infty}\frac{(3\nu-1)!}{(3\nu+1)!}x^\nu. \tag{C3}$$

All three are of the Saalschützian type. A Saalschützian series can be defined through a generalized hypergeometric function[11]

$${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_nx^n}{(d)_n(e)_nn!}, \tag{C4}$$

with

$$(a)_0=1,\quad (a)_n=\Gamma(a+n)/\Gamma(a)=a(a+1)\cdots(a+n-1). \tag{C5}$$

$(b)_n\cdots(e)_n$ are defined in the same way. If

$$a+b+c=-1+d+e, \tag{C6}$$

the series is called Saalschützian.

$S_1\cdots S_3$ can now be expressed through generalized hypergeometric functions in the following way:

$$S_1=\tfrac{1}{6}x\ {}_3F_2(1,1,\tfrac{3}{2};2,\tfrac{5}{2};x), \tag{C7}$$

$$S_2=1+\tfrac{1}{2}x\ {}_3F_2(1,1,1;3,1;x)=1+\tfrac{1}{2}xF(1,1;3;x), \tag{C8}$$

$$S_3=\tfrac{1}{12}x\ {}_3F_2(1,1,\tfrac{4}{3};2,7/3;x). \tag{C9}$$

Equation (C8) shows that S_2 can be expressed through an ordinary hypergeometric function. Its summation can thus be carried out by using the integral representation[11]

$$F(a,b;c;x)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\tau^{b-1}(1-\tau)^{c-b-1}(1-\tau x)^{-a}d\tau \quad \text{if}\ \operatorname{Re}c>\operatorname{Re}b>0,\ |\arg(1-x)|<\pi. \tag{C10}$$

Thus

$$S_2=1+x\int_0^1\frac{1-t}{1-xt}dt=2+\frac{1-x}{x}\ln(1-x), \tag{C11}$$

which confirms Eq. (44) of the text.

We will demonstrate 2 different methods to sum the series S_1 and S_3. Of course, either method could be applied in both cases.

From the definition (C4) one can directly derive the following relation between generalized and ordinary hypergeometric functions:

$${}_3F_2(1,a,b;a+1,b+1;x)=\frac{b}{b-a}F(1,a;a+1;x)+\frac{a}{a-b}F(1,b;b+1;x). \tag{C12}$$

Inserting the special values of the parameters as given by Eq. (C7) and using the integral representation (C10) yields

$$S_1=1-\frac{1}{2\sqrt{x}}\ln\frac{1+\sqrt{x}}{1-\sqrt{x}}-\tfrac{1}{2}\ln(1-x), \tag{C13}$$

which is the result used in Eq. (29) of the text.

To sum the series S_3 one can use the representation of the generalized hypergeometric function through

[11] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

Meyer's G function[11]

$$_3F_2(a,b,c;d,e;x)=\frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(c)}G_{3,3}{}^{1,3}\left(x\middle|\begin{matrix}1-a, & 1-b, & 1-c\\ 0, & 1-d, & 1-e\end{matrix}\right) \quad \text{(C14)}$$

together with the Euler transform[11]

$$G_{3,3}{}^{1,3}\left(x\middle|\begin{matrix}a, & b, & c\\ 0, & d, & e\end{matrix}\right)=\frac{1}{\Gamma(a-e)}\int_0^1 dy\, y^{-a}(1-y)^{a-e-1}G_{2,2}{}^{1,2}\left(xy\middle|\begin{matrix}b, & c\\ 0, & d\end{matrix}\right)$$

$$\text{if } |\arg x|<\pi,\quad \text{Re}d<\text{Re}a<1 \quad \text{(C15)}$$

and[11]

$$G_{2,2}{}^{1,2}\left(x\middle|\begin{matrix}a, & b\\ 0, & c\end{matrix}\right)=\frac{\Gamma(1-a)\Gamma(1-b)}{\Gamma(1-c)}\times F(1-a,1-b;1-c;x) \quad \text{(C16)}$$

to obtain

$$_3F_2(1,1,\tfrac{4}{3};2,7/3;x)={}_3F_2(1,1,\tfrac{4}{3};7/3,2;x)=\int_0^1 dy\, F(1,\tfrac{4}{3};7/3;xy). \quad \text{(C17)}$$

The exchange of parameters in the first equality of (C17) is necessary to avoid improper integrals which do not converge uniformly. Inserting the integral representation (C10) once again yields

$$S_3=\frac{1}{9}\int_0^x dz\int_0^1\frac{t^{1/3}}{1-tz}dt=1+\frac{1}{3a}\ln\frac{1-a}{[1+a+a^2]^{1/2}}-\frac{1}{a\sqrt{3}}\arctan\frac{2a+1}{\sqrt{3}}+\frac{1}{a\sqrt{3}}\frac{\pi}{6}-\tfrac{1}{3}\ln(1-x), \quad \text{(C18)}$$

where

$$a=x^{1/3}. \quad \text{(C19)}$$

If the argument in Eq. (C18) is pure imaginary it leads directly to Eq. (49) of the text.

APPENDIX D: ON THE COMPUTATION OF HIGHER ORDER GRAPHS IN e-ν_e SCATTERING

In order to obtain higher order corrections to e-ν_e scattering one has to compute $K(x-y)$ as defined by Eq. (37) of the text. To lowest order in G, $K(x-y)$ is given by

$$K(x-y)=\frac{iG}{\sqrt{2}}\langle 0|\psi_e{}^T(x)\mathcal{C}^{-1}(1+\gamma_5)\psi_\nu(x)\bar{\psi}_e(y)\times(1-\gamma_5)\mathcal{C}\bar{\psi}_\nu{}^T(y)|0\rangle+O(G^2)$$
$$=\frac{iG}{\sqrt{2}}[\mathcal{C}^{-1}(1+\gamma_5)]_{\alpha\beta}[(1-\gamma_5)\mathcal{C}]_{\gamma\delta}\times S_{\alpha\gamma}{}^+(x-y,m_e)S_{\beta\delta}{}^+(x-y,0)+O(G^2). \quad \text{(D1)}$$

By means of Eqs. (13) and (B4) this can be cast into the following form

$$K(x-y)=\frac{-iG}{\sqrt{2}}\frac{1}{(2\pi)^5}\int d^4k\,\Theta(k^0)\Theta(k^2-m_e{}^2)e^{-ik(x-y)}\times\frac{(k^2-m_e{}^2)^2}{k^2}+O(G^2). \quad \text{(D2)}$$

The Fourier transform of Eq. (D2) gives exactly the first term of Eq. (42) whereby the Θ functions are dropped. Note that they only restrict k to lie in the physical region of e-ν_e scattering.

The only graph contributing to second order in G is the one with 2 e-ν_e bubbles. In momentum space, it is simply the square of the first-order graph multiplied into the weight factor $\frac{2}{3}$. This gives the second term of Eq. (42).

Using the same technique as above, the contribution of the graphs shown in Fig. 1(b) to $K(x-y)$ can be written in the following form:

$$K^{(1b)}(x-y)=\frac{8\sqrt{2}G^3}{3i}\text{Tr}[(1+\gamma_5)\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_3}\gamma^{\nu_3}\gamma^{\mu_1}\gamma^{\nu_1}]\int d^4z_1d^4z_2\Delta_{;\nu_1}{}^+(x-z_1,m_e)\Delta_{;\nu}{}^+(z_1-z_2,m_e)\Delta_{;\nu_3}{}^+(z_2-y,m_e)$$
$$\times\Delta_{;\mu_1}{}^+(z_1-z_2,0)\Delta_{;\mu_2}{}^+(x-z_2,0)\Delta_{;\mu_3}{}^+(z_1-y,0). \quad \text{(D3)}$$

Inserting Eqs. (B3), (52), and (13) yields the Fourier transform of $K(x-y)$, defined by Eq. (38):

$$\tilde{K}^{(1b)}(p)=\frac{-i\sqrt{2}G^3}{144\pi^5}\text{Tr}[(1+\gamma_5)\gamma^{\mu_2}\gamma^{\nu_2}\gamma^{\mu_3}\gamma^{\nu_3}\gamma^{\mu_1}\gamma^{\nu_1}]\int d^4kd^4q\,\Theta(k^0)\Theta(k^0+p^0-q^0)\Theta(q^0-k^0)\delta(q^2-m_e{}^2)$$
$$\times\delta[(k+p-q)^2-m_e{}^2]\delta[(p-q)^2]\delta[(q-k)^2]q_{\nu_3}(k+p-q)_{\nu_2}(p-q)_{\mu_2}(q-k)_{\mu_3}\frac{(k^2-m_e{}^2)^2}{k^4}$$
$$\times\left[(k^2-m_e{}^2)\frac{g_{\mu_1\nu_1}}{4}+(k^2+2m_e{}^2)\frac{k_{\mu_1}k_{\nu_1}}{2k^2}\right]. \quad \text{(D4)}$$

Using the δ functions in the integrand of (D4) to obtain

$$\text{Tr}[(1+\gamma_5)(\boldsymbol{p}-\boldsymbol{q})\gamma^\mu(\boldsymbol{q}-\boldsymbol{k})(\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q})\gamma_\mu\boldsymbol{q}]=4(p^2-m_e{}^2)^2 \quad \text{(D5)}$$

and

$$\mathrm{Tr}[(1+\gamma_5)(p-q)k(q-k)(k+p-q)kq]=2(p^2-2m_e^2)(p^2k^2-m_e^4), \quad \text{(D6)}$$

and inserting Eqs. (D5) and (D6) in Eq. (D4) yields

$$\tilde{K}^{(1b)}(p)=\frac{-i\sqrt{2}G^3}{144\pi^5}\int d^4k d^4q\ \Theta(k^0)\Theta(k^0+p^0-q^0)\Theta(q^0-k^0) \times\frac{(k^2-m_e^2)^2}{k^4}\left[(k^2-m_e^2)(p^2-m_e^2)^2+(k^2+2m_e^2)(p^2-2m_e^2)\left(p^2-\frac{m_e^4}{k^2}\right)\right] \times\delta(q^2-m_e^2)\delta[(k+p-q)^2-m_e^2]\delta[(p-q)^2]\delta[(q-k)^2]. \quad \text{(D7)}$$

Since there is no dependence on the azimuth angle of $\bar{q}$, three of the δ functions in (D7) are sufficient to eliminate the integration over d^4q. The fourth δ function can then be used for the integration over dk^0. The remaining integral can be carried out in the rest frame of p. The result of this elementary integration is

$$\tilde{K}^{(1b)}(p)=\frac{-i\sqrt{2}G^3}{36}\frac{p^6}{(2\pi)^3}+O(m_e^2/k^2). \quad \text{(D8)}$$

When Eq. (D8) is inserted in Eq. (40), it has to be taken at

$$p^2=s\doteq(2q)^2, \quad \text{(D9)}$$

where q is the neutrino energy (in the c.m. system). Taking out the weight factor $2!/4!$ of the graph in Fig. 1(b) yields the result given in Eq. (48) of the text.

APPENDIX E: PROOF OF EQ. (54)

It is well known[9] that the Δ^+ function can be represented through a Hankel function of the first kind,

$$\Delta^+(x,m)=\frac{m}{8\pi\sqrt{z}}H_1^{(1)}(m\sqrt{z}), \quad \text{(E1)}$$

where

$$z=x^\mu x_\mu. \quad \text{(E2)}$$

For a small variation of m one gets (to first order in δm)

$$\Delta^+(x,m+\delta m)=\left(1+\frac{\delta m}{m}\right)\frac{m}{8\pi\sqrt{z}}\left\{H_1^{(1)}(m\sqrt{z})+\frac{\delta m}{\sqrt{z}}\frac{d}{dm}H_1^{(1)}(m\sqrt{z})\right\}. \quad \text{(E3)}$$

Using the relations[12]

$$\frac{d}{dx}H_1^{(1)}(x)=\frac{1}{x}H_1^{(1)}(x)-H_2^{(1)}(x) \quad \text{(E4)}$$

and

$$x^2H_2^{(1)}(x)=\int dx\, x^2H_1^{(1)}(x), \quad \text{(E5)}$$

yields, after some trivial rearrangement of integration variables,

$$\Delta^+(x,m+\delta m)=\Delta^+(x,m)\left\{1+2\frac{\delta m}{m}\right\}-\frac{\delta m}{m}x^2\int_0^m \kappa d\kappa\, \Delta^+(x,\kappa). \quad \text{(E6)}$$

If m is replaced by m^2, this is exactly Eq. (54) of the text.

[12] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II.