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Nonrelativistic Quark Model for Baryons

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A nonrelativistic quark model is proposed for baryons, according to which any two quarks are assumed to interact with each other through *p*-wave forces. Such forces are shown to be capable of producing strong binding in a three-quark system in a spatially antisymmetric state of angular-momentum unity, and making the model compatible with an extension of the [56] representation of SU_6. If the strength of the quark-quark force is adjusted to fit some central baryon mass (m_0), the model predicts a 2-quark bound state at a mass $\sim\frac{1}{2}(M+m_0)$, where M is the central mass of a quark. The validity of the nonrelativistic description is shown to depend on the smallness of a certain "inverse range parameter" β compared with the quark mass M, and this condition is shown to be fully compatible with the present experimental knowledge on baryon sizes, as measured by the charge radius of the proton. Further, using an SU_3-invariant interaction, an "equal interval rule" for the baryon masses is shown to follow *dynamically* from the assumption of a mass difference between the singlet and doublet quarks, under the same condition, $\beta\ll M$, as above. It is argued that a *p*-wave quark interaction, which leads more easily to the formation of antisymmetric spatial states than of symmetric ones, gives a sort of "saturated system" at the 3-quark level. This reduces considerably the (undesirable) prospects of very strong binding of a larger number of quarks, compared to the situation with *s*-wave forces (which facilitate the formation of symmetric states in multiquark systems with stronger and stronger binding as the number of quarks is increased). By ruling out the generally stronger *s*-wave forces as the main bond between two quarks, the model leaves scope for their action in quark-antiquark systems, which should require stronger binding in order to generate the (less massive) mesons.

INTRODUCTION

THE "quark" picture of baryons and mesons proposed independently by Gell-Mann[1] and Zweig[2] has proved of great value for attempts at dynamical formulations of a theory of elementary particles by many authors, using SU_3 symmetry and various relativistic generalization of SU_6 symmetry.[3] In a less fundamental way, several authors have tried to explore dynamical models of baryons and mesons with nonrelativistic quarks through appropriate Schrödinger equations.[4,5] Through such limited approaches it is already possible to understand the Gell-Mann–Okubo (GMO) mass formula, the equal interval rule for the baryon decouplet, the Schwinger mass formula for mesons[6] and the $-\frac{3}{2}$ ratio for the nucleon moments.[5,7] In spite of these successes, the assumption of nonrelativistic quarks would at first sight probably appear unjustified, because of the huge binding energies that must be required to offset the effect of the quark rest masses. However, it has been argued by Morpurgo[5] that this feature by itself need not prove an obstacle to a nonrelativistic description for the quarks. The more important part, as he shows by simple quantum-mechanical arguments, is played by the range of the Q-Q or Q-$\bar{Q}$ interactions,[8] which should not be much shorter than the inverse mass of, say, a vector meson, to make the huge binding energies compatible with nonrelativistic quarks. While a similar condition is not fulfilled in a Fermi-Yang model of the pion, there is *a priori* nothing against the assumption of a "long range" Q-Q or Q-$\bar{Q}$ force[9] (since very little is known about quarks anyway). If this idea is taken seriously, it is appealing enough to warrant investigations of a more detailed nature. In particular it may be interesting to ask if the available data can discriminate between certain types of Q-Q and Q-$\bar{Q}$ interactions that must be assumed in a composite quark model of baryons or mesons. This question has been the main motivation behind the present investigation.

One baryon model consists of three nonrelativistic quarks which interact in pairs. In the limit of SU_3 symmetry, these interactions are taken to be identical. Symmetry breaking is considered only to the extent of taking unequal masses M_2 and M_1 for the "strange" and "nonstrange" quarks, respectively, where the difference $\Delta M=M_2-M_1$ is small compared with the central mass $M=\frac{1}{3}(2M_1+M_2)$.

In Sec. 2, we discuss some evidences bearing on the nature of the input Q-Q interaction on the basis of available data. In particular we present some qualitative arguments favoring a *p*-wave interaction rather than a more conventional *s*-wave force. The final choice made to facilitate the treatment of the "three-body problem" at hand, is a separable *p*-wave Q-Q force. In Sec. 3, we calculate the binding energy of a $3Q$ system in the limit when the mass difference between the quarks is ignored, and obtain a relation between the masses of the $2Q$ and $3Q$ systems. The validity of the nonrelativistic descrip-

[1] M. Gell-Mann, Phys. Letters **8**, 214 (1964).
[2] G. Zweig; CERN Reports 8182/TH.401 and 8419/TH.412, 1964 (unpublished).
[3] See, e.g., Proceedings of the Coral Gables Conference on Symmetry Principles, 1965 (unpublished).
[4] N. N. Bogolubov *et al.*, Dubna JINR Reports D-1968, D-2075, P-2141, 1965 (unpublished).
[5] G. Morpurgo (to be published).
[6] J. Schwinger, Phys. Rev. Letters **12**, 237 (1964).
[7] B. Struminski, Dubna JINR Report P-1939 (1965, unpublished).
[8] We shall from now on use the abbreviations Q and $\bar{Q}$ for quarks and antiquarks, respectively.
[9] The term "long range" is used in a purely comparative sense: the inverse range must be compared with the mass of the quark, rather than with that of the baryon.

tion is shown to depend on the smallness of the parameter β/M, where β is an "inverse range" parameter of the interaction. In Sec. 4, and in the Appendix, the masses of the baryons are calculated for unequal quark masses, and an "equal-interval rule" is established under the assumptions $\beta \ll M$, $\Delta M \ll M$. In Sec. 5, the main conclusions are summarized and the model is also shown to be compatible with the right order of magnitude for a baryon size.

2. THE Q-Q INTERACTION

In the limit of SU_6 symmetry, the [56] representation which is a fully symmetric state in spin and unitary spin, is generally believed to be the one appropriate for baryons. In particular, this is the only representation which gives the correct $-\frac{3}{2}$ ratio for the nucleon moments,[5,7] though not their absolute values. If the quarks are assumed to obey Fermi statistics, it is clear that the [56] representation must be associated with a spatially antisymmetric state. Another possibility is to relax the requirement of Fermi statistics for the quarks, either using the ideas of parastatistics[10] or through the introduction of additional quantum numbers.[11] If such a relaxation is permissible, there need be no hindrance in principle to the assumption of a totally symmetric spatial state for a [56] $3Q$ system.

What do these respective requirements of symmetric or antisymmetric states imply in terms of an input Q-Q interaction? Normally one would expect a totally symmetric 3-particle state to be most easily constructed with the help of potentials which are *even* with respect to the interchange of coordinates (e.g., of the Serber type), i.e., potentials which are dominant in s waves. Similarly for a totally antisymmetric state one would require *odd* potentials, i.e., those which have p-wave dominance.

Thus we have essentially to choose one of these two alternatives and the question that must now be answered is whether there are any simple criteria which can distinguish one from the other. We feel that there are already some physical evidences bearing on this point. For example, if the idea of nonrelativistic quarks for baryons and mesons makes any sense at all, it must account for the fact that the mass of a baryon is much higher than that of a meson, which in turn implies that a Q-$\bar{Q}$ interaction must be substantially stronger than a Q-Q force. The simplest way to meet this requirement is to assign the odd (p-wave) interaction to the Q-Q system, and reserve the generally stronger even (s-wave) force for the Q-$\bar{Q}$ pair. A second consequence of the preference for a p-wave over an s-wave Q-Q force bears on systems involving an *even* number of quarks. A Q-Q force, whether p- or s-wave, designed to produce such a large binding energy in a $3Q$ system as nearly to offset the effect of the quark rest masses, must in general be strong enough to bind a $2Q$ system as well. Now if the Q-Q interaction strength is so adjusted as to reproduce the mass of a $3Q$ system at the level of the baryon mass, simple quantum-mechanical arguments suggest that the mass of a $2Q$ system should be of the order of a quark mass. Thus insofar as the mass of a quark may be supposed to be high enough for observation with present-day experimental facilities, a $2Q$ system could elude observation as well, and to that extent would not be able to discriminate between s- or p-wave Q-Q interactions. The case is, however, different with a $4Q$ system. For example, if four quarks are allowed, by statistics or otherwise, to be in a totally symmetric state (and this condition is particularly easy to fulfil with s-wave Q-Q forces), then by analogy with the famous α-particle case, the binding energy in a $4Q$ system will be relatively much larger than in a $3Q$ system, giving rise to the embarrassing prediction of a bound $4Q$ state with a mass much lower than the baryon, and fractional charge. With a p-wave Q-Q interaction, on the other hand, a totally symmetric $4Q$ state is clearly ruled out and such embarrassments avoided. As a matter of fact, a p-wave interaction gives rise to a kind of "saturated" antisymmetric state at the three-particle level, so that it is at least plausible that a larger number of quarks, under the action of p-wave forces, could have relatively a much higher energy than a $3Q$ system.

In this respect we cannot help recalling an analogy with the results of some recent calculations made with 3π systems[12,13] and 4π systems[14] as possible models for (ω,ϕ) and (η,X^0) resonances, respectively, through p-wave π-π interactions of the factorable type. It was found that with an intrinsically attractive interaction (called type a interaction in Ref. 14), a 3π state of $J^PI^G=1^-0^-$ was too tightly bound for the ω meson,[12] though other states like 0^-0^- had repulsive kernels. Further, a 4π state of 0^-0^+ with such π-π forces had a much higher energy than the sum of the pion masses. These results thus indicated that with p-wave forces the energy of a multiparticle state could depend quite strongly on the quantum numbers of the state involved. In particular, certain 3-particle states could be strongly attractive, while their 4-particle counterparts exhibited little attraction. This fact might be taken as suggestive of the present idea that a p-wave Q-Q interaction offers much less chance of a strongly bound $4Q$ system than does an s-wave force. Secondly, the prediction of a tightly bound antisymmetric 3π state with p-wave forces, which made such interactions look rather unphysical for the ω particle, *may be turned to advantage* in the present situation which demands very strong binding in a

[10] O. W. Greenberg, Phys. Rev. Letters **13**, 598 (1964).

[11] Such ideas have been used, particularly by the Dubna group. The author is indebted to Dr. A. N. Tavkhelidze for this information.

[12] A. N. Mitra, Phys. Rev. **127**, 1342 (1962); hereafter referred to as A.

[13] A. N. Mitra, Nuovo Cimento 33, 1235 (1964).

[14] A. N. Mitra and S. Ray, Phys. Rev. **137**, B982 (1965); hereafter referred to as B.

$3Q$ system. The fact that p-wave forces can in principle provide a strong binding, at least in certain quantum states, should be regarded as a particularly welcome feature in a context where precisely such forces are being looked for.

As for the shape of the p-wave Q-Q force we shall, as in B, choose the separable structure as a convenient device to treat a three-body problem. In the approximation of ignoring symmetry-breaking interactions, which makes the forces spin- and unitary-spin independent, only the spatial structure of the $3Q$ wave function comes into the dynamical picture. These spatial structures can of course be translated in terms of appropriate representations of the SU_3 or SU_6 groups. For example, with Fermi statistics for the quarks, a totally antisymmetric wave function must be associated with the representation [56]. Effects of orbital angular momentum in the spatial structure can also be included, on the lines of Mahanthappa and Sudarshan,[15] by an appropriate enlargement of the group from SU_6 to $SU_6 \otimes O_3$. For example, for a "vector" $3Q$ state, the desired extension of [56] is (56,3). However, since our basic object in this paper is to calculate a few dynamical effects of Q-Q forces, we shall not dwell any further on group structures, so that spin and unitary-spin functions will not explicitly enter into our calculations. It will thus be enough for our purposes to regard the quarks as nonrelativistic "scalar objects" which are identical except for a possible mass difference between the strange and nonstrange components of the triplet.

For a $2Q$ system with equal masses M, the Q-Q force is

$$\langle p|V|p'\rangle = -3\lambda M^{-1}v(p)v(p')\mathbf{p}\cdot\mathbf{p}', \qquad (2.1)$$

where, following B, the shape factor is chosen as

$$v^2(p)=\exp(-p^2\beta^{-2}). \qquad (2.2)$$

If the strength parameter λ is sufficiently large (as we expect it to be), the $2Q$ binding energy α_1^2/M can be deduced from the two-body Schrödinger equation,

$$(p^2+\alpha_1^2)\psi(\mathbf{p})=3\lambda v(p)\int d\mathbf{q}\;\mathbf{p}\cdot\mathbf{q}v(q)\psi(\mathbf{q}), \qquad (2.3)$$

in the standard manner[12,13] as

$$\lambda^{-1}=4\pi\int_0^\infty q^4dq\;e^{-q^2\beta^{-2}}[q^2+\alpha_1^2]^{-1}. \qquad (2.4)$$

On the assumption that β^2 is small compared with α_1^2, the integrand on the right of (2.4) has a sharp peak at $q^2\approx\frac{5}{2}\beta^2$ (see Appendix I of B), so that (2.4) reduces to

$$\tfrac{5}{2}\beta^2+\alpha_1^2\approx\tfrac{3}{2}\beta^2\sigma, \qquad (2.5)$$

where

$$\sigma=\lambda\pi^{3/2}\beta^3. \qquad (2.6)$$

The mass M^* of the $2Q$ system is then given by

$$2M-M^*\approx\tfrac{1}{2}M^{-1}\beta^2(3\sigma-5). \qquad (2.7)$$

3. THE 3Q PROBLEM WITH EQUAL MASSES

In the approximation where the mass difference between the strange and nonstrange quarks is neglected, and each Q-Q pair has an interaction of the form (2.1), the dynamical problem is almost identical to the formalism developed in Refs. 12 and 13, except for the fact that the nonrelativistic bound-state situation obtaining in the present case should provide a much better justification for some of the approximations made in them.

Let the momenta of the quarks be taken as $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, where

$$\mathbf{P}_1+\mathbf{P}_2+\mathbf{P}_3=0. \qquad (3.1)$$

The interaction between $Q(i)$ and $Q(j)$ in the over-all frame is given by

$$\langle\mathbf{P}_i\mathbf{P}_j|V_{ij}|\mathbf{P}_i'\mathbf{P}_j'\rangle=\delta(\mathbf{P}_k-\mathbf{P}_k')\langle\mathbf{p}_{ij}|V_{ij}|\mathbf{p}_{ij}'\rangle, \qquad (3.2)$$

where

$$\mathbf{P}_i+\mathbf{P}_j=-\mathbf{P}_k, \qquad (3.3)$$

$$2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j, \qquad (3.4)$$

and the two-body c.m. potential in (3.2) is as given by (2.1) and (2.2). As in A, the $3Q$ system obeys the Schrödinger equation

$$D_\alpha(\mathbf{P}_i)\Psi=-(\sum_{ij}V_{ij})\Psi, \qquad (3.5)$$

where

$$D_\alpha(\mathbf{P}_i)=\tfrac{1}{2}M^{-1}(P_1^2+P_2^2+P_3^2)+\alpha^2M^{-1}, \qquad (3.6)$$

and α^2/M is the $3Q$ binding energy related to the central mass m_0 of a baryon by

$$\alpha^2=3M^2-Mm_0. \qquad (3.7)$$

Substitution of (2.1) into (3.5) yields, in the usual way, an explicit structure for Ψ which, for a fully antisymmetrized axial-vector state is of the form (see A for details);

$$\Psi=MD_\alpha^{-1}(\mathbf{P}_i)\sum_{ijk}\mathbf{p}_{ij}\times\mathbf{P}_kv(p_{ij})F(P_k). \qquad (3.8)$$

The "spectator function" F satisfies the equation

$$[1-\lambda h(P^2)]F(P)=3\lambda\int d\mathbf{q}\;q^2\sin^2\theta v(\mathbf{P}+\tfrac{1}{2}\mathbf{q})v(\mathbf{q}+\tfrac{1}{2}\mathbf{P})\times[P^2+q^2+\mathbf{P}\cdot\mathbf{q}+\alpha^2]^{-1}F(q), \qquad (3.9)$$

where an integration has been carried out on the azimuth angle of $\hat{q}$ on the right-hand side of (3.9). The

[15] K. T. Mahanthappa and E. C. G. Sudarshan, Phys. Rev. Letters 14, 163 (1965).

function $h(P^2)$ is given by

$$\lambda h(P^2)=4\pi\lambda\int_0^\infty q^4\,dq\,e^{-q^2\beta^{-2}}[q^2+\tfrac{3}{4}P^2+\alpha^2]^{-1} \quad (3.10)$$

$$\approx\tfrac{3}{2}\sigma\beta^2[\tfrac{3}{4}P^2+\tfrac{5}{2}\beta^2+\alpha^2]^{-1}, \quad (3.11)$$

where, under the (extremely good) approximation $\beta^2\ll\alpha^2$, the integration in (3.10) has been carried out as in (2.4). Indeed, from (3.7) the condition $\beta^2\ll\alpha^2$ is a very generous one on β, and requires the range β^{-1} to be "large" only in comparison to M^{-1}, not the inverse baryon mass m_0^{-1}, so that it may even be "short" for all practical purposes.

For further manipulation of (3.9), we must make use of the approximations described in A and B, especially in the latter. Indeed, the presence of the huge binding energy term α^2 in the energy denominations of (3.9), (3.10), etc., greatly enhances the validity of the approximations that had been made in B for a "resonance problem" ($E>0$). Thus the neglect of angular correlations on the right of (3.9), an approximation which was already found to be good in B, should be very well justified in the present case. This gives

$$[1-\lambda h(P^2)]F(P)=8\pi\lambda\int_0^\infty q^4\,dqF(q)$$
$$\times\exp[-\tfrac{5}{8}\beta^{-2}(P^2+q^2)][P^2+q^2+\alpha^2]^{-1}. \quad (3.12)$$

The ansatz

$$F(P)=G(P)e^{-(5/8)P^2\beta^{-2}}[1-\lambda h(P^2)]^{-1}, \quad (3.13)$$

where $G(P)$ is assumed a slowly varying function, reduces Eq. (3.12) to

$$G(P)=8\pi\lambda\int_0^\infty q^4\,dq\,e^{-(5/4)q^2\beta^{-2}}G(q)[1-\lambda h(q^2)]^{-1}$$
$$\times[P^2+q^2+\alpha^2]^{-1}. \quad (3.14)$$

To solve this equation, it is now merely necessary to make use of the approximation that had led from (3.10) to (3.11), since for $\alpha^2\gg\beta^2$, the integrand in (3.14) has a sharp peak at

$$q^2\approx q_m^2=\tfrac{5}{2}\times\tfrac{4}{5}\beta^2=2\beta^2, \quad (3.15)$$

and the factor $1-\lambda h(q^2)$ which is a slowly varying function, does not have any zero in the entire region $0\leq q\leq\infty$. This approximation which is again good in the present case, leads finally to the explicit formula

$$\alpha^2+4\beta^2-\tfrac{3}{2}\sigma\beta^2\approx3\sigma\beta^2\tfrac{4}{5}^{5/2}, \quad (3.16)$$

whence the central baryon mass m_0 is given by

$$3M-m_0\approx1.05\beta^2(3\sigma-3.8)/M. \quad (3.17)$$

Both the equations (2.7) and (3.17) show that to make the quark masses compatible with the 10-BeV region, the dimensionless "strength parameter" σ must be extremely large, so that $3\sigma-5\approx3\sigma-3.8$. Eliminating β between (2.7) and (3.17) then yields the relation

$$M^*\approx0.53M+0.48m_0, \quad (3.18)$$

which predicts the mass of the "di-quark" to be somewhat larger than half the quark mass.[16]

To obtain some ideas on the nonrelativistic assumption, we note that the two basic momenta p_{ij}^2 and P_k^2 have distributions governed *essentially* by the functions $v^2(p)$ and $F^2(P)$, respectively. Since both these functions are Gaussian (or nearly so) with parameters β^2 and $\frac{4}{5}\beta^2$, respectively, $\langle p_{ij}^2\rangle$ and $\langle P_k^2\rangle$ are both of order β^2. The condition for the nonrelativistic approximation is then merely

$$\beta^2\ll M^2, \quad (3.19)$$

which is also the condition for the approximation of "peaked integrands" that we have extensively used in the derivation of the mass formula.

4. MASS FORMULA WITH UNEQUAL MASS QUARKS

We next consider the case when the common mass M_1 of the two nonstrange quarks is taken to be different from the mass M_2 of the strange quark. The "central mass" of the quark may be defined as

$$M=\tfrac{1}{3}(2M_1+M_2) \quad (4.1)$$

and the mass differences expressed in terms of the dimensionless parameter

$$\epsilon=\Delta M/M=(M_2-M_1)/3M, \quad (4.2)$$

which we expect to be a small quantity.

The kinematics are somewhat more involved in this case, but a procedure essentially similar to Sec. 3 is valid for the calculation of the various baryon masses. The masses of the quarks which are involved in the internal structures of the various baryons are summarized in Table I. For convenience both the baryon SU_3 representations [8] and [10] are treated simultaneously in the present analysis. This is not to suggest that the interaction parameters β and λ in our potential (2.1) are the same for the two cases. Indeed, within SU_3 symmetry there is no reason to take them as identical for

TABLE I. Masses of the quarks involved in the internal structures of the various baryons.

Baryon	N; N^*	(Σ,Λ); Y^*	Ξ; Ξ^*	Ω^-
Q masses	$3M_1$	$2M_1+M_2$	M_1+2M_2	$3M_2$
Mass differences	$-3\Delta M$	0	$3\Delta M$	$6\Delta M$
Dynamical corrections	$+4\beta^2\epsilon/M$	0	0	$-8\beta^2\epsilon/M$

[16] E.g., with $M=10$ and 15 BeV, $M^*\approx5.7$ and 8.4 BeV, respectively.

the treatment of the [8] and [10] baryons. Since the various combinations of the quark masses involved in Table I already give the "equal interval" rule, one would, within SU_3 symmetry, expect the present scheme to work much better for the decouplet than for the octet case (which requires an SU_3-violating interaction to account for the Σ-Λ mass difference). This expectation is of course based on the assumption that the "equal interval rule" for baryons will not be affected by the "dynamical corrections," provided the basic Q-Q interaction is SU_3-invariant. This result is of course known from perturbation calculations and group-theoretical considerations. What we want to show here is that the conclusion will be maintained also by our dynamical calculations on the basis of a Schrödinger equation which takes account of the quark mass differences. We shall also obtain explicit expressions for the dynamical corrections to the equal interval rule and show how their smallness is related to the inverse range parameter β of the Q-Q interaction.

As our basic interaction is taken to be SU_3-invariant, we shall use the same expression as (2.1) for the Q-Q potential, the quantity M now representing the central mass (4.1). However, the expression for the kinetic energy will now depend on the number and types of the quarks involved, and can easily be written down from Table I. The cases $3M_1$ and $3M_2$ which are particularly simple, can be directly adapted to the formalism of Sec. 3, with the modifications

$$P_1^2+P_2^2+P_3^2 \to (1+\epsilon)(P_1^2+P_2^2+P_3^2), \quad (4.3)$$

$$P_1^2+P_2^2+P_3^2 \to (1-2\epsilon)(P_1^2+P_2^2+P_3^2), \quad (4.4)$$

in the operator $D_\alpha(\mathbf{P}_i)$ of Eq. (3.6), for the two cases $3M_1$ and $3M_2$, respectively. Proceeding exactly as in Sec. 3, one now obtains the following expressions for the binding energy parameters for the $Y=1$ and $Y=-$ baryons:

$$\alpha^2(N^*)+4\beta^2(1+\epsilon)\approx 3.15\beta^2\sigma, \quad (4.5)$$

$$\alpha^2(\Omega^-)+4\beta^2(1-2\epsilon)\approx 3.15\beta^2\sigma, \quad (4.6)$$

for which the corresponding masses are

$$m(N^*)=m_0-3\Delta M+4\beta^2\epsilon/M, \quad (4.7)$$

$$m(\Omega^-)=m_0+6\Delta M-8\beta^2\epsilon/M, \quad (4.8)$$

the "central mass" m_0 being given by

$$m_0=3M-3.15\beta^2(\sigma-3.8)/M. \quad (4.9)$$

These formulas bring out the explicit dependence of the "dynamical corrections" on the parameters β and ϵ.

The cases of $Y=0$, -1 baryons are somewhat more complicated, since these involve the unequal mass combinations $2M_1+M_2$ and $2M_2+M_1$, respectively. These mass differences will produce "symmetry-breaking terms" through the kinetic energy operator:

$$T=\tfrac{1}{2}(P_1^2+P_2^2)M_1^{-1}+\tfrac{1}{2}P_3^2M_3^{-1}. \quad (4.10)$$

This will bring in two "spectator functions" $F(P)$ and $F_3(P)$ instead of a single function $F(P)$ as in the previous cases. However, under the assumption $\epsilon=\Delta M/M\ll 1$ it is clear that the difference $F_3-F\equiv\Delta F$ must also be of $O(\epsilon)$, so that a perturbative procedure for the calculation of the binding energy can be developed on this basis, in addition to the approximation techniques already used in Sec. 3. The necessary details are sketched in the Appendix, where it is shown that to $O(\epsilon)$, the binding energy is *unchanged* from the "zeroth order" estimate (3.17) of Sec. 3. The complementary case of $Y=-1$ baryons which needs the combination M_1+2M_2 obviously admits of an identical treatment, except for the replacement $\epsilon\to-\epsilon$, so that to $O(\epsilon)$ this binding energy also remains unchanged.

The dynamical corrections to the binding energies of the various baryons are listed in Table I. It is seen that the corrections are all of order

$$\Delta m/m_0\equiv\beta^2\epsilon/Mm_0, \quad (4.11)$$

so that they are small for the two simultaneous reasons $\beta/M\ll 1$ and $\epsilon\ll 1$. We, therefore, find that for the "equal spacing rule" to hold to a high order of accuracy, the parameter β (inverse range), need only be small compared with the quark mass, but not necessarily compared with the baryon mass.

5. SUMMARY AND CONCLUSIONS

It appears that our model has several desirable features. First, it has a built-in justification for the nonrelativistic approximation to the quarks, in spite of a binding energy comparable to their masses. Secondly, by employing p-wave potentials, it largely eliminates the undesirable prospects of strongly bound nQ states with $n\geq 4$. Further, while it cannot rule out a bound $2Q$ state, the mass of such a state is not too low to be easily observable. Next, a mass difference between the singlet and doublet quarks, which can easily be accommodated in the model leads in a simple way to the "equal interval rule" with an SU_3 invariant interaction, the dynamical corrections to the mass being very small. In this respect the model works somewhat better for the decouplet than for the octet baryons, insofar as an SU_3-breaking interaction (absent in the model) is needed to account for the Σ-Λ mass difference.

The quantity β, which plays the role of an "inverse range" of the interaction, is restricted to be small compared with the quark mass M, in order that the nonrelativistic model be qualitatively valid. Within this restriction, it may be interesting to see if its magnitude can be compatible with the "size" of a baryon. In this connection, a reasonable estimate may be provided by the charge radius of the proton which, according to the estimates of Hofstadter *et al.*,[17] is given by

$$a_{1p}=0.85\ \text{F}\approx 0.6m_\pi^{-1}. \quad (5.1)$$

[17] R. Hofstadter *et al.*, Phys. Rev. Letters **6**, 293 (1961).

To connect this quantity with the range parameter β^{-1} in a satisfactory way, one must calculate the baryon form factors $F_B(q^2)$ on this model and pick up the coefficient of $(-\frac{1}{6}q^2)$ in their expansion in powers of q^2. While a detailed calculation of the baryon form factors[18] on this model is beyond the scope of this paper, simple considerations suggest that the size of the baryon will be of the order of β^{-1}.[19] An estimate of β which is consistent with (5.1) is then given by

$$\beta \sim a_{1p}^{-1} \approx 1.6 m_\pi, \qquad (5.2)$$

a value well within the limitations on this parameter, so that our model is fully compatible with the right order of magnitude for the baryon sizes.

Finally for completeness we list the binding energies for $2Q$ and $3Q$ (totally symmetric) systems under the action of s-wave forces of the type

$$\langle p | V_s | p' \rangle = -\lambda_s M^{-1} g(p) g(p'), \qquad (5.3)$$

where

$$g^2(p) = e^{-p^2\beta^{-2}}. \qquad (5.4)$$

The procedure which is identical to the one described in Sec. 3, yields for the masses M^* and m_0 of the $2Q$ and $3Q$ systems, respectively, the following values

$$2M - M^* = \beta^2(\sigma - 1.5)/M, \qquad (5.5)$$

$$3M - m_0 = \beta^2 M^{-1}[\sigma - 2.4 + 2\sigma_{\frac{4}{5}}^{3/2}], \qquad (5.6)$$

so that

$$M^* \approx 0.76M + 0.41 m_0. \qquad (5.7)$$

A comparison with (3.18) shows that (5.7) allows a somewhat higher value for M^*, but this advantage is more than offset by the price to be paid in terms of strongly bound nQ states with $n \geq 4$, and, perhaps, a repudiation of Fermi statistics for the quarks. It looks more reasonable to reserve these (stronger) s-wave forces for Q-$\bar{Q}$ systems which need larger binding to produce the meson masses.

Applications to electromagnetic properties like magnetic moments and form factors will be the subject of a future paper.

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APPENDIX

We sketch here an outline of the steps for the calculation of the binding energy of a $Y=0$ system. The case of $Y=-1$ requires merely the replacement $\epsilon \to -\epsilon$ in all the steps.

Let the momenta $\mathbf{P}_1$ and $\mathbf{P}_2$ be associated with the two equal masses (M_1), so that the various relative momenta in this case are given by

$$\begin{aligned}(M_1+M_2)\mathbf{q}_{31} &= M_1\mathbf{P}_3 - M_2\mathbf{P}_1,\\ (M_1+M_2)\mathbf{q}_{23} &= M_2\mathbf{P}_2 - M_1\mathbf{P}_1,\\ 2\mathbf{q}_{12} &= \mathbf{P}_1 - \mathbf{P}_2,\end{aligned} \qquad (A1)$$

and of course,

$$P_1 + P_2 + P_3 = 0.$$

The energy denominator corresponding to (3.6) is now

$$D_B(\mathbf{P}_i) = \tfrac{1}{2}M_1^{-1}(P_1^2 + P_2^2) + \tfrac{1}{2}M_2^{-1}P_3^2 + B_y, \qquad (A2)$$

where B_y is the relevant binding energy. When these definitions are inserted in the Schrödinger equation corresponding to (3.5) the latter yields, instead of (3.8), the following wave function:

$$MD_B(\mathbf{P}_i)\Psi = v(q_{23})\mathbf{q}_{23} \times \mathbf{P}_1 F(P_1) + v(q_{31})\mathbf{q}_{31} \times \mathbf{P}_2 F(P_2) + v(q_{12})\mathbf{q}_{12} \times \mathbf{P}_3 F_3(P_3), \qquad (A3)$$

where we have made use of the antisymmetry in $\mathbf{P}_1$ and $\mathbf{P}_2$ only, without taking a corresponding liberty on the third particle (of unequal mass). Substitution of (A3) back into the Schrödinger equation, now yields two coupled integral equations of the type (3.9), connecting F and F_3 and these reduce to a single equation only in the limit $M_1 = M_2$, when $F_3 = F$. Using the approximations described in Sec. 3 of the text and retaining terms up to the first order in ϵ, these coupled equations are

$$\begin{aligned}[1-\lambda h_1(P^2)]F(P) &= \int_0^\infty 4\pi\lambda q^4\, dq \exp[-\tfrac{1}{2}\beta^{-2}(P^2+q^2)(\tfrac{5}{4}-\tfrac{3}{4}\epsilon)]\\ &\quad\times F(q)[M\beta_y + q^2(1-\tfrac{1}{2}\epsilon) + P^2(1-\tfrac{1}{2}\epsilon)]^{-1}\\ &\quad+4\pi\lambda\int_0^\infty q^4\, dq\, F_3(q) \exp[-\tfrac{1}{2}\beta^{-2}P^2(\tfrac{5}{4}+\tfrac{3}{4}\epsilon) - \tfrac{5}{8}\beta^{-2}q^2]\\ &\quad\times[MB_y + q^2(1-\tfrac{1}{2}\epsilon) + P^2(1+\epsilon)]^{-1},\end{aligned} \qquad (A4)$$

$$\begin{aligned}[1-\lambda h_2(P^2)]F_3(P) &= 8\pi\lambda\int_0^\infty q^4\, dq\, F(q)[MB_y + P^2(1-\tfrac{1}{2}\epsilon) + q^2(1+\epsilon)]^{-1}\\ &\quad\times\exp[-\tfrac{5}{8}\beta^{-2}P^2 - \tfrac{1}{2}\beta^{-2}q^2(\tfrac{5}{4}+\tfrac{3}{4}\epsilon)],\end{aligned} \qquad (A5)$$

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[18] For an excellent article on three-body form factors, see L. I. Schiff, Phys. Rev. **133**, B802 (1964).

[19] This is opposite to the situations for a *loosely bound* system, like a deuteron, whose size is measured essentially by the binding energy parameter α_D through the asymptotic form $\exp(-\alpha_D r)$ of the wave function. In the present case of a *tightly bound* system, the binding-energy parameter is far larger than the inverse-range parameter, so that the asymptotic form is governed mainly by the range of the potential.

where

$$\lambda h_1(P^2)\approx\tfrac{3}{2}\beta^2\sigma[MB_y+\tfrac{5}{2}\beta^2(1-\tfrac{1}{2}\epsilon)+\tfrac{3}{4}P^2(1+\tfrac{1}{2}\epsilon)]^{-1}, \quad (A6)$$

$$\lambda h_2(P^2)\approx\tfrac{3}{2}\beta^2\sigma[MB_y+\tfrac{5}{2}\beta^2(1+\epsilon)+\tfrac{3}{4}P^2(1-\epsilon)]^{-1}. \quad (A7)$$

To solve these simultaneous equations, we note that the mass-difference parameter ϵ appears in (a) the potentials and (b) the energy denominators. For calculating the corrections to the binding energy up to $O(\epsilon)$, it is most convenient to consider these two effects separately. We note also that $\Delta F\equiv F_3-F$ is of order (ϵ).

A. ΔM Effects in the Potentials

To calculate these effects we can set $\epsilon=0$ in all the energy denominators, so that $h_1(P^2)\approx h_2(P^2)\approx h(P^2)$, [see Eq. (3.11)]. Next, the ansatz

$$\begin{pmatrix}F(P)\\F_3(P)\end{pmatrix}=[1-\lambda h(P^2)]^{-1}e^{-(5/8)P^2\beta^{-2}}\begin{pmatrix}G(P)\\G_3(P)\end{pmatrix}, \quad (A8)$$

where G, G_3 are slowly varying functions, reduces (A4) and (A5) to

$$G_3(P)=2\int_0^\infty dq\,K(P^2,q^2)G(q)[1-\tfrac{3}{8}\epsilon q^2\beta^{-2}], \quad (A9)$$

$$G(P)=\int_0^\infty dq\,K(P^2,q^2)[(1-\tfrac{3}{8}\epsilon P^2\beta^{-2})G_3(q)+(1+\tfrac{3}{8}\epsilon q^2\beta^{-2}+\tfrac{3}{8}\epsilon P^2\beta^{-2})G(q)], \quad (A10)$$

where

$$K(P^2,q^2)=4\pi\lambda q^2e^{-(5/4)q^2\beta^{-2}}[1-\lambda h(q)]^{-1}\times[MB_y+P^2+q^2]^{-1}. \quad (A11)$$

From the last two equations we find immediately that to $O(\epsilon)$ the quantity

$$G_0(P)=G_3(P)+2G(P) \quad (A12)$$

satisfies the equation

$$G_0(P)=\int_0^\infty K(P^2,q^2)G_0(q)\,, \quad (A13)$$

which is identical with Eq. (3.14), and hence yields the same solution as Eq. (3.17) for B_y:

$$B_y=3M-m_y=1.05\beta^2(3\sigma-3.8)/M. \quad (A14)$$

Thus the correction to the binding energy, arising from the "potential terms," vanishes to $O(\epsilon)$.

B. ΔM Effects in the Energy Denominators

In this case, we put $\epsilon=0$ in the potential terms, and then proceed by defining

$$F(P)=G(P)e^{-5P^2/8\beta^2}[1-\lambda h_1(P^2)]^{-1}, \quad (A15)$$

$$F_3(P)=G_3(P)e^{-5P^2/8\beta^2}[1-\lambda h_2(P^2)]^{-1}. \quad (A16)$$

If now in the resultant equations for G and G_3 we make use of the approximation of "peaked integrands", as in Sec. 3, these reduce eventually to the coupled *algebraic* equations

$$C=\tfrac{1}{2}B[C(1+\eta\epsilon)(A-\tfrac{1}{2}\epsilon\beta^2)^{-1}+C_3(A+\epsilon\beta^2)^{-1}], \quad (A17)$$

$$C_3=CB[1-\eta\epsilon](A-\tfrac{1}{2}\epsilon\beta^2)^{-1}, \quad (A18)$$

where

$$A=MB_y+4\beta^2-\tfrac{3}{2}\beta^2\sigma, \quad (A19)$$

$$B=\tfrac{3}{2}\beta^2\sigma, \quad (A20)$$

$$\eta=\tfrac{3}{2}\beta^2(MB_y+4\beta^2)^{-1}. \quad (A21)$$

To solve these equations, we note that according to the "zeroth order" solution ($\epsilon=0$) of Sec. 3, $A=B$. We therefore, set $A=B+\delta B$ in the above equations which are then found to yield

$$\delta B/B\approx 0 \quad (A22)$$

to order ϵ, so that we obtain once again, the zeroth order solution (3.17) or (A14).

Combining (A14) and (A22) we conclude that the dynamical corrections to the binding energy of the $Y=0$ baryons vanish *completely* to $O(\epsilon)$. An identical result holds for the $Y=-1$ baryons (through $\epsilon\rightarrow-\epsilon$).