$\eta\pi\pi$ Model of X^0 and a Possible η-π Resonance

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An $\eta\pi\pi$ model of the X^0 meson at 960 MeV is proposed through factorable π-π and η-π s-wave forces. The over-all dynamics is described by a three-body Schrödinger equation under a nonrelativistic approximation which is justified by the relatively low Q value ($\sim$120 MeV) of the particle with respect to decay into an $\eta\pi\pi$ system. The π-π interaction, which is used as input information, is taken to be of two types: (i) a smooth attractive force which falls short of the requirements of a π-π bound state (called ABC-type), and (ii) a force characterized by a long-range barrier surrounding an inner well, which is capable of producing a π-π resonance at several hundred MeV (called σ-type interaction). Using three-body techniques proposed earlier by one of us (A.N.M.), the nature and strength of the η-π interaction is deduced from a knowledge of the energy of the $\eta\pi\pi$ system at the experimental position of the X^0-mass. It is found that a σ-type π-π interaction, designed to produce a π-π resonance at 400–600 MeV, requires the existence of a bound η-π system. On the other hand, an ABC-type π-π interaction demands an η-π resonance in the BeV region (via a σ-type η-π interaction). The physical implications of these results are discussed.

1. INTRODUCTION

THE recently discovered X^0 meson at 960 MeV, with a width $\Gamma\sim12$ MeV,[1–3] is a unique particle, both from the experimental and from the theoretical point of view. Experimentally, it is found to decay almost exclusively to $\eta+\pi^++\pi^-$, a mode whose electromagnetic character is ruled out by the total absence of simultaneous 3π modes. Its most likely quantum numbers (J^PI^G) turn out as 0^-0^+ on the basis of (i) isotropic angular distribution of the decay particles, (ii) absence of any rise for small or large $M^2(\pi^+\pi^-)$, and (iii) the observation of the $\pi^+\pi^-\gamma$ mode.[4] Theoretically, the particle is believed to be an SU_3 singlet (perhaps even an SU_6 singlet) in contrast to η which is merely the SU_2 singlet member of an SU_3-octet. Such unique features lend this particle a particularly interesting status as an object of study for its internal structure and dynamics, without the risks of predicting dynamical properties of other allied particles within a common group-theoretical representation.

A study of the internal structure of X^0 can be carried out on the basis of any one of several alternative hypotheses. For example, if it is looked upon as built up entirely of pions, the simplest possibility consistent with its quantum numbers is a 4π state, though the latter may not be the principal mode of decay. Indeed, such a point of view was pursued recently in a semiquantitative way,[5] leading to certain physical conclusions on the nature of the (p-wave) π-π interaction. Thus it was found that an *effective* p-wave interaction which is repulsive at low energies but becomes attractive at higher energies (termed type b interaction in Ref. 5), could lead to a 4π state of 0^-0^+ at an energy rather close to that of X^0. On the other hand, an intrinsically attractive π-π interaction (called type a in Ref. 5) did not seem to produce any resonance anywhere near the mass of X^0.

A second possibility would be to regard this particle as a strongly bound quark-antiquark composite,[6] in the spirit of a Fermi-Yang theory. If this hypothesis is found to be the only correct one, it would of course yield direct information on the quark-antiquark force in an SU_3-singlet state, but would not be of much value in correlating other physical data which might have a possible bearing on the structure of this particle.

A third and perhaps more interesting possibility is provided by the assumption that X^0 is a three-particle resonant state of $\eta\pi\pi$ (to which it is known to decay almost entirely). While there is no *a priori* reason to believe that the "decay channel" must also play the dominant role in the formation of the resonance, it can at least provide a reasonable working hypothesis for purposes of further scrutiny. Granting such a possibility, the study of X^0 as an $\eta\pi\pi$ system has several simplifying features in its theoretical formulation. In the first place, the rather low Q value of this channel ($Q\sim120$ MeV) warrants an almost nonrelativistic situation for the $\eta\pi\pi$ system. Secondly, the quantum numbers 0^-0^+ of X^0 require only s-wave interactions in all the pairs of particles involved. One might, therefore, be able to describe the internal dynamics of this particle in terms of a nonrelativistic three-body Schrödinger equation with potentials due to π-π and η-π interactions in s-waves. In

[1] G. R. Kalbfleisch *et al.*, Phys. Rev. Letters **12**, 527 (1964).
[2] M. Goldberg *et al.*, Phys. Rev. Letters **13**, 249 (1964).
[3] G. R. Kalbfleisch, O. I. Dahl, and A. Rittenburg, Phys. Rev. Letters **13**, 349a (1964).
[4] See, e.g., S. Ya. Nikitin, in *Proceedings of the International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965).
[5] A. N. Mitra and S. Ray, Phys. Rev. **137**, B982 (1965).
[6] M. Gell-Mann, Phys. Letters **8**, 214 (1964); G. Zweig, Cern Report, 1964 (unpublished).

this manner an $\eta\pi\pi$ model for X^0 could, from the physical point of view, be regarded as a tool for probing the s-wave η-π interaction in relation to the corresponding π-π interaction.

The model has some points of similarity to the old three-body problem of the ω meson regarded as the bound state of one stable particle (pion) and a resonance in two others (ρ meson).[7,8] However, the essentially nonrelativistic features of the present problem, compared with the relativistic situation for the ω meson, should give a more realistic touch to a three-body investigation of this problem within an acceptable dynamical framework like a Schrödinger equation, until a satisfactory relativistic three-body formulation becomes available.[9] Another difference from the ω problem (apart from unequal mass kinematics in the present case) concerns the physical mechanism for a three-body resonance. Whereas in the ω problem the p-wave centrifugal barrier provides a natural mechanism for a narrow resonance, the absence of a centrifugal barrier in the present case must be compensated by a suitable alternative description in keeping with the requirements of an s-wave resonance. A possible mechanism, which was suggested some time ago by one of us[10] in connection with the mysterious σ particle,[11] relies on the existence of a relatively low but thick wall enclosing an attractive well. The mechanism which was proposed through a nonlocal factorable interaction predicted a low-energy repulsion, but could give a fairly narrow resonance at a sufficiently high energy by making its range-parameter sufficiently "long." On the other hand, a smooth attractive π-π s-wave interaction (below a certain minimum strength for binding) would give merely a positive scattering length—an ABC—type effect.[12] In the following, we shall describe these two types of s-wave interaction between a π-π or an η-π pair as σ type and ABC type, respectively.[13] Our procedure would then consist in postulating an input π-π interaction (ABC or σ type) in the $\eta\pi\pi$ Schrödinger equation for a given total energy of the system, and deducing the nature and strength of the η-π interaction from the condition of a three-body resonance at the above energy.

In Sec. 2, we collect, for convenience, the results of a phenomenological relativistic formulation of the π-π interaction as given in Ref. 10. We also give extension of this formalism to the (unequal mass) case of relativistic η-π interaction, to facilitate an easy interpretation of the output data on the latter, in terms of η,π bound states or resonances (especially at relativistic energies).

In Sec. 3 we give a formulation of the $\eta\pi\pi$ system as an "eigenvalue problem"[7] for the 3-body resonance, on the lines of similar investigations done earlier,[14] viz., through a 3-body Schrödinger equation.[15] Section 4 describes the results of numerical solution of the "eigenvalue equation" for various choices of the input π-π parameters. The main conclusion is that a σ-type π-π interaction requires the existence of a π-η bound state, whereas, an ABC-type interaction predicts an η-π resonance at a rather high energy ($\sim$1.5 BeV), with a manifestation of low-energy repulsion.

2. THE $\pi\pi$ AND $\eta\pi$ SYSTEMS

We consider a π-π interaction of the form

$$\langle p|V_{\pi\pi}|p'\rangle = -\lambda\mu^{-1}g(p)g(p')\,, \tag{2.1}$$

$$g(p)=(\beta^2+p^2)^{-1}. \tag{2.2}$$

The sign of the strength parameter λ distinguishes an ABC-type interaction ($\lambda>0$) from a σ-type one ($\lambda<0$). For a nonrelativistic Schrödinger equation

$$(p^2-k^2)\psi(\mathbf{p})=\lambda\int g(p)g(q)\psi(\mathbf{q})d\mathbf{q} \tag{2.3}$$

yields a π-π scattering length as[16]

$$+a_{\pi\pi}^{-1}=\lim_{k\to 0} k\cot\delta$$

$$=-\tfrac{1}{2}\beta+\tfrac{1}{2}\beta^4/\pi^2\lambda\,, \tag{2.4}$$

the condition for a bound or an unbound state being, respectively,

$$\lambda< \text{ or } >\lambda_c\equiv\beta^3\pi^{-2}. \tag{2.5}$$

On the other hand, $\lambda<0$ predicts a π-π s-wave resonance at a generally relativistic energy. This case, which can be formally described by Eq. (2.3) *looked upon* as a relativistic equation, was discussed in Ref. 10, according to the results of which the resonance momentum k_R and full width Γ_R (at resonance) are expressible in the following form:

$$\beta^{-2}k_R^2=P=\tfrac{1}{2}\sigma-1+(\tfrac{1}{4}\sigma^2-2\sigma)^{1/2}, \tag{2.6}$$

$$\Gamma_R=4\sigma P^{1/2}\beta^2(\mu^2+k_R^2)^{-1/2}(\sigma^2-8\sigma)^{-1/2}, \tag{2.7}$$

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[7] A. N. Mitra, Phys. Rev. **127**, 1342 (1962).

[8] L. I. Schiff, Phys. Rev. **125**, 777 (1962).

[9] See, e.g., F. Coester, Helv. Phys. Acta **38**, 7 (1965). More recently, other attempts have been made for relativistic formulation of three-body problems. Some references are S. Mandelstam, Phys. Rev. **140**, B375 (1965); R. Blankenbecler and R. Sugar, *ibid.* this issue, **142**, 1051 (1966); C. Lovelace *et al.* (to be published).

[10] A. N. Mitra, Nuovo Cimento **32**, 506 (1964).

[11] The first evidence for this particle was suggested by N. Samios *et al.*, Phys. Rev. Letters **9**, 139 (1962). However, its fate has undergone many vicsissitudes during the years, and present experimental evidence (in terms of direct observation) is generally against it (see Ref. 4).

[12] See, e.g., T. N. Truong, Phys. Rev. Letters **6**, 308 (1961).

[13] This is not to suggest that we necessarily believe in the existence of the σ particle, or for that matter, its more recent comparison, called ϵ_0 at $\sim$720 MeV. See, e.g., L. M. Brown, Phys. Rev. Letters **14**, 836 (1965).

[14] For a detailed set of references on this subject, see C. Lovelace, Phys. Rev. **135**, B1225 (1964), which gives a full exposition of Faddeev's theory.

[15] While a 3-body Schrödinger equation is completely equivalent to the Faddeev theory (see Ref. 14) when separable potentials are used, we prefer to keep the former as the basis of our formulation.

[16] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).

where the (positive) dimensionless strength parameter σ is defined by

$$\sigma = -\pi^2\lambda\beta^{-3}. \tag{2.8}$$

For the η-π system, the unequal masses of the two particles introduce some complications. The η-π interaction in the local center of mass may be taken as

$$\langle p|V_{\eta\pi}|p'\rangle = -\lambda_1\mu^{-1}v(p)v(p'), \tag{2.9}$$

where

$$v(p) = (\beta_1^2+p^2)^{-1}. \tag{2.10}$$

The nonrelativistic Schrödinger equation for the η-π system is

$$\tfrac{5}{8}\mu^{-1}(p^2-k^2)\phi(\mathbf{p}) = \lambda_1\mu^{-1}v(p)\int v(q)\phi(\mathbf{q})d\mathbf{q}, \tag{2.11}$$

where for simplicity we have taken

$$m_\eta \approx 4m_\pi \equiv 4\mu, \tag{2.12}$$

as a reasonably good approximation. As before, for $\lambda_1>0$, the scattering length $a_{\eta\pi}$ and the bound-state condition are, respectively, given by

$$a_{\eta\pi}^{-1} = -\tfrac{1}{2}\beta_1 + \tfrac{5}{16}\beta_1^4/\pi^2\lambda_1, \tag{2.13}$$

$$\lambda_1 > \lambda_{1c} \equiv \tfrac{5}{8}\beta_1^3\pi^{-2}. \tag{2.14}$$

Again for $\lambda_1<0$, in analogy with the π-π case,[10] we expect an η-π resonance to occur at relativistic energies. However, unlike Eq. (2.3), the unequal mass kinematics in this case require a modification of Eq. (2.11) before it can be given a relativistic meaning. A reasonable relativistic modification [which is consistent with (2.3) for equal masses] is an equation of the form

$$(W_p^2-W_k^2)\phi(\mathbf{p}) = -\int \langle p|K|q\rangle\phi(\mathbf{q})d\mathbf{q}, \tag{2.15}$$

$$W_p = (m_\eta^2+p^2)^{1/2} + (m_\pi^2+p^2)^{1/2}, \tag{2.16}$$

which, in the language of dispersion relations, is at least consistent with Macdowell's reflection symmetry.[17] A comparison of the nonrelativistic form of (2.15) with (2.11), then gives

$$\langle p|K|p'\rangle \approx 10\mu\langle p|V_{\eta\pi}|p'\rangle. \tag{2.17}$$

The energy of an $\eta\pi$ resonance for the case of $\lambda_1<0$ can now be calculated with the help of Eq. (2.15) in conjunction with (2.17). Indeed, an approximation which is rather good for a high-energy resonance (a situation which will be justified *a fortiori* from the output results for the $\eta\pi$ interaction) helps us to treat the $\eta\pi$ case in close analogy to the treatment of Ref. 10. To obtain the desired approximation, we note the formula[18]

$$p^2 = \tfrac{1}{4}W_p^2 + \tfrac{1}{4}W_p^{-2}(m_\eta^2-\mu^2)^2 - \tfrac{1}{2}(m_\eta^2+\mu^2), \tag{2.18}$$

so that, using (2.12),

$$\tfrac{1}{4}(W_p^2-W_k^2) = p^2-k^2+\tfrac{15}{16}\mu^4(W_p^2-W_k^2)W_p^{-2}W_k^{-2} \tag{2.19}$$

$$\approx (p^2-k^2)[1+(15/4)\mu^4W_p^{-2}W_k^{-2}], \tag{2.20}$$

where in the last step, use has been made of the smallness of the second term in (2.19) for reasonably large values of k. If we now replace the last factor in (2.20) by its value "on the energy shell," viz.,[19]

$$f(k) = 1+(15/4)\mu^4W_k^{-4}, \tag{2.21}$$

we finally obtain

$$W_p^2-W_k^2 \approx 4f(k)(p^2-k^2), \tag{2.22}$$

so that Eq. (2.15) becomes formally identical with Eq. (2.3), except for the replacements

$$\lambda \to (-\lambda_k) \equiv \tfrac{5}{2}\lambda_1 f^{-1}(k) \tag{2.23}$$

and $\beta \to \beta_1$.

The resonance momentum k_0 and width Γ_0 of a possible η-π resonance for $\lambda_1<0$ which can now be calculated on the lines of Ref. 10, are expressible by the approximate formulas

$$k_0^2 \approx \beta_1^2[\tfrac{1}{2}\sigma_{k_0} - 1 + (\tfrac{1}{4}\sigma_{k_0}^2 - 2\sigma_{k_0})^{1/2}], \tag{2.24}$$

$$\sigma_k = \pi^2\lambda_k/\beta_1^3, \tag{2.25}$$

and

$$\Gamma_0 = 8\sigma_{k_0}\beta_1 k_0 f(k_0) W_{k_0}^{-1}(\sigma_{k_0}^2 - 8\sigma_{k_0})^{-1/2}, \tag{2.26}$$

where some use has been made of (2.22).

Equations (2.13)–(2.14) or (2.24)–(2.26), as the case may be, are the necessary tools for interpretation of the numerical results to be obtained for the η-π interaction, using an $\eta\pi\pi$ model for X^0.

3. $\eta\pi\pi$ FORMALISM

Consider a system of two pions and an η-meson with respective momenta $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$ in the over-all c.m. frame, so that

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3 = 0. \tag{3.1}$$

The momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ of $(\pi_1\eta)$ and $(\pi_2\eta)$ in their respective c.m. frames are

$$\mathbf{p}_1 = \mathbf{P}_1+\tfrac{1}{5}\mathbf{P}_2, \quad \mathbf{p}_2 = \mathbf{P}_2+\tfrac{1}{5}\mathbf{P}_1. \tag{3.2}$$

The π-π interaction in the over-all c.m. frame is expressible as

$$\langle \mathbf{P}_1\mathbf{P}_2|V_{\pi\pi}|\mathbf{P}_1'\mathbf{P}_2'\rangle = -\delta(P-P')\lambda\mu^{-1}g(p)g(p'), \tag{3.3}$$

where

$$\mathbf{P} = \mathbf{P}_1+\mathbf{P}_2, \tag{3.4}$$

$$2\mathbf{p} = \mathbf{P}_1-\mathbf{P}_2. \tag{3.5}$$

Similarly the $(\pi_1\eta)$ interaction may be written as

$$\langle P_1P_3|V_{\eta\pi}|P_1'P_3'\rangle = -\delta(P_2-P_2')\lambda_1\mu^{-1}v(p_1)v(p_1'), \tag{3.6}$$

[17] S. W. Macdowell, Phys. Rev. **116**, 774 (1959).

[18] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

[19] As the factor $f(k)$ differs little from unity, the error in this approximation is probably small.

with a corresponding expression for the $(\pi_2\eta)$ interaction. The primed momenta have definitions identical to (3.2), (3.4), and (3.5), The three-particle wave function Ψ is governed by the Schrödinger equation[20,21,22]

$$D_E(\mathbf{P}_1\mathbf{P}_2)\Psi(\mathbf{P}_1,\mathbf{P}_2)$$
$$=\lambda_1 v(p_1)\int d\mathbf{p}_1' v(p_1')\Psi(\mathbf{P}_1',\mathbf{P}_2)$$
$$+\lambda_1 v(p_2)\int d\mathbf{p}_2' v(p_2')\Psi(\mathbf{P}_1,\mathbf{P}_2')$$
$$+\lambda g(p)\int d\mathbf{p}' g(p')\Psi((\tfrac{1}{2}\mathbf{P}+\mathbf{p}'),(\tfrac{1}{2}\mathbf{P}-\mathbf{p}')), \quad (3.7)$$

where certain δ-function integrations have been carried out, and

$$p_1'=P_1+\tfrac{1}{5}P_2', \quad p_2'=P_2+\tfrac{1}{5}P_1', \quad (3.8)$$

$$D_E(\mathbf{P}_1\mathbf{P}_2)=\tfrac{5}{8}(P_1^2+P_2^2)+\tfrac{1}{4}\mathbf{P}_1\cdot\mathbf{P}_2-E\mu. \quad (3.9)$$

The solution of Eq. (3.7) is dependent on the type of boundary conditions that one may choose for the function Ψ. As we are interested in the formation of a resonant state, the problem is analogous to a bound-state one,[20] except for a complex energy eigenvalue. Such a requirement is obviously met by solving Eq. (3.7) as a homogeneous equation. Keeping this point in view, the solution of (3.7) is expressible as

$$\Psi=D_E^{-1}(\mathbf{P}_1\mathbf{P}_2)[v(p_1)F(P_2)$$
$$+v(p_2)F(P_1)+g(p)G(P)], \quad (3.10)$$

where F and G are as yet undetermined functions of single arguments, and the effect of Bose statistics has been included. The isospin effects which have been explicitly omitted, may be seen to appear only through the pions, so that the requirement of $T=0$ is trivially met by taking the potential $V_{\pi\pi}$, defined by Eq. (3.3) above, to be the one appropriate to $\mathrm{T}=0$. Further, according to the interpretations given in Ref. 19, $F(P_1)$ is the wave function of π_1 with respect to the center of mass of η and π_2, and $G(P)$ is that of η with respect to the center of mass of π_1 and π_2. Insertion of (3.10) in (3.7) leads in the usual way[20] to the following integral equations for F and G:

$$F(\mathbf{P}_2)[\lambda_1^{-1}-k(E\mu-\tfrac{3}{5}P_2^2)]$$
$$=\int d\mathbf{q}\, v(\mathbf{q}+\tfrac{1}{5}\mathbf{P}_2)v(\mathbf{P}_2+\tfrac{1}{5}\mathbf{q})F(\mathbf{q})[q^2+\tfrac{3}{8}P_2^2-E\mu]^{-1}$$
$$+\int d\mathbf{Q}\, v(\mathbf{Q}+\tfrac{4}{5}\mathbf{P}_2)g(\mathbf{P}_2+\tfrac{1}{2}\mathbf{Q})G(\mathbf{Q})$$
$$\times[\tfrac{5}{8}Q^2+\mathbf{Q}\cdot\mathbf{P}_2+P_2^2-E\mu]^{-1}, \quad (3.11)$$

$$G(\mathbf{P})[\lambda^{-1}-h(E\mu-\tfrac{3}{8}P^2)]$$
$$=2\int d\mathbf{q}\, g(\mathbf{q}+\tfrac{1}{2}\mathbf{P})v(\mathbf{P}+\tfrac{4}{5}\mathbf{q})F(\mathbf{q})$$
$$\times[q^2+\mathbf{q}\cdot\mathbf{P}+\tfrac{5}{8}P^2-E\mu]^{-1}, \quad (3.12)$$

where

$$h(z)=\int g^2(p)d\mathbf{p}(p^2-z)^{-1}, \quad (3.13)$$

$$k(z)=\int v^2(p)d\mathbf{p}(\tfrac{5}{8}p^2-z)^{-1}. \quad (3.14)$$

Substitution of (3.12) in (3.11) gives a single integral equation for $F(\mathbf{P}_2)$. However, before writing this down, it is convenient to carry out certain angular integrations, noting that for the treatment of a homogeneous equation like the present one, the vectors involved in $G(\mathbf{P})$ and $F(\mathbf{P}_2)$ have no "sense of direction" (i.e., they are merely functions of P and P_2, respectively). Thus with the definitions[21]

$$4\pi H(x,y,E)=\int d\Omega\, g(\mathbf{x}+\tfrac{1}{2}\mathbf{y})v(\mathbf{y}+\tfrac{4}{5}\mathbf{x})$$
$$\times[\tfrac{5}{8}y^2+\mathbf{x}\cdot\mathbf{y}+x^2-E\mu]^{-1}, \quad (3.15)$$

$$4\pi J(q,p,E)=4\pi J(p,q,E)=\int d\Omega\, v(\mathbf{q}+\tfrac{1}{5}\mathbf{p})v(\mathbf{p}+\tfrac{1}{5}\mathbf{q})$$
$$\times[\tfrac{5}{8}p^2+\tfrac{5}{8}q^2+\tfrac{1}{4}\mathbf{p}\cdot\mathbf{q}-E\mu]^{-1}, \quad (3.16)$$

the integral equation for $F(p)$ is

$$F(p)[\lambda_1^{-1}-k(E\mu-\tfrac{3}{8}p^2)]$$
$$=4\pi\int_0^\infty q^2dq\, F(q)M(p,q,E), \quad (3.17)$$

where

$$M(p,q,E)=M(q,p,E)$$
$$=J(p,q,E)+8\pi\int_0^\infty Q^2dQ\, H(p,Q,E)H(q,Q,E)$$
$$\times[\lambda^{-1}-h(E\mu-\tfrac{3}{8}Q^2)]^{-1}. \quad (3.18)$$

Equation (3.17) is the required eigenvalue equation in the complex energy E. It is however, more convenient to regard it as an eigenvalue equation in λ_1^{-1} for a given value of E. If the width of the three-body resonance is assumed to be small (as is actually the case for the physical particle X^0 under consideration), the above equation can be considerably simplified by ignoring the imaginary part of all the functions H, J, h, and k. This amounts to evaluating these functions as "principal value" integrals.

For purposes of numerical evaluation, it is more convenient to recast Eq. (3.17) in the form

$$\lambda_1^{-1}F_1(p)=\int_0^\infty dq\, L(q,p,E)F_1(q), \quad (3.19)$$

[20] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).

[21] The case of unequal masses in a three-body problem has also been discussed by other authors. See, e.g., I. Sh. Vashakidze and G. A. Chilasvilli Dokl. Akad. Nauk. SSSR **157**, 557 (1964) [English transl.: Soviet Phys.—Doklady **9**, 576 (1965)]; and J. Hetherington and L. Schick, Phys. Rev. **139**, B1164 (1965).

[22] See, also, A. N. Mitra, Phys. Rev. **139**, B1472 (1965).

where

$$F_1(p)=pF(p) \tag{3.20}$$

and

$$L(q,p,E)=4\pi pqM(p,q,E)+k(E\mu-\tfrac{3}{5}p^2)\delta(p-q) \tag{3.21}$$

$$=L(p,q,E). \tag{3.22}$$

This brings out explicitly how the kernel of the "eigenvalue equation" (3.19) is symmetric in the two variables p and q. The expressions for the various functions appearing in the kernel L are given in the Appendix.

4. NUMERICAL RESULTS AND DISCUSSION

Equation (3.19) was used for an evaluation of the quantity λ_1^{-1} for several values of the input parameters β, β_1, λ and E, on the CDC 3600 Computer situated at the University of Wisconsin Computing Center. For the evaluation of the position of the resonance, which was the main object of this investigation, the feed-back effects of the imaginary part of the kernel were neglected. This neglect was based on the argument that, if our claims to the model as a possible mechanism for a resonance at the position of X^0 were justified at all, the (experimentally) small width of X^0 ($\sim$10 MeV) as well as its relatively low Q value ($\sim$120 MeV) should bear out the validity of the above approximation. This would reduce the eigenvalue problem in Eq. (3.19) to one involving only *real* values of λ_1^{-1}, which could then be interpreted in terms of the results of Sec. 2 (for η-π bound states or resonances).

The choice of the input parameters was motivated by the following physical considerations. There are two possible mechanisms for holding a three-body resonance. First, if we believe in the existence of a π-π s-wave resonance somewhere between 400 and 700 MeV, for which evidences have been periodically appearing and disappearing, our model would give a π-π potential well with a repulsive barrier, within which the η particle should find itself occasionally (if it has an attractive interaction with a pion), and this would suffice for an η-π resonance. Such a situation would be covered by taking rather small values of the parameter β associated with negative values of λ for a reasonably narrow π-π resonance, according to the results of Ref. 10 and Sec. 2.

Secondly, if the π-π interaction is a smooth ABC type one, we must rely on a potential well inside a repulsive barrier, arising this time from the η-π, rather than the π-π interaction. This case is described by taking $\lambda>0$ and β of the order of a pion mass or two. In this case one would expect to find negative output values of λ_1^{-1} especially if rather small input values of β_1 are used.

In order to estimate the importance of the reactive effects on the qualitative trend of the results, two different input values of E were chosen, (i) $E=0.864\,\mu$ corresponding to the experimental position of X^0, and (ii) $E=0$ for which the reactive effects are precisely zero. A broad similarity of the results, if obtained for the two cases, would provide an *a fortiori* justification for the neglect of reactive effects for $E>0$, at least for a qualitative assessment of our model as a candidate for X^0.

The results of evaluation of λ_1^{-1} (in units of μ^{-3}) for some typical values of β, β_1 and λ^{-1} at the experimental X^0-mass are listed in Tables I and II. The numbers in parentheses represent the corresponding values obtained for a just-bound $\eta\pi\pi$ state, viz., $E=0$. In Table I, the input values of β and λ^{-1} for a σ-type π-π interaction correspond to the following π-π resonance parameters:

$$\text{(i)}\quad E_{\pi\pi}=397\text{ MeV},\quad \Gamma_{\pi\pi}=92\text{ MeV},\quad \beta/\mu=0.2,\quad \lambda^{-1}=-43.80\,\mu^{-3}; \tag{4.1}$$

$$\text{(ii)}\quad E_{\pi\pi}=397\text{ MeV},\quad \Gamma_{\pi\pi}=127\text{ MeV},\quad \beta/\mu=0.25,\quad \lambda^{-1}=-32.78\,\mu^{-3}; \tag{4.2}$$

$$\text{(iii)}\quad E_{\pi\pi}=626\text{ MeV},\quad \Gamma_{\pi\pi}=100\text{ MeV},\quad \beta/\mu=0.2,\quad \lambda^{-1}=-12.50\,\mu^{-3}. \tag{4.3}$$

In Table II, for an ABC-type interaction, the parameters β and λ^{-1} have been chosen to give the following scattering length:

$$a_{\pi\pi}=1.3\mu\pm0.3\mu, \tag{4.4}$$

according to the analysis of Hamilton *et al.*[23] The output values of λ_1^{-1} can most easily be interpreted in terms of a scattering length or the position and width of a π-η resonance, according to the formulas of Sec. 2, and these are listed in Table III for the cases of interest.

Our main conclusions can be summarized as follows. For σ-type π-π interaction, the η-π force is attractive, its strength increasing with the energy of the π-π reso-

TABLE I. Output values of λ_1^{-1}, in units of μ^{-3} for σ-type π-π interaction. The quantities in parentheses represent the corresponding values when the $\eta\pi\pi$ state is just bound.

		$E_{\pi\pi}=397$ MeV		$E_{\pi\pi}=626$ MeV
β/μ	β_1/μ	$\lambda^{-1}=-43.80\,\mu^{-3}$	$\lambda^{-1}=-32.78\,\mu^{-3}$	$\lambda^{-1}=-12.50\,\mu^{-3}$
0.2	2.0	1.083(0.796)		1.116
0.2	1.5	1.546		1.754
0.25	2.0		1.325(0.825)	
0.25	1.5		1.758(1.061)	

TABLE II. Output values of λ_1^{-1} in units of μ^{-3} for ABC-type π-π interaction. The quantities in brackets represent the corresponding values when the $\eta\pi\pi$ state is just bound.

β/μ	β_1/μ	$\lambda^{-1}=2.206\,\mu^{-3}$	$\lambda^{-1}=5.933\,\mu^{-3}$
1.5	0.3		−1.856(−2.004)
1.5	0.4		−3.222(−3.370)
1.5	0.46		−3.682
1.5	0.52		−3.964
2.0	0.3	−1.857(−2.002)	
2.0	0.4	−3.204(−3.318)	
2.0	0.46	−3.641	
2.0	0.52	−3.923	

[23] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

nance (as a sort of compensation for the weaker π-π force that is associated with a higher energy resonance). This is indicated by the dimensionless strength parameter $\sigma_1=\lambda_1\pi^2\beta_1^{-3}$. For all the cases considered, it is found that $a_{\eta\pi}<0$, suggesting the existence of a bound η-π state. This qualitative conclusion seems to be a general feature of the results of this investigation. and may be taken as evidence against the existence of a π-π s-wave resonance like σ or ϵ_0, in so far as a bound η-π state, being a charged particle, would have been fairly easy to observe if it had existed.

For the ABC-type π-π interaction, our model seems to predict the existence of an η-π resonance in the BeV region, with an appreciable width. Now in so far as this energy range is still far from fully explored, and many more meson resonances are still being discovered in this range, it may be too early to rule out the existence of such a particle. On the other hand, the low-energy manifestation of such an η-π resonance would be a negative scattering length, indicating an η-π repulsion.

For both types of interaction, the corresponding results for $E=0$ are fully in accord with the above conclusions, thus confirming our earlier belief that the reactive effects for $E>0$ are small enough to maintain these conclusions.

We have made no attempt in this paper to calculate the width of the X^0 resonance. However, it is possible to give some qualitative arguments to suggest that this width is small. With either of the two proposed mechanisms, the $\eta\pi\pi$ resonance may be regarded as a bound state of a stable and a resonant particle, viz., $\eta+\sigma$ and $\eta^*+\pi$ for the σ-type and ABC-type π-π interactions, respectively. Now as a composite of $\eta^*+\pi$ it is a very tightly bound system as is clear from the low Q value ($\sim$120 MeV) of the particle in relation to the high mass of the η^*. This fact should make it rather difficult for the pion to be away from the η-π potential well for any appreciable amount of time. This will necessarily make the resonance a "long-lived" one, giving rise to a small width. On the other hand, with a π-π resonance like σ in the range 400–600 MeV, a composite of $(\sigma+\eta)$ at the position of π would not be a strongly bound system, and could, therefore, allow a much larger width. Our neglect of the reactive effects would, therefore, seem to be justified at least for the $(\eta^*+\pi)$ mechanism for X^0, which we regard as a more plausible candidate for an understanding of the internal structure of this particle.

TABLE III. Interpretation of λ_1^{-1} in terms of η-π scattering or resonance parameters.

$\mu^3\lambda_1^{-1}$	β_1/μ	σ_1 $(=\lambda_1\beta_1^{-3}\pi^2)$	$\mu^{-1}a_{\eta\pi}^{-1}$	$\mu^{-1}E_{\eta\pi}$	$\mu^{-1}\Gamma_{\eta\pi}$
1.083	2.0	1.14	−0.452		
1.325	2.0	0.93	−0.329		
1.541	1.5	1.90	−0.503		
1.758	1.5	1.66	−0.467		
−1.858	0.3	−196		14.42	1.10
−3.222	0.4	−47.5		10.29	1.34
−3.682	0.46	−27.6		9.35	1.47
−3.964	0.52	−17.7		8.89	1.62

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APPENDIX

We list here the explicit expressions for the various functions needed for a numerical solution of the $\eta\pi\pi$ integral equation.

From (2.2), (2.10), (3.13), and (3.14), we have

$$h(z)=\pi^2\beta^{-1}[\beta+(-z)^{1/2}]^{-2}, \qquad \mathrm{Re}z<0 \tag{A1}$$

$$=\pi^2(\beta^2-z)\beta^{-1}(\beta^2+z)^{-2}, \qquad \mathrm{Re}z>0; \tag{A2}$$

$$k(z)=(8/5)\pi^2\beta_1^{-1}[\beta_1+(-8z/5)^{1/2}]^{-2}, \quad \mathrm{Re}z<0 \tag{A3}$$

$$=(8/5)\pi^2(\beta_1^2-(8/5)z)\beta_1^{-1}(\beta_1^2+(8/5)z)^{-2}, \qquad \mathrm{Re}z>0. \tag{A4}$$

This is in accord with our approximation of neglecting reactive effects for $\mathrm{Re}E>0$, (see Sec. 4). Further, with the definitions

$$A_1=x^2+\tfrac{1}{4}y^2+\beta^2, \quad A_2=\tfrac{5}{8}y\tfrac{1}{5}+\tfrac{2}{5}x^2+\tfrac{5}{8}\beta_1^2, \quad A_3=\tfrac{5}{8}y^2+x^2-E\mu; \tag{A5}$$

$$B_1=\tfrac{5}{2}q^2+\tfrac{1}{10}p^2+\tfrac{5}{2}\beta_1^2, \quad B_2=\tfrac{5}{2}p^2+\tfrac{1}{10}q^2+\tfrac{5}{2}\beta_1^2, \quad B_3=\tfrac{5}{2}(p^2+q^2)-4E\mu; \tag{A6}$$

the expressions for H and J are

$$32\pi H(x,y,E)=5\sum_{ijk}(A_i-A_j)^{-1}(A_i-A_k)^{-1} \times\ln|(A_i+xy)/(A_i-xy)|, \tag{A7}$$

$$4\pi J(p,q,E)=25\sum_{ijk}(B_i-B_j)^{-1}(B_i-B_k)^{-1} \times\ln|B_i+pq)/(B_i-pq)|. \tag{A8}$$

Here the indices i, j, k run over the values 1, 2, 3 and $i\neq j\neq k$.