Perturbed Bound-State Poles in Potential Scattering. I*

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The perturbation formula of Dashen and Frautschi is studied in nonrelativistic potential scattering. A formalism based on the Jost function is presented and is shown to be equivalent to the Dashen-Frautschi approach. It is shown for both cases that unless the Born approximation is satisfied for the unperturbed problem, the mass shift cannot be calculated in a simple way. It is pointed out that the Born approximation is not expected to be valid if the bound state is assumed to exist. The formalism is illustrated by the use of a soluble square-well potential. The infrared problem is also discussed. It is pointed out that the infrared divergence is automatically eliminated at the binding energy and is not "spurious."

I. INTRODUCTION

THE recent calculation of the proton-neutron mass difference by Dashen and Frautschi[1] and their subsequent work on the octet enhancement[2] are based on the assumption that the strongly interacting particle can be regarded as a bound state of two particles. Their results depend crucially on the relativistic analog of the perturbation formula in nonrelativistic potential scattering. Since this formula is the key to their numerical results which have drawn considerable current interest, it seems extremely important to understand correctly the perturbation formalism in the potential scattering.

In this paper, we examine first the first-order correction to the binding energy using the Jost-function formalism. We then show that the Jost-function approach is equivalent to the formulas used by Dashen and Frautschi. It is shown that the input function in the Dashen-Frautschi formula depends on the unperturbed potential and the unperturbed wave function unless the Born approximation is valid for the original problem. It is pointed out also that the Born approximation is not expected to be valid if the bound state is assumed to exist. We then use the soluble square-well potential to illustrate the Jost-function formalism. The infrared problem is also examined.

In Sec. II, we introduce first the unperturbed and perturbed potentials, wave functions, and Jost functions. We then derive the first-order correction to the position of the bound-state pole in terms of the Jost functions. In Sec. III, the Jost-function formula of the preceding section is shown to be equivalent to the formula derived and used by Dashen and Frautschi. It is shown that unless the Born approximation is valid for the unperturbed problem, the mass formula does not have general applicability. It is pointed out also that the validity of the Born approximation is incompatible with the existence of the bound state. In Sec. IV, we illustrate the Jost-function approach using a soluble square-well potential. We show that the result of the Jost-function calculation and that of the conventional perturbation theory coincide for a weak perturbing Yukawa potential. The infrared problem connected with the Coulomb-potential limit of the Yukawa potential is examined in detail. It is shown that the bound-state condition eliminates the term causing the infrared divergence.

II. JOST-FUNCTION APPROACH

We consider an attractive potential $V_0(r)$ and call this unperturbed potential. Then the Schrödinger equation for the lth partial wave can be written as

$$\varphi_l(r)=j_l(kr)+(1/k)\times\int_0^r[n_l(kr)j_l(kr')-j_l(kr)n_l(kr')]\times V_0(r')\varphi_l(r')dr', \quad (1)$$

where φ_l is the lth radial wave function. We define $j_l(x)$ and $n_l(x)$ as

$$j_l(x)=(\tfrac{1}{2}\pi x)^{\frac{1}{2}}J_{l+\frac{1}{2}}(x),$$
$$n_l(x)=(\tfrac{1}{2}\pi x)^{\frac{1}{2}}N_{l+\frac{1}{2}}(x).$$

If we add a small perturbing potential $\delta V(r)$, then the perturbed wave function $\Psi_l(r)$ satisfies the equation

$$\Psi_l(r)=j_l(kr)+(1/k)\times\int_0^r[n_l(kr)j_l(kr')-j_l(kr)n_l(kr')]\times[V_0(r')+\delta V(r')]\Psi_l(r')dr'. \quad (2)$$

The perturbed wave function satisfies also the "full-Green's-function" integral equation

$$\Psi_l(r)=\varphi_l(r)+(1/k)\int_0^r\varphi_l(r-r')\delta V(r')\Psi_l(r')dr', \quad (3)$$

which, to lowest order in $\delta V(r)$, takes the form

$$\Psi_l(r)=\varphi_l(r)+(1/k)\int_0^r\varphi_l(r-r')\delta V(r')\varphi(r')dr'. \quad (4)$$

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[1] R. F. Dashen and S. C. Frautschi, Phys. Rev. **135**, B1190 (1964); R. F. Dashen, *ibid.* **135**, B1196 (1964).
[2] R. F. Dashen and S. C. Frautschi, Phys. Rev. **137**, B1318, B1331 (1965).

We now restrict ourselves to the S wave and drop the subscript l. The generalization to higher partial waves seems to be trivial. Then the integral equations in Eqs. (1), (2), and (4) take the following familiar forms:

$$\varphi(r)=\sin kr+(1/k)\int_0^r[\sin k(r-r')]\times V_0(r')\varphi(r')dr', \quad (5)$$

$$\Psi(r)=\sin kr+(1/k)\int_0^r[\sin k(r-r')]\times[V_0(r')+\delta V(r')]\Psi(r')dr', \quad (6)$$

$$\Psi(r)-\varphi(r)=(1/k)\int_0^r\varphi(r-r')\delta V(r')\varphi(r')dr'. \quad (7)$$

In order to obtain the S matrix from the above solutions, we next introduce the Jost functions $f_0(\pm k)$ and $f(\pm k)$, respectively, for the perturbed and unperturbed problems.

$$f_0(\mp k)=1+(1/k)\int_0^\infty[\exp(\pm ikr)]V_0(r)\varphi(r)dr,$$

$$f(\mp k)=1+(1/k)\int_0^\infty[\exp(\pm ikr)]\times[V_0(r)+\delta V(r)]\Psi(r)dr. \quad (8)$$

In terms of these Jost functions we can now write the phase shifts η_0 and η for the original and perturbed problems, respectively.

$$\exp[2i\eta_0]=f_0(k)/f_0(-k),$$
$$\exp[2i\eta]=f(k)/f(-k).$$

In the following discussions we will be led to study the property of the Jost functions in the complex energy plane. We thus use x as the energy variable, that is,

$$x=k^2,$$

and adopt the following notation for the Jost functions.

$$\begin{aligned} f_0^{(+)}(x)&=f_0(k), \quad & f^{(+)}(x)&=f(k),\\ f_0^{(-)}(x)&=f_0(-k), \quad & f^{(-)}(x)&=f(-k). \end{aligned} \quad (9)$$

Let us now assume that the unperturbed problem has a bound state at $x=-x_0$, and therefore

$$f_0^{(-)}(-x_0)=0. \quad (10)$$

For the perturbed system,

$$f^{(-)}(-x_0-\delta x_0)=0, \quad (11)$$

where δx_0 is the shift in the binding energy.

In order to calculate δx_0, we note from Eq. (9) that $f^{(\mp)}(x)$ can be written as

$$f^{(\mp)}(x)=f_0^{(\mp)}(x)+\delta f^{(\mp)}(x), \quad (12)$$

where

$$\delta f^{(\mp)}(x)=(1/k)\int_0^\infty[\exp(\pm ikr)]V_0(r)[\Psi(r)-\Phi(r)]dr+(1/k)\int_0^\infty[\exp(\pm ikr)]\delta V(r)\Phi(r)dr.$$

If the Born approximation is valid for the unperturbed problem, both $\Psi(r)$ and $\varphi(r)$ become $\sin kr$, and $\delta f^{(\mp)}(x)$ takes the following simple form:

$$\delta f^{(\mp)}(x)=(1/k)\int_0^\infty[\exp(\pm ikr)]\delta V(r)\sin krdr.$$

If, on the other hand, the Born approximation is not valid, we have to use Eq. (7) for $[\Psi(r)-\varphi(r)]$, and the above $\delta f^{(\mp)}(x)$ becomes

$$\delta f^{(\mp)}(x)=(1/k)\int_0^\infty[\exp(\pm ikr)]\delta V(r)\varphi(r)dr+(1/k^2)\int_0^\infty[\exp(\pm ikr)]V_0(r)dr\times\int_0^r\varphi(r-r')\delta V(r')\varphi(r')dr'. \quad (13)$$

We return now to the bound-state condition of Eq. (11), which can be written as

$$f_0^{(-)}(-x_0-\delta x_0)+\delta f^{(-)}(-x_0-\delta x_0)=0.$$

By taking only the first-order terms in δx_0 and $\delta V(r)$, we arrive at the following expression for δx_0:

$$\delta x_0=\delta f^{(-)}(-x_0)/f_0^{(-)\prime}(-x_0). \quad (14)$$

This is the first-order correction to the binding energy. In a later section we shall show, using the square-well potential, that the energy shift calculated in this way is the same as the result in the conventional perturbation theory. Before carrying out this task, let us examine whether the above formula is equivalent to the mass formula given by Dashen and Frautschi.

III. DASHEN-FRAUTSCHI FORMULA

For completeness we derive here the Dashen-Frautschi formula for the energy shift. The S-wave scattering amplitude $A_0(x)$ is assumed to have a bound-state pole at $x=-x_0$. Thus near this pole, the amplitude will be of the form

$$A_0(x)=R/(x+x_0), \quad (15)$$

where R is the residue of the bound-state pole.

If we add a perturbing potential, both the binding energy x_0 and the binding strength R will receive corrections δx_0 and δR, respectively. The difference between the perturbed and unperbed amplitudes will

then be

$$\delta A(x)=\frac{-R\delta x_0}{(x+x_0)(x+x_0+\delta x_0)}+\frac{\delta R}{x+x_0+\delta x_0}, \quad (16)$$

near $x=-x_0$. The perturbed pole position $x=-(x_0+\delta x_0)$ is assumed to be very near that of the original pole. It is important to retain δx_0 in the denominators of the above expression since we are not excluding the possibility of $|x+x_0|$ being much smaller than δx_0, which is assumed to be a nonvanishing quantity.

Next, we multiply both sides of the above equation by $[f_0^{(-)}(x)]^2/(x+x_0)$ and perform a Cauchy integral along the counter-clockwise contour that encloses the poles at $x=-x_0$ and $x=-(x_0+\delta x_0)$ only. Then, to lowest order in δx_0,

$$\delta x_0=-\frac{1}{R}\left[\frac{1}{f_0^{(-)\prime}(-x_0)}\right]^2\frac{1}{2\pi i}\times\int_C\frac{[f_0^{(-)}(z)]^2\delta A(z)dz}{z+x_0}, \quad (17)$$

where the contour C is described above. This contour can equivalently enclose all other singularities in the z plane clockwise. This is the Dashen-Frautschi formula for the first-order energy shift.

Before discussing the applicability of this relation, let us show that the above mass formula is equivalent to that of the previous section. We write the amplitudes $A(x)$ and $A_0(x)$ as

$$A(x)=\frac{1}{2ik}\left\{\frac{f^{(+)}(x)}{f^{(-)}(x)}-1\right\}, \quad (18)$$

and

$$A_0(x)=\frac{1}{2ik}\left\{\frac{f_0^{(+)}(x)}{f_0^{(-)}(x)}-1\right\}.$$

The mass formula then can be written as

$$\delta x_0=-\frac{1}{R}\left[\frac{1}{f_0^{(-)\prime}(-x_0)}\right]^2\frac{1}{2\pi i}\int_C\frac{dz}{z+x_0}\frac{f_0^{(-)}(z)}{f^{(-)}(z)}\times\left\{\frac{f^{(+)}(z)f_0^{(-)}(z)-f^{(-)}(z)f_0^{(+)}(z)}{2ik}\right\}.$$

Now, in the approximation of retaining only the lowest order terms in $\delta V(x)$, the above formula is simplified to

$$\delta x_0=-\frac{1}{R}\left[\frac{1}{f_0^{(-)\prime}(x_0)}\right]^2\frac{1}{2\pi i}\int_C\frac{dz}{z+x_0}\times\frac{\delta f^{(+)}(z)f_0^{(-)}(z)-\delta f^{(-)}(z)f_0^{(+)}(z)}{2ik}. \quad (19)$$

By taking the residue at $z=-x_0$ and by relating the quantity R to the Jost functions, one can show that the right-hand side is $\delta f^{(-)}(-x_0)/f_0^{(-)\prime}(-x_0)$. This is just the expression we derived in the preceding section.

We can of course evaluate the above integral in the spirit of Dashen and Frautschi by enclosing all other singularities clockwise. But, unless the Born approximation is satisfied along the integration contour, in which case the input information takes the following simple form

$$(1/sik)\{\delta f^{(+)}(x)f_0^{(-)}(x)-\delta f^{(-)}(x)f_0^{(+)}(x)\}=-(1/k^2)\int_0^\infty\delta V(r)\sin^2krdr, \quad (20)$$

there does not seem to be any easy way of using the mass formula of Eq. (19). As one can illustrate using various simple forms for the unperturbed potential, the Born approximation is not expected to be valid if the bound state is to exist. Unfortunately, therefore, the mass formula of Eq. (17) does not seem to have general applicability in the potential scattering.[3]

IV. SOLUBLE SQUARE-WELL POTENTIAL AND INFRARED PROBLEM

In order to illustrate the formalism discussed in the preceding sections we now study the soluble square-well potential. We calculate the first-order correction due to a weak Yukawa potential using Eq. (14), which has been shown to be equivalent to the Dashen-Frautschi formula, and point out the result is exactly the same as that of the first-order energy shift in the conventional perturbation theory. It is well known that the result in the conventional theory is finite even if the perturbing Yukawa potential is replaced by a Coulomb potential. We shall show also that the result in the Jost-function formalism is finite and, in fact, coincides with that of the conventional calculation.

We consider first the unperturbed square-well potential

$$V_0(r)=-V_0 \quad \text{for} \quad 0<r<a,$$
$$=0 \quad \text{for} \quad a<r,$$

where V_0 is positive. Then the scattering-state solution for this unperturbed potential is

$$\varphi(r)=k\sin Kr/K \quad \text{for} \quad 0<r<a$$
$$=k\sin Ka\sin(kr+\eta)/K\sin(ka+\eta) \quad \text{for} \quad a<r \quad (21)$$

with the relations

$$K=(V_0+k^2)^{\frac{1}{2}},$$
$$K\cot Ka=k\cot(ka+\eta).$$

The unperturbed Jost function is

$$f_0^{(-)}(x)=[\exp(ika)/2ik]\times[(K-k)\exp(iKa)+(K+k)\exp(-iKa)]. \quad (22)$$

[3] By "general applicability" we mean the case where δx_0 is a functional of $\delta V(r)$ and $f_0^{(-)}(x)$ only and can be determined from them in a straightforward way.

Now, for the perturbing potential,

$$\delta V(r)=\alpha[\exp(-\lambda r)]/r,$$

we write the perturbed wave function as

$$\Psi(r)=\varphi(r)+\alpha\int_0^r \frac{\exp(-\lambda r')}{kr'}\varphi(r-r')\varphi(r')dr'$$

to lowest order in $\delta V(r)$. After a straightforward calculation, we obtain

$$\Psi(r)-\varphi(r)=\frac{\alpha k}{K}\int_\lambda^\infty \frac{d\mu}{\mu^2+4K^2}\{\sin Kr[1+\exp(-\mu r)] - (2K/\mu)\cos Kr[1-\exp(-\mu r)]\} \quad (23)$$

for $r<a$. The wave function $\Psi(r)$ can also be obtained for other values of r, but is not necessary for the present purpose. Using the above expression we can now calculate $\delta f^{(-)}(x)$:

$$\delta f^{(-)}(x)=-(V_0/k^2)\int_0^a \exp(ikr)[\Psi(r)-\varphi(r)]dr$$

$$+\alpha\int_0^a \frac{\exp(ikr)\exp(-\lambda r)\sin Kr}{Kr}dr$$

$$+\alpha\sin Ka\int_a^\infty \frac{\exp(ikr)\exp(-\lambda r)\sin(kr+\eta)}{Kr\sin(ka+\eta)}dr. \quad (24)$$

After evaluating the above integrals we arrive at the final expression for δx_0.

$$\delta x_0=\delta f^{(-)}(-x_0)/f_0^{(-)\prime}(-x_0)$$

$$=-\frac{4\alpha k_0K_0^2}{1+ak_0}\int_\lambda^\infty \frac{d\mu[1-\exp(-\mu a)]}{\mu(\mu^2+4K_0^2)}, \quad (25)$$

where $k_0=(x_0)^{\frac{1}{2}}$, and $K_0=(V_0-k_0^2)^{\frac{1}{2}}$.

Using the bound-state solutions for this problem, one can obtain δx_0 in the conventional perturbation theory. The result turns out to be the same as above.

As for the infrared problem, we first note that the expression in Eq. (25) remains finite in the limit $\lambda\to 0$. The only possible source of divergence is in the last term of Eq. (24) containing the integral

$$\int_a^\infty dr\exp(ikr)\exp(-\lambda r)\frac{\sin(kr+\eta)}{r\sin(ka+\eta)}$$

which can be written as

$$\int_a^\infty dr\frac{\exp(ikr)\exp(-\lambda r)}{r} \times\left\{\frac{\exp(ikr)-\exp(-2i\eta)\exp(-ikr)}{\exp(ika)-\exp(-2i\eta)\exp(-ika)}\right\}. \quad (26)$$

In the above integrand, the exponential factor $\exp(ikr)$ could cause the divergence. But at $x=-x_0$, the exponential factor $\exp[-2i\eta]$ vanishes, and the term causing the infrared divergence is eliminated.

The above mechanism illustrates the fact that the bound-state wave function is localized and that the infrared divergence is eliminated by this localization effect. Dashen and Frautschi[1] attempt to treat the infrared problem by introducing a fictitious photon mass and by subtracting an infinite quantity using an unjustified method. Their treatment of this problem is therefore incorrect.

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