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Final-State Interactions among Three Particles. II. Explicit Evaluation of the First Rescattering Correction

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The lowest approximation to the three-body final-state interaction problem is a simple superposition of 3 two-body interaction terms, as given by the Watson final-state formula. The first rescattering correction corresponds to the "triangle graph" of perturbation and dispersion theory; for example, the process $W \to 2+(13) \to 1+(23) \to 1+2+3$, i.e., where first particles 1 and 3 interact strongly, then separate in such a way that 2 and 3 come together and interact strongly. For the case in which the (13) interaction is dominated by a resonance, we discuss the kinematic conditions under which such a rescattering correction will contribute appreciably to the total transition amplitude. We then evaluate this term exactly in the nonrelativistic (N.R.) case; that is, we evaluate the N.R. triangle graph. An analogous expression is proposed for the relativistic case, with suitable reinterpretation of the variables. The method is generalized to include spin and angular momentum, and to incorporate (23) rescattering to all orders, for the cases in which this can be parametrized either by a scattering length or by a resonance. Specific application is made to the $\pi\pi N$ case.

1. INTRODUCTION

THE general problem of analyzing three-body final-state interactions is of great present interest, both from the experimental and the theoretical point of view. This is particularly the case when there are two-body resonances accessible for one or more pairs of the three particles.

To be general, we consider the reaction

$$b+t \to W \to 1+2+3,$$

where the symbols b, t, 1, 2, 3 refer to "particles" (which term includes complex nuclei). Here W need not have a true existence as a compound state, but it does have a unique four-momentum and mass. (Of course, we can also consider the case where W is a true decaying particle, e.g., $K \to 3\pi$.) In Fig. 1 we show possible intermediate states leading to structure in the final three-body state. Figure 1(i) contains no final-state interaction effects (FSI), and if we ignore all dependence on momentum-transfer variables (e.g., the b-to-3 momentum transfer, etc.) it gives a constant matrix element for fixed beam energy. Figures 1(iia–c) show FSI in only 1 two-body channel, and the appropriate matrix elements are fairly well understood (cf. the work of Watson,[1] and Omnès[2]; but also the rather different results obtained by Phillips, Griffy, and Biedenharn[3]). In these figures the complete (reiterated) two-body scattering is to be incorporated, which leads to a Breit-Wigner form in the case of a fairly narrow resonance.

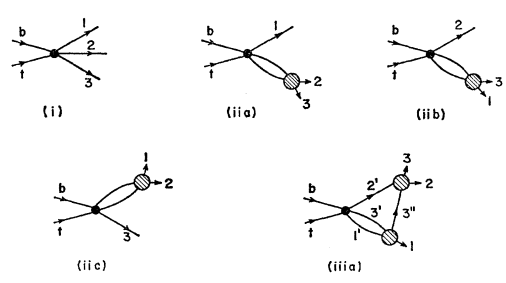

FIG. 1. Classification of three-body final-state interactions processes.

Finally Fig. 1(iiia) shows the beginning of the (infinite) set of rescattering corrections. In this diagram the final (23) rescattering is again to be understood as reiterated to all orders. This diagram is our present interest. In the particular case that the (13) interaction is dominated by a resonance R [which would already be evident in Fig. 1(iia)], it has been proved by the present authors[4] that in a well-defined approximate sense it is valid to replace the process shown in Fig. 1(iiia) by that shown in Fig. 2. In this figure R represents a particle with a discrete complex mass m_R (the imaginary part being related to the width of the resonance). R is necessarily unstable, i.e., $\mathrm{Re}\, m_R > m_1 + m_3$.[5]

The fundamental property of the process shown in Fig. 2 is that, for a suitable range of incident beam energies, the process can take place as a sequence of

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[1] K. Watson, Phys. Rev. 88, 1163 (1952).
[2] R. Omnès, Nuovo Cimento 8, 316 (1958); also N. I. Muskhelishvili, *Singular Integral Equations* (Even P. Noordhoff, Groningen, Netherlands, 1953).
[3] G. C. Phillips, T. A. Griffy, and L. C. Biedenharn, Nucl. Phys. 21, 327 (1960).
[4] I. J. R. Aitchison and C. Kacser, Phys. Rev. 133, B1239 (1964).
[5] Even when the (12) channel is not dominated by a resonance, it is always possible to represent the process of Fig. 1(iiia) by a superposition of those shown in Fig. 2; specifically as a weighted integral over all real m_R such that $m_1+m_3 \le m_R < \infty$. However, since all m_R occur, there is no *single* graph like Fig. 2 for this case.

real intermediate states.[6,7] It follows that the perturbation-theory evaluation of Fig. 2 has a singularity in the physical region, which should be observable experimentally for suitable processes. In order to develop criteria of choice of "suitable" processes, it is therefore essential to have a closed form analytic expression for the matrix element. This is the primary purpose of the present paper.

For the general relativistic case, dispersion-integral representations have been known for some time, being first given by Barton and Kacser[8] (see also Fronsdal and Norton[9]). In fact the method of evaluation of the exact relativistic matrix element for the most general triangle graph has been given by Wu.[10] However, Wu's result is not very useful for practical *numerical* calculations.

Similarly, methods have been given[11,12] for numerical integration of the dispersion-integral representation. The drawback here is that one has no general result at the end of computations made for a specific process.

Since Wu's method of evaluation, while *exact*, leads to a form of final result which is so unmanageable from the practical point of view, it is clear that one needs some sort of *approximate* analytic expression for the matrix element. We emphasize that we are not simply asking for the dominant, or singular, part of the matrix element near the rescattering singularity, but need also the regular part of the matrix element. The former is of course easy to determine, but it turns out to be a very weak singularity of the logarithmic type, and hence the regular part of the matrix element can well be comparable to the singular part. Of course only the singular part varies rapidly near the singularity, but one still needs the regular part of the matrix element.[13]

For want of anything better, we propose the use of the nonrelativistic (N.R.) matrix element, but evaluated numerically using relativistic kinematics where appropriate. Thus the main part of this paper consists of a discussion of the nonrelativistic matrix element. We obtain the latter in several ways, to demonstrate their equivalence to each other, and their relation to the relativistic approximate calculation. Nonrelativistic Feynman graphs have been discussed and evaluated fairly extensively in the Russian literature, in particular by Blokhintsev, Dolinskiĭ, and Popov,[14] Komarov and Popova,[15] and by Shapiro.[16] However, most of the Russian work has been based on a single method of evaluation, which while elegant and powerful, does not easily show its relationship to the more usual formulations of perturbation-theory and dispersion-theoretic methods. Further, this method cannot be easily generalized to the case when the particles have spin, or when angular momenta other than S waves are relevant. We treat this general case (albeit rather sketchily) in Sec. 5A and in Appendix C.

It may well be, however, that nuclear physics will furnish the best experimental tests of this whole theory,[17] because at present the accuracy attainable there is so much greater (the effect we are looking for is probably only a 10% one). In this case our N.R. graph is perfectly appropriate. In fact, the first suggestion of the possibility of seeing such rescattering effects in nuclear physics was made some time ago, using a quite different method, by Fox,[18] who, however, considered only the static limit $m_1 \gg m_3$, m_2. Recently the experiment $d+C^{12} \rightarrow n+N^{13*} \rightarrow (n+p)+C^{12}$ has been performed, in which the np rescattering appears to have been observed.[19]

2. KINEMATICS

A. Kinematics of the Boundary Curve

We consider

$$b+t \rightarrow W \rightarrow 1+2+3 \qquad (2.1)$$

and let the "particles" have masses m_b, m_t, m_1, m_2, and m_3. Also let W denote the total invariant mass in the over-all center-of-mass (c.m.) system. Here and throughout the *kinematics*, E will denote a total energy, and T a kinetic energy; also N.R. approximations will have an "arrow" rather than an "equal sign" (that is, the arrow is to be read as "equals in the N.R. limit"). Further, in the relativistic equations we set $c=1$, but show c explicitly in the N.R. equations. Finally we use subscripts on momenta and energies to denote the relevant particle.

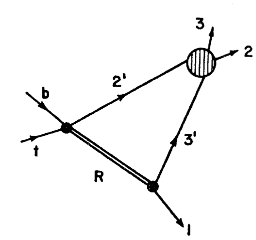

FIG. 2. The triangle graph with internal resonance.

[6] C. Kacser, Phys. Letters **12**, 269 (1964).

[7] R. E. Norton, Phys. Rev. **135**, B1381 (1964). See also S. Coleman and R. E. Norton, Nuovo Cimento **38**, 438 (1965).

[8] G. Barton and C. Kacser, Nuovo Cimento **21**, 593 (1961).

[9] C. Fronsdal and R. E. Norton, J. Math. Phys. **5**, 100 (1964).

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[11] I. J. R. Aitchison, Phys. Rev. **133**, B1257 (1964).

[12] Y. F. Chang and S. F. Tuan, Phys. Rev. **136**, B741 (1964).

[13] We should warn the reader that, as shown in Ref. 4, there is always a (1′3′) "continuum" contribution to the process of Fig. 1(iiia), as well as the resonance contribution of Fig. 2. However, this continuum background (defined more fully in Ref. 4) does not depend on the resonance in any way, and may safely be ignored in most cases near the rescattering singularity.

[14] L. D. Blokhintsev, E. I. Dolinskiĭ, and V. S. Popov, Zh. Eksperim. i Teor. Fiz. **42**, 1636 (1962) [English transl.: Soviet Phys.—JETP **15**, 1136 (1962)].

[15] V. V. Komarov and A. M. Popova, Zh. Eksperim. i Teor. Fiz. **45**, 214 (1963) [English transl.: Soviet Phys.—JETP **18**, 151 (1964)].

[16] I. S. Shapiro, in *Selected Topics in Nuclear Theory, lectures at the International Summer School held at Tatra, 1962*, edited by F. Janouch (International Atomic Energy Agency, Vienna, 1963), and references cited therein.

[17] For more details, see C. Kacser and I. J. R. Aitchison, Rev. Mod. Phys. **37**, 350 (1965).

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[19] J. Lang, R. Müller, W. Wölfli, R. Bösch, and P. Marmier, Phys. Letters **15**, 248 (1965).

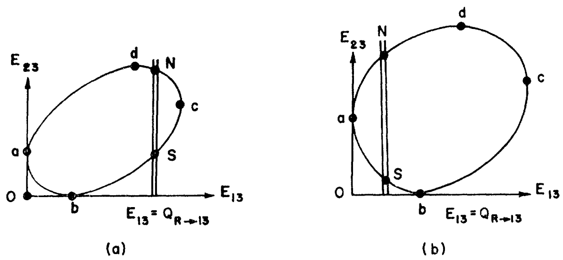

FIG. 3. The nonrelativistic kinematic conditions for *real* rescattering. In Fig. 3(a), real rescattering occurs corresponding to the point S, while in Fig. 3(b), no real rescattering can occur.

The usual relativistic variables used to describe the three-body final state (but ignoring correlations between the initial and final state other than energy-momentum conservation) are given by

$$s=(E_2+E_3)^2-(\mathbf{p}_2+\mathbf{p}_3)^2 = m_2^2+m_3^2+2(E_2E_3-\mathbf{p}_2\cdot\mathbf{p}_3) \quad (2.2)$$

and similarly for t and u, so that

$$s+t+u=W^2+m_1^2+m_2^2+m_3^2. \quad (2.3)$$

We turn now to the boundary of the *allowed* physical region for fixed W^2, with necessarily $W>m_1+m_2+m_3$. We go to the (23) c.m. frame, in which the initial system $(b+t)=W$ has three-momentum $\mathbf{p}$, as has particle 1, while particles 2 and 3 have three-momenta $\mathbf{q}$ and $-\mathbf{q}$, respectively. From now on, p and q without suffices denote the magnitudes of these momenta in the (23) c.m. system. One finds[20]

$$p^2=[(W+m_1)^2-s][(W-m_1)^2-s]/(4s)\rightarrow 2m_1m_{123}(Q-E_{23})/m_{23}, \quad (2.4)$$

$$q^2=[s-(m_2+m_3)^2][s-(m_2-m_3)^2]/(4s)\rightarrow (s-m_{23}^2)\mu_{23}/m_{23}\rightarrow 2\mu_{23}E_{23}. \quad (2.5)$$

Here $Q=(W-m_1-m_2-m_3)c^2$. Also

$$t=W^2+m_2^2-2E_WE_2+2pq\cos\theta, \quad (2.6)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$. Notice that in the N.R. limit p^2 and q^2 are linear in E_{23} (or s); this is the main feature that enables us to evaluate the N.R. graph. Also in this limit,

$$p^2\rightarrow (2m_1m_{123}Q/m_{23})-m_1m_{123}q^2/(m_2m_3). \quad (2.7)$$

The boundary of the allowed region occurs when all three particles have collinear momenta in the (23) c.m. system, or, equivalently, in the over-all c.m. system. The equation of the boundary curve is[21,22]

$$s^2t+st^2-st(W^2+m_1^2+m_2^2+m_3^2) +s(W^2-m_2^2)(m_1^2-m_3^2)+t(W^2-m_1^2)(m_2^2-m_3^2) +(W^2m_3^2-m_1^2m_2^2)(W^2-m_1^2-m_2^2-m_3^2)=0. \quad (2.8)$$

In the N.R. limit Eq. (2.8) becomes

$$[m_{13}m_{23}(E_{23}+E_{13})-Qm_{123}m_3]^2 -4E_{13}E_{23}m_{13}m_{23}m_1m_2\rightarrow 0. \quad (2.9)$$

Equation (2.9) can be factorized and solved for E_{23} in terms of E_{13}, or vice versa. One finds

$$\frac{2m_{13}m_{23}E_{23}}{m_1m_2m_3}\rightarrow\left[\left(\frac{2E_{13}}{m_3}\right)^{1/2}\pm\left(\frac{2(Q-E_{13})m_{123}}{m_1m_2}\right)^{1/2}\right]^2. \quad (2.10)$$

B. Kinematics of the Rescattering Condition

Consider Fig. 2, and in particular the intermediate process $R+2'\rightarrow 1+3'+2'$, in the over-all c.m. system. In this system, if $2'3'$ rescattering is to occur (neglecting any "finite size" of the "particles"), the velocity $\mathbf{v}_3'$ of $3'$ must be parallel to $\mathbf{v}_2'$, and also $v_3'>v_2'$. But if $\mathbf{v}_3'$ is parallel to $\mathbf{v}_2'$, it follows that $\mathbf{p}_1$, $\mathbf{p}_2'$, and also $\mathbf{p}_3'$ are all *collinear*. This implies that a necessary condition for the rescattering is that $\mathbf{p}_1$, $\mathbf{p}_2'$, and $\mathbf{p}_3'$ correspond to a point on the boundary of the phase space. Furthermore, since $R\rightarrow 1+3'$ and R is a real intermediate particle, it follows that

$$m_R^2=(E_1+E_3')^2-(\mathbf{p}_1+\mathbf{p}_3')^2\equiv t'. \quad (2.11)$$

Here we have associated the variable t' with the configuration of 1, $2'$ and $3'$, exactly as in the previous section. If we also define

$$s'\equiv(E_2'+E_3')^2-(\mathbf{p}_2'+\mathbf{p}_3')^2, \quad (2.12)$$

it then follows from the phase-space boundary condition, that s' is given in terms of $t'=m_R^2$ (which is fixed and known), exactly by the equations of the previous section, in particular (2.8).

In the N.R. limit we can express the conditions in terms of $E_{2'3'}$ and $E_{13'}$. If we define $Q_{R\rightarrow 13}=(m_R-m_1-m_3)c^2$, then $E_{13'}\rightarrow Q_{R\rightarrow 13'}$, and then using (2.10) we see that

$$E_{2'3'}\rightarrow\tfrac{1}{2}\mu_{23}\left[\left(\frac{2m_1Q_{R\rightarrow 13}}{m_3m_{13}}\right)^{1/2}\pm\left(\frac{2(Q-Q_{R\rightarrow 13})m_{123}}{m_{13}m_2}\right)^{1/2}\right]^2. \quad (2.13)$$

Equation (2.13) can be derived more directly, and each term has a physical significance: namely, the square roots are precisely the velocities of particles 2

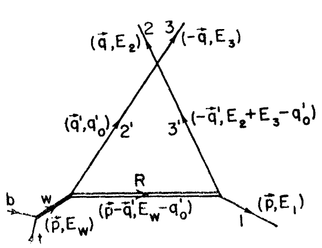

FIG. 4. Detailed energies and momenta used in the evaluation of the triangle graph M_Δ.

[20] $m_{ij}=m_i+m_j$, $m_{123}=m_1+m_2+m_3$, $\mu_{ij}=m_im_j/m_{ij}$.

[21] T. W. B. Kibble, Phys. Rev. **117**, 1159 (1960).

[22] P. Nyborg, H. S. Song, W. Kerman, and R. H. Good, Jr., Phys. Rev. **140**, B914 (1965).

and 3, respectively, before rescattering, in the rest frame of the (13) resonance.

Equation (2.13) is a necessary condition, in that it is needed to ensure that $\mathbf{v}_3'$ and $\mathbf{v}_2'$ are collinear. In order that they also be parallel, as opposed to antiparallel, one readily finds that only the smaller of the two values of $E_{2'3'}$ is suitable. However, the final necessary condition is $v_3'>v_2'$. This requires the first square root in (2.13) to be larger than the second. Thus we find:

necessary and sufficient condition in N.R. case for $2'3'$ rescattering:

$$\frac{2m_1Q_{R\to13}}{m_3m_{13}}\geq\frac{2m_{123}(Q-Q_{R\to13})}{m_{13}m_2} \tag{2.14}$$

in which case:

$$E_{2'3'}\to\tfrac{1}{2}\mu_{23}\left[\left(\frac{2m_1Q_{R\to13}}{m_3m_{13}}\right)^{1/2}-\left(\frac{2m_{123}(Q-Q_{R\to13})}{m_{13}m_2}\right)^{1/2}\right]^2. \tag{2.15}$$

We show these N.R. results in Fig. 3, in which is shown the boundary of the allowed E_{23} versus E_{13} plot, with a vertical line at $E_{13}=Q_{R\to13}$. In Fig. 3(a), condition (2.14) is satisfied, the $E_{23}=Q_{R\to13}$ line being to the right of the point b. Then of the two values of E_{23} given by the points N and S, *only the one at S permits rescattering.* In Fig. 3(b) condition (2.14) is not satisfied, and neither N nor S permits rescattering.

In the relativistic case very similar results apply.[6] The only domain permitting rescattering corresponds to the arc bc of Fig. 3, substituting s and t for E_{23} and E_{13}.

The above are the necessary and sufficient conditions on the intermediate state $12'3'$ so that $2'$ and $3'$ will rescatter. For a given 13 resonance mass m_R, there is a unique value of $s_{2'3'}$ or $E_{2'3'}$ *before* rescattering. Corresponding to this, particle 1 has a unique speed in the over-all c.m. system, but in an arbitrary direction. In thc rescattering, energy and momentum are conserved, i.e.,

$$E_2+E_3=E_{2'}+E_{3'},\quad \mathbf{p}_2+\mathbf{p}_3=\mathbf{p}_{2'}+\mathbf{p}_{3'}.$$

Hence s is unchanged in the rescattering, and therefore $E_{23}=E_{2'3'}$; *that is, the energy of relative motion of 2 and 3 (i.e., the excitation energy of the 23 system) is unchanged in the final rescattering, and is given by (2.15).* However, in the rescattering, the ultimately observed particles 2 and 3 can travel in arbitrary directions compatible with energy and momentum conservation. Thus t (or E_{13}) can take on *all* allowed values. This last point, while straightforward, cannot be overemphasized.

In our subsequent work, we will need to refer to both the s or E_{23} value at the S intercept on Fig. 3, and that at the N intercept. We hence define generally

$$\begin{aligned} s_N&=\max(s_1(t=m_R^2),\,s_2(t=m_R^2)),\\ s_S&=\min(s_1(t=m_R^2),\,s_2(t=m_R^2)),\end{aligned} \tag{2.16}$$

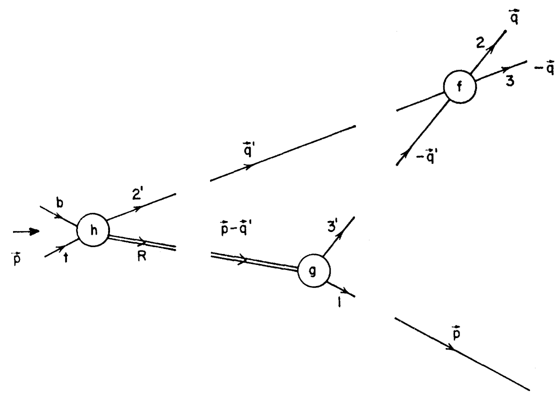

FIG. 5. The triangle graph M_Δ as it arises in third-order nonrelativistic perturbation theory.

where s_1 and s_2 are the two roots of (2.8) for a given t, and correspondingly

$$\begin{aligned} E_{23N}&\to\tfrac{1}{2}\mu_{23}\left[\left(\frac{2m_1Q_{R\to13}}{m_3m_{13}}\right)^{1/2}+\left(\frac{2m_{123}(Q-Q_{R\to13})}{m_{13}m_2}\right)^{1/2}\right]^2,\\ E_{23S}&\to\tfrac{1}{2}\mu_{23}\left[\left(\frac{2m_1Q_{R\to13}}{m_3m_{13}}\right)^{1/2}-\left(\frac{2m_{123}(Q-Q_{R\to13})}{m_{13}m_2}\right)^{1/2}\right]^2.\end{aligned} \tag{2.17}$$

In Appendix A we give the relativistic matrix element M_Δ corresponding to Fig. 4. We analyzed its singularities in Ref. 4, with the conclusion that except for the normal threshold singularity of $q(s)$ at $s=m_{23}^2$, the only other singularity which affects the *physical* amplitude is s_S. This occurs if and only if the above conditions are satisfied, i.e., if and only if s_S and $t=m_R^2$ intersect on the arc bc of the Dalitz plot boundary as in Fig. 3. *Thus the amplitude M_Δ has a physically observable singularity if and only if rescattering is possible.*

To examine such effects one has to evaluate M_Δ. As we remark in Appendix A, this cannot be done in convenient form in the relativistic case. However, in the N.R. limit, M_Δ can be integrated exactly, and we now discuss that limit. The corresponding graph is shown in Fig. 5.

3. EVALUATION OF THE NONRELATIVISTIC MATRIX ELEMENT

The nonrelativistic third-order perturbation-theory matrix element for the process shown in Fig. 5, and

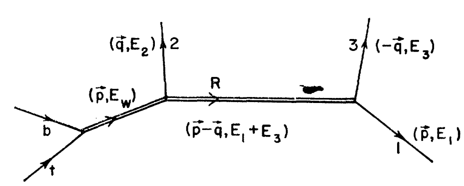

FIG. 6. The resonance graph M_R.

derived in Appendix A, is

$$M_\Delta=\frac{(-i)(-2\pi i)}{(2\pi)^4 2m_{13}2m_2 2m_3c^3}\int d^3q'\,fgh\frac{1}{[\mathbf{q}^2/(2\mu_{23})-\mathbf{q}'^2/(2\mu_{23})+i\epsilon]}\times\frac{1}{[\mathbf{p}^2/(2m_W)+Q_{W\to R2}-\mathbf{q}'^2/(2m_2)-(\mathbf{p}-\mathbf{q}')^2/2m_R+i\epsilon]}. \quad (3.1)$$

Here

$$Q_{W\to R2}=(W-m_2-m_R)c^2=Q-Q_{R\to 13}, \qquad Q_{R\to 13}=(m_R-m_1-m_3)c^2. \quad (3.2)$$

Also in the future we may let $m_R\to m_{13}$, $m_W\to m_{123}$ in kinetic-energy terms. The effective coupling constants f, g, and h are defined more precisely with normalization in Eqs. (A1) to (A3) in a general relativistic notation. We will discuss f more fully in Sec. 4 and only remark here that f is dimensionless, and that g is defined in terms of the resonance width in Appendix A; finally h is best understood by giving the matrix element for the graph of Fig. 6 (cf. equation), which is *by definition*

$$M_R=\frac{-hg}{2m_Rc^2[\mathbf{p}^2/(2m_W)-\mathbf{q}^2/(2m_2)+Q_{W\to R2}-(\mathbf{p}-\mathbf{q})^2/(2m_R)+i\epsilon]}. \quad (3.3)$$

Our aim is to evaluate Eq. (3.1). We have done this in three ways, all of which we believe to be instructive and capable of further application. But since they all (fortunately) give the same answer, we relegate two of them to Appendix B, presenting here only the most straightforward one.

This method simply evaluates the d^3q' integral in Eq. (3.1) directly. The angular integration over $d\Omega_{q'}$ affects only the second denominator. We take axes along $\mathbf{p}$, so that $\mathbf{p}\cdot\mathbf{q}'=pq'\cos\theta$, and then θ is precisely the same as in Sec. 2. Then

$$M_\Delta=\frac{+(-i)(-2\pi i)f}{(2\pi)^4 2m_2 2m_3c}\int_0^\infty \tfrac{1}{2}q'\frac{d(q'^2)2\mu_{23}P(p,q')}{q'^2-q^2-i\epsilon}, \quad (3.4)$$

where

$$P(p,q')=(4\pi)\frac{1}{2}\int_{-1}^{1}d\cos\theta(-hg)\left\{2m_Rc^2\left[\frac{p^2}{2m_{123}}-\frac{q'^2}{2m_2}+Q_{W\to R2}-\frac{(\mathbf{p}-\mathbf{q}')^2}{2m_{13}}+i\epsilon\right]\right\}^{-1}. \quad (3.5)$$

Comparing (3.5) with (3.3) we see that $P(p,q')$ is very closely related to the S-wave projection of M_R, except that in (3.5) $\mathbf{q}'$ is of arbitrary magnitude (to be integrated over), while in (3.3) $\mathbf{q}'$ is replaced by $\mathbf{q}$, both $\mathbf{p}$ and $\mathbf{q}$ being on the "energy shell" and given by Eqs. (2.4), (2.5), and (2.7). Thus $P(p,q')$ is very similar to, but *not* exactly the same thing as, the S-wave projection of M_R. Thus in fact this method of evaluation is somewhat related to the dispersion relation method, which is, in fact, our third method of evaluation.[23] (The second uses a Feynman parameter.) See Appendix B.

For the moment, however, let us just evaluate $P(p,q')$ and hence M_Δ by brute force. We readily find that

$$P(p,q')=\frac{+hg\pi}{pq'c^2}\ln\left\{\left[\left(q'+\frac{m_2p}{m_{123}}\right)^2-\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}-i\epsilon\right]\Big/\left[\left(q'-\frac{m_2p}{m_{123}}\right)^2-\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}-i\epsilon\right]\right\}. \quad (3.6)$$

We now substitute (3.6) into (3.4) and perform the q' integration. Notice that p is a *constant* in (3.6), and further that $P(p,q')$ is invariant under $q'\rightleftarrows -q'$ (the ln is on its principal sheet). Furthermore $q'^2(q'^2-q^2-i\epsilon)^{-1}$ is invariant under $q'\rightleftarrows -q'$. Finally, the behavior of the full integrand at ∞ is $(q')^{-2}$. Hence we can convert the q' integration to one going from $-\infty$ to ∞, and then close the contour in a suitable half-plane provided it avoids the branch cuts of the ln. Now the argument of the ln contains an explicit $i\epsilon$, and the $i\epsilon$ has the same sign as an imaginary part which would arise if we let $m_R\to m_R-i\Gamma/2$, for then $Q_{W\to R2}\to Q_{W\to R2}+i\Gamma/2$. Thus even for positive $Q_{W\to R2}$ the branch points of (3.6) are fully specified.[24] We hence can factorize the ln in the q' plane into parts with singularities either only in the upper half-plane, or the lower half-plane, and close the contour accordingly. Thus from (3.4)

[23] We should warn the reader that when q' is off the energy shell, there are two *in*equivalent forms of $P(p,q')$. The one given explicitly in (3.5) uses the energy difference between W and $(2'R)$; another form would contain the energy difference between $13'$ and R. When $q'\to q$, then these two differences are equal. This must be the case, as is readily seen explicitly from Eq. (2.7). While there are two inequivalent $P(p,q')$, yet they of course ultimately lead to the same expression for M_Δ.

[24] Of course, another guaranteed method of specifying the appropriate branch points would be to use analytic continuation in the upper half-plane in W^2, or equivalently in $Q_{W\to R2}$, from the region where the decay $W\to R2$ is forbidden. This leads to the same conclusion as the explicit $i\epsilon$.

to (3.6)[24a]

$$M_\Delta=\frac{(-i)(-2\pi i)fgh\pi}{(2\pi)^4 2m_{23}pc^3 4}\int_{-\infty}^{\infty}dq'\left[\frac{1}{q'-q-i\epsilon}+\frac{1}{q'+q+i\epsilon}\right]$$

$$\times\left\{\ln\left\{\left[q'+\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\Big/\left[q'-\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\right\}\right.$$

$$\left.+\ln\left\{\left[q'+\frac{pm_2}{m_{123}}-\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}-i\epsilon\right]\Big/\left[q'-\frac{pm_2}{m_{123}}-\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}-i\epsilon\right]\right\}\right\}. \quad (3.7)$$

In (3.7) the first ln has a branch cut of *finite* extent lying completely in the lower half-plane, so that for it we close the contour in the upper half-plane, and only pick up the residue at $q+i\epsilon$. Similarly the second ln picks up the residue only at $-q-i\epsilon$, but since the sense of the contour loop is reversed, there is an over-all minus sign with respect to the first term. Thus in fact both terms give the same contribution, and we finally obtain

$$M_\Delta=\frac{-i\pi^3 fgh}{(2\pi)^4 m_{23}pc^3}\ln\left\{\left[q+\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\Big/\left[q-\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\right\}. \quad (3.8)$$

The singularities of M_Δ arise where either the numerator or the denominator of the argument of the logarithm vanishes.[25] This occurs when

$$\left(\frac{pm_2}{m_{123}}\right)^2=\left[\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+q\right]^2=\frac{m_2^2}{m_{123}^2}\left[\frac{2m_1m_{123}Q}{m_{23}}-\frac{m_1m_{123}q^2}{m_2m_3}\right], \quad (3.9)$$

where we have used (2.7). But this can be rearranged to give

$$\left[q\left(\frac{m_{13}m_{23}}{m_{123}m_3}\right)^{1/2}+\left(\frac{2Q_{W\to R2}m_2m_3}{m_{23}}\right)^{1/2}\right]^2-\frac{2m_1m_2^2}{m_{123}m_{23}}(Q-Q_{W\to R2})=0. \quad (3.10)$$

Now $q^2=2\mu_{23}E_{23}$, $Q_{W\to R2}=Q-Q_{R\to 13}$, and $Q-Q_{W\to R2}=Q_{R\to 13}$, so that finally we see that M_Δ has singularities in q at those momenta corresponding to the energies E_{23S}, E_{23N} of Eq. (2.17) namely, at[26]

$$q_S=+(2\mu_{23}E_{23S})^{1/2},\quad q_N=-(2\mu_{23}E_{23N})^{1/2},$$

that is, at

$$\begin{Bmatrix}q_N\\ q_S\end{Bmatrix}=\pm\left(\frac{2m_1m_2^2m_3Q_{R\to 13}}{m_{13}m_{23}^2}\right)^{1/2}-\left(\frac{2m_{123}m_2m_3^2Q_{W\to R2}}{m_{13}m_{23}^2}\right)^{1/2}. \quad (3.11)$$

Correspondingly, we introduce the associated quantities p_N, p_S by[26] [cf. Eq. (2.7)]

$$\begin{Bmatrix}p_N\\ p_S\end{Bmatrix}=\left(\frac{2m_{123}^2m_1m_3Q_{R\to 13}}{m_{13}m_{23}^2}\right)^{1/2}\pm\left(\frac{2m_{123}m_1^2m_2Q_{W\to R2}}{m_{13}m_{23}^2}\right)^{1/2}, \quad (3.12)$$

from which it follows that

$$\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}=-q_S+\frac{m_2}{m_{123}}p_S=-q_N-\frac{m_2}{m_{123}}p_N. \quad (3.13)$$

[24a] The ln of (3.6) is necessarily on its principal sheet since it is defined by (3.5), and we are working in that range of $Q_{W\to R2}$ and Q such that the kinematic conditions *are* satisfied. Hence the possible distortions of the $\cos\theta$ contour associated with the second type singularity *do not* occur (see C. Kacser, Phys. Rev. **132**, 2712 (1963), also the paper cited in Ref. 33). It is then easy to prove that the factorization in (3.7) is valid, with each ln{ } being separately on its principal sheet; i.e., with separate cuts lying only in one half plane.

[25] There is also the possibility of a second type singularity at $p=0$ on the lower edge of the physical cut—see Ref. 4.

[26] The sign convention is artificial but convenient for later use; see also Appendix B2.

Thus we can rewrite Eq. (3.8) in the two equivalent forms:

$$M_\Delta=\frac{-i\pi^3 fgh}{(2\pi)^4 m_{23}pc^3}\ln\left[\frac{(q-q_N)+(m_2/m_{123})(p-p_N)}{(q-q_S)-(m_2/m_{123})(p-p_S)}\right], \tag{3.14}$$

$$M_\Delta=\frac{-i\pi^3 fgh}{(2\pi)^4 m_{23}pc^3}\ln\left[\frac{q+(m_2/m_{123})p-q_S+(m_2/m_{123})p_S}{q-(m_2/m_{123})p-q_S+(m_2/m_{123})p_S}\right]. \tag{3.15}$$

Equations (3.14) and (3.15) can be extended throughout the whole complex q plane. The physical region is near the positive q axis, so that q_S is near, but q_N far from the physical region. Correspondingly, E_{23S} is on the second E sheet reached by going through the physical cut from above, and is hence near the physical region; while E_{23N} is on the second E sheet but reached from below the physical cut (cf. Fig. 7). Thus only q_S is of physical significance, and is precisely the momentum in the rescattering condition of Eq. (2.17).

4. GENERALIZATIONS

As a general relativistic approximation, we propose the use of Eq. (3.15), where p and q are to be determined from relativistic kinematics, i.e., (2.4) and (2.5); and q_S is to be obtained from s_S and put into the proper quadrant as indicated in Fig. 7; similarly q_N is to be obtained from s_N. It might be argued that m_{123} should be replaced by W, but this is only a guess, and will not affect numerical conclusions greatly. Note that by using Eq. (3.15) we are ensuring that M_Δ has no singularity at $p=0$ on the physical sheet. (The use of (3.14) with approximate relativistic expressions could introduce a spurious singularity.)

There are two further generalizations to be made to this result. The first has to do with general angular momentum factors, and the second with the final rescattering amplitude f. We discuss each of these briefly, treating the possible angular factor first. A concrete angular-momentum example is treated in Appendix C.

A. Angular-Momentum Factors

In Eqs. (A1) to (A3) we defined "coupling constants" for the various vertices. In actual fact these will not be constants but rather "form factors" with some energy dependence. This can also be seen from the nonrelativistic third-order perturbation theory formulation

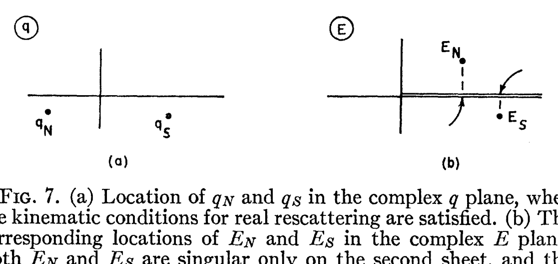

FIG. 7. (a) Location of q_N and q_S in the complex q plane, when the kinematic conditions for real rescattering are satisfied. (b) The corresponding locations of E_N and E_S in the complex E plane. Both E_N and E_S are singular only on the second sheet, and the arrows indicate the shortest path to each of these singularities.

implicit in Eq. (A6) and the following discussion. The most constant of the constants is g, provided we do not go too far off the mass shell for R, 1, and 3. Since the dominant contribution to M_Δ arises precisely when all particles are on their mass shells, this is not a serious worry. However, the form g is meaningful if and only if R, 1, and 3 are all spinless particles.

In the general case of particles with spin, polarization vectors must enter, suitably coupled with momenta which give rise to both angular factors and angular-momenta barrier factors. Hence g changes from being a constant to a well-defined and fairly straightforward function of the three-momenta of 1 and 3, and the various polarization vectors. Naturally the simplest form arises in the c.m. frame of R, but this can easily be transformed into the frame specified by the momenta of Fig. 5, at least when $q'=q$ in magnitude.

Turning next to h, this implies that R and 2 come off in an S wave in the over-all c.m. energy W, but since W has been kept constant throughout, this does not affect us. What ought to be generalized even for spinless particles is the angular momentum in the intermediate $R2$ state (of course this is linked to the initial state). Furthermore, the possibility of particles with spin should be allowed for. Again, however, both these generalizations of h are straightforward and can be expressed in terms of the momenta appearing in Fig. 5, together with certain "polarization" vectors defining the initial $|bt\rangle$ state, and also the polarization states of R and 2.

Finally in precisely the same way we should generalize f. As we will see in Sec. 4B, f is properly a function of q^2, and this q^2 dependence can be taken into account *a posteriori*. This q^2 dependence is not our concern here. Rather, we want to allow for more than pure S-wave 23 scattering, and also allow for spin. This introduces a simple generalization of f.

We do not write down the most general form of f, g, and h, since the formalism would become too cumbersome to be useful. Rather we treat a specific example in detail in the Appendix. Hence here we continue simply sketching the procedure.

With f, g, and h suitably generalized, the first step is to sum over all polarization states of R, $2'$, and $3'$. Assume this has been done, and let f, g, and h stand now for the correct expressions after this sum has been performed. Then Eq. (3.1) remains valid. The next step is the $d\Omega_{q'}$ integration. If the 23 scattering has an orbital angular momentum L, then f will contain a $P_L(\hat{q}\cdot\hat{q}')$, where P_L is the regular Legendre polynomial. Hence

(3.5) becomes modified with fgh in the numerator, and the effect of f is to extract the Lth partial-wave projection of M_R. Thus (3.6) will become modified, and contain the same ln as previously, but now multiplied by an algebraic function of p and q'; and also other purely algebraic terms.

Finally the dq' integral can then be done using methods similar to those in Sec. 3. If R has spin 1 or greater, an apparent difficulty is that the dq' integral can diverge, depending on the initial angular momentum. This is a well-known difficulty in partial-wave dispersion relations in which particles are exchanged with spin ≥ 1. It arises also in the relativistic case, and in fact the nonrelativistic approximations we have used increase the degree of divergence. The standard technique is then to introduce a "suitably chosen cutoff," and hope that the answer is not too dependent on it. Alternatively, one can use a form factor "determined from experiment."[27] In the case we consider explicitly in the Appendix, namely $\pi N \to \pi\pi N$ going from an initial D-wave state, divergence problems do not occur. We have not investigated the general case fully.

B. Inclusion of the Full (23) Rescattering

The complete amplitude, to the order we are considering, will be a sum of the terms represented by Figs. 1(i), 1(iia–c), and Fig. 1(iiia) (and two similar ones obtained by cyclic interchange). In this section we include the full (23) interaction in Fig. 1(iiia) so as to obtain Fig. 2.

The easiest way to do this is to examine the integral equations which sum up the complete chain of rescatterings of the type of Fig. 8. This was done in Ref. 28 for the simple case of identical spinless particles; Fig. 2 then emerged as the *first iteration* of these equations.

Following the method of Ref. 28, the full amplitude can be written as a sum of three terms

$$F(s,t,u)=\Phi(s)+\Psi(t)+\chi(u),$$

where

$$\Phi(s)=\frac{C_1}{D_{23}(s)}+\frac{1}{D_{23}(s)}\frac{1}{\pi}\int\frac{g_{23}D_{23}\bar{\Psi}ds'}{s'-s}+\frac{1}{D_{23}(s)}\frac{1}{\pi}\int\frac{g_{23}D_{23}\bar{\chi}ds'}{s'-s}\qquad(4.1)$$

and two similar equations for Ψ and χ. Here

$$\bar{\Phi}=\frac{1}{2}\int_{-1}^{1}d\cos\theta\,\Phi(t(s,\cos\theta))$$

[cf. Eq. (2.6)] and D_{ij} is the Omnès function

$$D_{ij}(s)=\exp\left[-\frac{1}{\pi}\int\frac{ds'\delta_{ij}(s')}{s'-s}\right],\qquad(4.2)$$

δ_{ij} being the s-wave (ij) phase shift. The C_i are three parameters representing different production probabilities for the three two-body channels.

The question then arises: Do the equations have a unique solution or do arbitrary parameters have to be introduced? This question has not been answered yet, and we do not wish to discuss it in detail here. Suffice it to say that it is usually the case, with such equations, that the answer lies in the asymptotic behavior of the inhomogeneous terms and kernels of (4.1). The problem, therefore, has to do with subtractions.

For our present purpose, however, we can skirt these difficulties altogether. For we are only going to take the first iteration of Eq. (4.1), so that we shall in any case neglect all higher iterations, and assume that they produce some slowly varying—essentially constant—background. Then we can absorb subtraction terms into the same background, and work with Eq. (4.1) as it stands, together with a fourth (over-all background) constant G, which represents a renormalized Fig. 1(i).

Consider first Fig. 1(iib). In this paper we have considered the case in which the (13) final-state interaction is dominated by a resonance. The D_{13} function for this case can be obtained most simply by choosing the N.R. parametrization

$$\rho_{13}\cot\delta_{13}=(q_R^2-q_{13}^2)\frac{m_{13}c^2}{\gamma\mu_{13}}\leftarrow\frac{m_R^2c^4-t}{\gamma},$$

where

$$\rho_{13}=2q_{13}c/t^{1/2}\to 2q_{13}c/m_{13}.$$

[Note ρ_{13} is dimensionless, and γ has units of (energy)^2.] Also q_R^2 is given in Eq. (2.5), with m_R^2 and s and m_1 for m_2. We find, using Eq. (4.2) subtracted at threshold,

$$D_{13}(t)=[m_R^2c^4-t-2i\gamma q_{13}/(m_{13}c)]\mu_{13}/(m_{13}c^2q_R^2).$$

The first term in the equation [like Eq. (4.1)] for Ψ may then be identified with M_R [Fig. 1(iib)] in a unitarized form; namely

$$M_R=C_2'(e^{i\delta_{13}}\sin\delta_{13}/\rho_{13}),\qquad(4.3)$$

$$e^{i\delta_{13}}\sin\delta_{13}/\rho_{13}=\gamma/(m_R^2c^4-t-i\gamma\rho_{13}).\qquad(4.4)$$

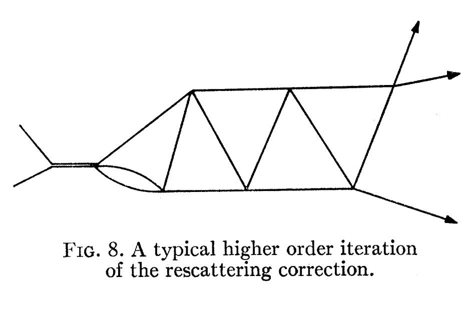

FIG. 8. A typical higher order iteration of the rescattering correction.

[27] D. Z. Freedman, C. Lovelace, and J. Namyslowski, CERN Report 65/980/5-TH.575, 1965 (unpublished).
[28] I. J. R. Aitchison, Nuovo Cimento 35, 434 (1965).

In a similar way, the constant C_1 in Eq. (4.1) for Φ must be chosen to conform to Fig. 1(iic). Here, of course, it depends what type of 2–3 interaction we take, and we shall confine ourselves to two cases.

Case A. Scattering-Length Parametrization for (23) Interaction

We obtain the $D_{23}(s)$ for this case by using Eq. (4.2) subtracted at threshold and, again N.R.

$$e^{i\delta_{23}}\sin\delta_{23}=\frac{2aq/(m_{23}c)}{1-2iaq/(m_{23}c)},\quad q=q_{23}.$$

(Here a is dimensionless.) Then

$$D_{23}(s)=1-2iaq/(m_{23}c).$$

The first term in Eq. (4.1), written as in Eq. (4.3), is

$$\frac{C_1'a}{1-2iaq/(m_{23}c)}.$$

The first iteration of Eq. (4.1) is now got by inserting for $\bar{\Psi}$ the form given by Eq. (4.3). However, as far as this is concerned, it is a good approximation to replace q_{13} by q_R,[4] taking

$$\bar{\Psi}=\frac{1}{2}\int_{-1}^{1}d\cos\theta\frac{C_2'\gamma}{m_R^2c^4-t}.$$

So $\bar{\Psi}$ is just $P(p,q)/4\pi$, P being defined by Eq. (3.5), with $C_2'\gamma=gh$, and to this order the second term in Eq. (4.1) becomes

$$\frac{-16\pi a}{[1-2iaq/(m_{23}c)]}\left\{\frac{1}{\pi}\int\frac{-\pi^2q'}{(2\pi)^4 2m_{23}c}\frac{P(p',q')}{(q'^2-q^2)}dq'^2\right\}$$

which is just $-M_\Delta$ of Eq. (B3) if we make the replacement

$$\frac{f}{16\pi}\sim\frac{a}{1-2iaq/(m_{23}c)}=g_{23}\frac{m_{23}c}{2q}=\frac{g_{23}}{\rho_{23}},$$

where $g_{23}=e^{i\delta_{23}}\sin\delta_{23}$.

The sum of Figs. 1(i), 1(iic), Fig. 6, and Fig. 2 will then be

$$G+\frac{C_1'a}{1-2iaq/(m_{23}c)}\left\{1+\frac{i\gamma C_2'}{C_1'm_{23}p}\ln\left[\frac{q-q_s+(m_2/m_{123})(p+p_s)}{q-q_s-(m_2/m_{123})(p-p_s)}\right]\right\}+\frac{C_2'\gamma}{m_R^2c^4-t-i\gamma q_{13}/(m_{13}c)}.\qquad(4.5)$$

In a similar way, the first-order contribution to Φ from resonant interactions in the 1–2 channel could be included, giving a second logarithm contribution.

We may compare Eq. (4.5) with Eq. (4.8) of Ref. 28. In that reference an attempt was made to find a simple approximate form for M_Δ (in the present work we have evaluated it *exactly* in the N.R. limit). It may easily be verified that the two expressions are consistent. In terms of our Eq. (4.5), the approximations of Ref. 28 amount to taking the momentum p to be constant, and evaluating it at p_S (which was there called p_b). Also, the difference $(p-p_N)$ was neglected.

Case B. A Resonance for g_{23}

This case is in fact little different, and will only be sketched here. The first term in Eq. (4.1) is now

$$\frac{C_1'\beta}{m_0^2c^4-s-2i\beta q/(m_{23}c)}$$

and the second term is again just $-M_\Delta$ if we now make the replacement

$$\frac{f}{16\pi}\sim\frac{\beta}{m_0^2c^4-s-2i\beta q/(m_{23}c)}=\frac{g_{23}}{\rho_{23}}$$

as before. For this case the sum of the graphs in Figs. 1(i), 1(iic), Fig. 6, and Fig. 2 will be

$$G+\frac{C_1'\beta}{m_0^2c^4-s-2i\beta q/(m_{23}c)}\left\{1+\frac{iC_2'\gamma}{C_1'm_{23}p}\ln\left[\frac{q-q_s+(m_2/m_{123})(p+p_s)}{q-q_s-(m_2/m_{123})(p-p_s)}\right]\right\}+\frac{C_2'\gamma}{m_R^2c^4-t-2i\gamma q_{13}/(m_{13}c)}.\qquad(4.6)$$

This equation can again be modified to include the χ channel.

We see that the essential result in either case A or B is that the (23) final-state interaction form—constant/$D_{23}(s)$—is modified by a factor representing the triangle process, which has the nearby singularity at $q=q_S$. In a similar way, triangle processes in the other channels will modify the single final-state interaction terms $D_{13}^{-1}(t)$, $D_{12}^{-1}(u)$. We also note that, again in either case, the (23) interaction is included by merely making the replacement

$$f/16\pi\sim g_{23}/\rho_{23}. \tag{4.7}$$

5. DISCUSSION

Equation (4.7) is the final generalization to complete our discussion of M_Δ. Of course, it is the forms given in Eqs. (4.5) and (4.6) for the *complete* matrix element (in this lowest order of rescattering iteration) which give the experimentally observed transition probability. We see that to this order, the rescattering corrections lead to a very definite modification of the individual two-body final-state interaction terms.

Of course, the rescattering process which ends with particles 2 and 3 rescattering after previous (13) resonant interaction has the form of a (23) final-state interaction matrix element. Thus our detailed consideration of the process of Fig. 1(iiia) has in fact led to a term of the form of the process 1(iic). However, in the latter process the initial "production" vertex was assumed to be a constant. The correction corresponding to Fig. 1(iiia) effectively leads to a specific "form factor" correction to this production vertex. This form factor has a very characteristic behavior near s_S (or E_{23S}) which is very different from a typical "smoothly varying" form factor. Hence it is essential to consider the rescattering corrections explicitly as we have done.

This remains true even if we were to include all higher order corrections—for example, by using the integral equations, Eq. (4.1). These equations contain all rescattering corrections, provided we ignore the possibility of higher partial waves occurring in the dispersion integrands. This is a reasonable approximation below the four-particle production threshold, in which case we can lump the virtual contributions from higher mass intermediate states into an over-all subtraction constant.[29] Complete solutions of these equations have recently been obtained by one of us,[30] and by Duck and Khanna,[31] by numerical iteration. The higher order corrections seem to have the over-all effect of modifying the D function. In the resonance case, the resonance is broadened and shifted; in the "one scattering length+two resonances" case, rescattering corrections in the scattering-length channel can be large, and depend very much on the particular process considered. The important point for our purposes is that it is *only* the first iteration—our M_Δ—that produces anything other than a *smooth* variation in s. An ansatz which allows for the higher order iterations to some extent is thus simply to replace the C_i's in "direct" terms [i.e., those like Eq. (4.3)] by new complex constants (cf. also Ref. 32). Thus the effect of the higher iterations is such as to leave *un*determined the relative phases of the "direct" competing two-body interactions. Hence finally we write for only the s dependence of the full matrix element:

Scattering-length case:
$$\frac{Ca}{1-2iaq/(m_{23}c)}\left\{K+\frac{i\gamma}{m_{23}p}\ln\left[\frac{(q-q_s)+(m_2/m_{123})(p+p_s)}{(q-q_s)-(m_2/m_{123})(p-p_s)}\right]\right\}, \tag{5.1}$$

Resonance case:
$$\frac{C\beta}{m_0{}^2c^4-s-i2\beta q/(m_{23}c)}\left\{K+\frac{i\gamma}{m_{23}p}\ln\left[\frac{(q-q_s)+(m_2/m_{123})(p+p_s)}{(q-q_s)-(m_2/m_{123})(p-p_s)}\right]\right\}, \tag{5.2}$$

where C and K are constants.

We should perhaps emphasize that our discussion is meaningful only for that range of incident beam energies which correspond to physically allowed rescattering. That is, the kinematic conditions of Sec. 2B must be satisfied. Of course our nonrelativistic evaluation of the graph of Fig. 2 is exact, and hence always valid. However, even when the (13) interaction is completely dominated by a narrow resonance, the graph of Fig. 2 does not give the major contribution to the process of Fig. 1(iiia), once s_S has moved away from the lower edge of the physical cut on the second sheet. Hence the kinematic conditions are essential. On the other hand, for any endothermic production process, there is always a range of beam energies for which these conditions are satisfied.

APPENDIX A. REDUCTION OF THE RELATIVISTIC GRAPH TO N.R. FORM

We write down the Feynman graph of Fig. 4 in the (23) c.m. frame, and the appropriate four-momenta are as indicated in Fig. 4. The notation is exactly as in Sec. 2A.

[29] N. N. Khuri and S. B. Treiman, Phys. Rev. **119**, 1115 (1960). If this approximation is not made, one must include two-dimensional Mandelstam-type dispersion integrals in Eq. (4.1), and then complex singularities necessarily arise, c.f., G. Barton and C. Kacser, Nuovo Cimento **21**, 988 (1961).

[30] I. J. R. Aitchison (to be published).

[31] I. Duck and F. C. Khanna, Rice University report (to be published).

[32] V. V. Anisovich and L. G. Dakhno, Phys. Letters **10**, 221 (1964).

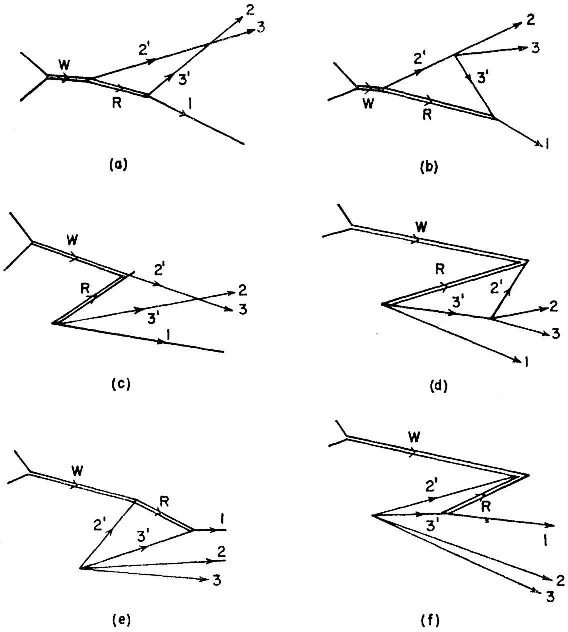

FIG. 9. The six different nonrelativistic time-ordered graphs corresponding to the one relativistic Feynman graph of Fig. 2.

We associate "coupling constants" with the various vertices as follows:

$$\langle 1|j_3|R\rangle = [1/(2E_1 2E_R)^{1/2}]g, \quad \text{(A1)}$$

$$\langle 2|j_3|2'3'\rangle_{S\text{ wave}} = [1/(2E_2 2E_{2'} 2E_{3'})^{1/2}]f, \quad \text{(A2)}$$

and

$$\langle t|j_b|R2\rangle_{S\text{ wave}} = [1/(2E_t 2E_R 2E_2)^{1/2}]h. \quad \text{(A3)}$$

We think of f, g, and h, as *constants* and ignore the problems of the off-shell nature of the matrix elements.

We first discuss the simpler graph of Fig. 6, which forms a part of Fig. 4, to show the meaning of h. The invariant matrix element M_R for this graph is

$$M_R = -\frac{hg}{m_1^2+m_3^2+2E_1E_3+2\mathbf{p}\cdot\mathbf{q}-m_R^2+i\epsilon}. \quad \text{(A4)}$$

Here m_R^2 is to be understood as including an imaginary part $-im_R\Gamma$, where Γ is the full width of the resonance at half-height. This expression is not unitary. However, we showed in Ref. 4 that this no real drawback as far as using it to describe an *internal* resonance line in a diagram such as Fig. 2. Only when we want it to describe the single final-state interaction in the (23) channel is it necessary to make it unitary. The change is effected by making the replacement

$$\Gamma m_R \sim 2\gamma q/(m_{13}c).$$

Since $\Gamma = (g^2/4\pi)q_R/2m_R^2$, where q_R is given by Eq. (2.5) with m_R^2 for s and m_1 for m_2, it follows that h also has a $\Gamma^{1/2}$ dependence, ensuring that (A4) is never singular even in the limit $\Gamma \to 0$.

With these preliminaries out of the way, we can write down the full invariant matrix element M_Δ for the triangle graph of Fig. 4 as follows:

$$M_\Delta = \frac{-i}{(2\pi)^4} fgh \int d^3q' dq_0' \frac{1}{[(E_W-q_0')^2-(\mathbf{p}-\mathbf{q}')^2-m_R^2+i\epsilon]} \frac{1}{[(E_2+E_3-q_0')^2-\mathbf{q}'^2-m_3^2+i\epsilon][q_0'^2-\mathbf{q}'^2-m_2^2+i\epsilon]}. \quad \text{(A5)}$$

For the reasons given in the Introduction, we now try to simplify (A5) by approximating the propagators by their N.R. limits.

With $E = T + mc^2$, we have quite generally

$$\frac{1}{E^2-p^2c^2-m^2c^4+i\epsilon} \to \frac{1}{[2mT-p^2+i\epsilon]c^2} + \frac{1}{[2m\tilde{T}-p^2+i\epsilon]c^2},$$

where $\tilde{T} = -2mc^2 - T$. Since the energy integration in (A5) runs from $-\infty$ to $+\infty$, *each* of T and $\tilde{T}$ runs over this range, and so we obtain *two* nonrelativistic propagators $c^{-2}[2mT-p^2+i\epsilon]^{-1}$ for each relativistic propagator. As is well known this simply means that nonrelativistic graphs have an *explicit* time sense, so that each different temporal sequence of interactions gives a separate matrix element. On the other hand, these contributions are summed up in one relativistic Feynman graph. (This point is not treated fully in Refs. 14, 15, and 16.)

Turning to our process of Fig. 2, we should properly consider the six separate graphs shown in Fig. 9. Recall that $m_Rc^2 = m_1c^2 + m_3c^2 + Q_{R\to 13}$, $(m_W - m_2 - m_R)c^2 = Q - Q_{R\to 13}$. Independently of whether $R \to 1+3$ and $W \to 2+R$ are exothermic or endothermic, i.e., independent of the signs of $Q_{R\to 13}$ and $Q - Q_{R\to 13}$, provided these Q values are such that $|Q| \ll mc^2$, we can classify these graphs by the number of "nearly energy conserving" vertices. Graph 9(a) always has three such vertices, graphs (b) and (c) have only one such vertex, while graphs (d), (e), and (f) have none. In terms of nonrelativistic perturbation theory, this means that the energy denominators arising from intermediate states will both be of order Q for graph 9(a); for graphs (b) and (c) one such denominator will be of order Q, but one will be of order m; while for graphs (d), (e), and (f), both denominators will be of order m. Hence in a nonrelativistic treatment to lowest order in Q/m, only graph (a) need be considered.

Graph 9(a) can be obtained from Fig. 4 by the replacements $E_W \to W + T_W$, $q_0' \to m_2c^2 + T_2'$, $E_1 \to m_1c^2 + T_1$ etc., since the sense of the arrows in Fig. 4 is the same as that in Fig. 9(a). Care must be taken in collapsing the propagators, however. Thus, for a line with mass m, total energy E, and momentum p as given in Fig. 4 and Eq. (A5), one writes $c^{-2}[2m(E-m)-p^2]^{-1}$; for example,

$$(E_W-q_0')^2-(\mathbf{p}-\mathbf{q})^2-m_R^2\rightarrow(E_W-q_0')^2-(\mathbf{p}-\mathbf{q})^2c^2-m_R^2c^4\rightarrow 2m_Rc^2(E_W-q_0'-m_Rc^2)-(\mathbf{p}-\mathbf{q})^2c^2.$$

The $T'(q_0')$ integration can now be performed immediately by contour integration. The $i\epsilon$ in each term shows that the only T' pole in the lower half-plane is at $T'=q^2/(2m_2)-i\epsilon$. Since the integrand $\sim T'^{-3}$ as $T'\rightarrow\infty$ we can close the contour in the lower half-plane and get immediately

$$M_\Delta\rightarrow-\frac{ifgh}{(2\pi)^4}\frac{(-2\pi i)}{2m_R2m_22m_3c^3}\int d^3q'\frac{1}{[T_W+c^2(W-m_2-m_R)-q'^2/(2m_2)-(\mathbf{p}-\mathbf{q}')^2/(2m_R)+i\epsilon]}\times\frac{1}{[T_2+T_3-q'^2/(2m_3)-q'^2/(2m_2)+i\epsilon]}. \quad \text{(A6)}$$

We have now obtained exactly the same result as standard nonrelativistic third-order perturbation theory (cf. Fig. 5) with a sum over two sets of intermediate states.

APPENDIX B. ALTERNATIVE EVALUATIONS OF M_Δ

1. The Feynman Parameter Method

This method makes use of the Feynman trick

$$\frac{1}{ab}=\int_0^1\frac{d\alpha}{[\alpha a+(1-\alpha)b]^2},$$

and is closely related to the method of Ref. 15. We have

$$\left[\frac{p^2}{2m_{123}}+Q_{W\rightarrow R2}-\frac{q'^2}{2m_2}-\frac{p^2}{2m_{13}}+\frac{2\mathbf{p}\cdot\mathbf{q}'}{2m_{13}}-\frac{q'^2}{2m_{13}}+i\epsilon\right]^{-1}\left[\frac{q^2}{2\mu_{23}}-\frac{q'^2}{2\mu_{23}}+i\epsilon\right]^{-1}$$

$$=\left(\frac{2m_2m_{13}}{m_{123}}\right)(2\mu_{23})\left[q'^2-2\mathbf{p}\cdot\mathbf{q}'\frac{m_2}{m_{123}}+p^2\frac{m_2^2}{m_{123}^2}-\frac{2Q_{W\rightarrow R2}m_2m_{13}}{m_{123}}-i\epsilon\right]^{-1}[q'^2-q^2-i\epsilon]^{-1}$$

$$=\frac{4\mu_{23}m_2m_{13}}{m_{123}}\int_0^1\frac{d\alpha}{\{[\mathbf{q}'-\alpha\mathbf{p}(m_2/m_{123})]^2-X-i\epsilon\}^2} \quad \text{(B1)}$$

with

$$X=(2\alpha m_2m_{13}/m_{123})Q_{W\rightarrow R2}+(1-\alpha)q^2-\alpha(1-\alpha)p^2(m_2^2/m_{123}^2).$$

Following standard methods we find

$$M_\Delta=\frac{-i(-2\pi i)}{(2\pi)^42m_{13}2m_22m_3}\frac{(fgh)4\mu_{23}m_2m_{13}}{m_{123}c^3}(i\pi^2)\int_0^1d\alpha\,X^{-1/2}.$$

Since X is a quadratic in α, the integration is elementary, and leads to Eq. (3.8) of the text.

2. The Dispersion-Relation Method

We now give our final derivation of this result. We remarked below Eq. (3.5) that we were very close to a dispersion relation for M_Δ. However in a true dispersion relation, the spectral function should be a function only of q', and not of q. $P(p,q')$ had p given as a function of q, and further $P(p,q')$ corresponded to an *off*-energy-shell variant of the S-wave projection of M_R.

We re-examine (3.4) and (3.6). In order to simplify the analysis we now imagine that $Q_{W\rightarrow R2}<0$, and will *finally* analytically continue $Q_{W\rightarrow R2}$ in the upper half-plane to $Q_{W\rightarrow R2}>0$ (cf. Footnote 24). We keep $Q=c^2(m_W-m_1-m_2-m_3)>0$, however. In that case $P(p,q')$ is real for all q' in $0\leq q'\leq\infty$ and is invariant under $q'\rightleftarrows -q'$. Further (3.4) shows that M_Δ is analytic in q^2 except for a cut from $0\leq q^2\leq\infty$. It follows that M_Δ has a dispersion representation in q^2, in which the spectral function is just the imaginary part of M_Δ. But from (3.4)

$$\mathrm{Im}M_\Delta(q^2)=-\pi^2f[(2\pi)^42m_{23}c]^{-1}qP(p,q), \quad \text{(B2)}$$

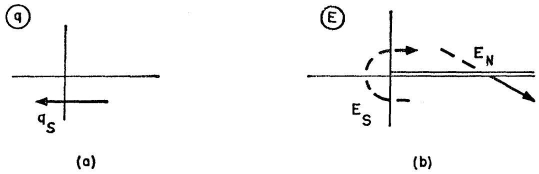

FIG. 10. (a) The motion of q_S in the complex q plane as W is increased. (b) The associated motion of E_S and E_N in the E_{23} plane.

where $P(p,q)$ is obtained from (3.6) by replacing q' by q. We see that $P(p,q)$ is precisely the S-wave projection of M_R, with both p and q related to s, so that M_R is on its energy shell. (Hence Ref. 23 does not apply to it). Hence

$$M_\Delta(q^2)=\frac{1}{\pi}\int_0^\infty \frac{dq'^2(-\pi^2)fq'P(p',q')}{[q'^2-q^2-i\epsilon](2\pi)^4 2m_{23}c}. \quad \text{(B3)}$$

(This N.R. dispersion relation agrees exactly with the N.R. reduction of the standard relativistic expression.[8,9])

Our final task is to evaluate (B3). Two remarks are in order. (i) The q'^2 integration goes to ∞, though the "physical region" for the process is limited by $Q\geq E_{23}$, i.e., $q^2\leq 2\mu_{23}Q$, in order that, in the over-all c.m. frame, $T_1+T_{(23)}\geq 0$, where T_1 is the kinetic energy of 1, and $T_{(23)}$ is the kinetic energy of the (23) c.m. motion. (ii) Related to (i), we see from Eq. (2.7) that p'^2 becomes negative for $q'^2\geq 2\mu_{23}Q$, so that p' becomes imaginary. The integrand is even under $q'\rightleftarrows -q'$, and/or $p'\rightleftarrows -p'$, so that we can convert the integral with respect to q' from $-\infty$ to $+\infty$. Further, in the q' plane, we take the cuts of p' as

$$-\infty\leq q'\leq -(2\mu_{23}Q)^{1/2} \quad\text{and}\quad +(2\mu_{23}Q)^{1/2}\leq q'\leq \infty\,,$$

and work on the sheet with $\mathrm{Re}\,p'\geq 0$. On this sheet $p'(-q)=+p(+q')$. Further $p'=0$ is not a singularity.

In order to locate the positions of the singularities of P with respect to the contour of integration, we recall that we commence with $Q_{W\to R2}<0$ continued in the upper half-plane, and $Q>0$.[33] Algebraically we proceed as in Sec. 3, and find the singularities to be at $\pm q_S$, $\pm q_N$ [cf. (3.11)]. With $Q_{W\to R2}<0$, both q_N and q_S are in the lower half-plane. Hence we can manipulate (B3) into the form

$$M_\Delta=\frac{1}{\pi}\frac{1}{2}\int_{-\infty}^{\infty} dq'\left(\frac{1}{q'-q-i\epsilon}+\frac{1}{q'+q+i\epsilon}\right)\frac{(-\pi^3 fgh)}{p'(2\pi)^4 2m_{23}c^3}$$
$$\times\left\{\ln\left[\frac{[q'+(m_2/m_{123})p']-[-q_S+(m_2/m_{123})p_S]}{[q'-(m_2/m_{123})p']-[-q_N-(m_2/m_{123})p_N]}\right]+\ln\left[\frac{[q'+(m_2/m_{123})p']-[q_N+(m_2/m_{123})p_N]}{[q'-(m_2/m_{123})p']-[q_S-(m_2/m_{123})p_S]}\right]\right\}. \quad \text{(B4)}$$

[Recall $p(q)=p(-q)$, so $p(-q_S)=p_S$, etc.] Here the first ln has singularities only in the upper half-plane, and the second ln only in the lower half-plane. Further the entire integrand is independent of $p'\rightleftarrows -p'$, and the q poles do not lie on the p' cuts, so that we can ignore the formal p' cuts. Then closing the contours appropriately as we did after Eq. (3.7), we obtain Eq. (3.14) of Sec. 3 after some algebra.

The above has been analyzed for $Q_{W\to R2}<0$, $Q_{R\to 13}>0$. The continuation in $Q_{W\to R2}$ is quite straightforward, and leads to the prescription $q_S-i\epsilon$. (This is when we neglect any possible negative imaginary part associated with m_R.) When $m_R\to m_R-i\Gamma/2$, we see that q_{S-} gets a finite negative imaginary part, since $Q_{W\to R2}\to Q_{W\to R2}+i\Gamma/2$, $Q_{R\to 13}\to Q_{R\to 13}-i\Gamma/2$, and both the prescription $Q_{W\to R2}+i\epsilon$ and the $m_R-i\Gamma/2$ prescription lead to the same result. As $Q_{W\to R2}$ increases past the limit given by (2.14), the imaginary part of q_S remains negative on either prescription, while the real part goes negative. The effect of this on E_{23S} is as shown in Fig. 10. It remains on the second E sheet, but circles around the normal threshold at $E_{23}=0$, and hence becomes "far" from the physical region. Thus once again we see the importance of the conditions (2.14) and (2.15). A similar analysis for q_N (for complex m_R) shows that E_{23N} moves as shown in Fig. 10(b) and that q_N and E_{23N} are never of physical significance.

APPENDIX C. AN EXAMPLE WITH SPIN AND ANGULAR MOMENTUM

For concreteness we treat $\pi+N\to\pi+\pi+N$ with R corresponding to the N^* spin-$\frac{3}{2}$ πN resonance. For simplicity we ignore the charge states and hence isotopic spin. We assume that in the process $\pi+N\to\pi+N^*$, the initial state is $D_{3/2}$, which implies S-state production of the N^*. The $N^*\to\pi N$ breakup of course is P wave. Finally we assume that the $\pi\pi$ rescattering is pure S wave.[34]

We proceed in a fully nonrelativistic manner. Thus we represent the initial and final N spin states by two component spinors $u(i)$ and $u(f)$, and ignore the spinor transformations induced in transforming from one frame of reference to another. Further we use Galilean rather than Lorentz transformations.

[33] Notice that even if m_R has a finite negative imaginary part, this prescription would agree with that on $Q_{W\to R2}$; see also J. B. Bronzan and C. Kacser, Phys. Rev. **132**, 2703 (1963). Hence $(Q_{W\to R2})^{1/2}$ is always to be taken in the upper half-plane, and $(Q_{R\to 13})^{1/2}$ is always positive.

[34] This example is not without relevance to the real world. Most low-energy pion-production data can be fitted in terms of the simple isobar model [see M. Olsson and G. B. Yodh, Phys. Rev. Letters **10**, 353 (1963), also to be published]. However the $\pi^-p\to\pi^-\pi^+n$ channel shows an "anomalous" $\pi\pi$ effective mass spectrum, which Anisovich and Dakhno (Ref. 32) have suggested may be due to a rescattering effect. The angular momenta which we assume are those most appropriate to test this hypothesis. A detailed numerical test is under way.

The various polarization states must be referred to some quantization axis, which we take to be the incident π beam direction. Let $\mathbf{b}$ be the three-momentum of the pion beam in the over-all c.m. system; then $\mathbf{b}$ is parallel to the quantization axis. The initial $D_{3/2}$ state is made up of an $L=2$ orbital part associated with the angles of $\hat{b}$ (a unit vector along $\mathbf{b}$), coupled with the nucleon to total spin $j=\frac{3}{2}$ and projection $j_z=m$ along $\mathbf{b}$. Here m is the sum of the z projections of L_{initial} and of s_{initial}, and hence for a given initial state it is fixed. The intermediate πN^* state couples the N^* directly to the $D_{3/2}$ initial state, there being no orbital angular momentum. The N^* also has magnetic quantum number m.

Turning next to the N^* decay, we go to the N^* rest frame and assume the polarization state is *un*changed in this transformation. Then this $(\frac{3}{2},m)$ state decays into $\pi+N$, with a P-wave orbital part depending on the angle of $\mathbf{p}'$, where $\mathbf{p}'$ is the final nucleon momentum in the N^* c.m. frame. This over-all coupling can then be written schematically as

$$(s_{\text{initial}},Y_2(\hat{b}))_{3/2} \quad (s_{\text{final}},Y_1(\hat{p}'))_{3/2}\,,$$

where suitable Clebsch-Gordan coefficients are implied. But this has the *unique* recoupling

$$(s_{\text{initial}},s_{\text{final}})_1 \quad (Y_2(\hat{b}),Y_1(\hat{p}'))_1\,.$$

Since futher $(s_{\text{initial}},s_{\text{final}})_1\sim u^*(f)\boldsymbol{\sigma}u(i)$, $Y_1(\hat{p}')\sim\hat{p}'$, and $Y_2(\hat{b})\sim\delta_{jk}-3\hat{b}_j\hat{b}_k$, we see that the complete $\pi N\rightarrow\pi N^*\rightarrow\pi\pi N$ amplitude has angular factors

$$[u^*(f)\sigma_j u(i)][\delta_{jk}-3\hat{b}_j\hat{b}_k]\hat{p}_k'\,.$$

Furthermore centrifugal barrier factors of b^2 and p' must appear in the over-all matrix element. Hence the full M_R matrix element for $\pi N\rightarrow\pi N^*\rightarrow\pi\pi N$ takes the form

$$M_R=\frac{\zeta}{2m_Rc^2}\frac{b^2(\boldsymbol{\sigma}\cdot\mathbf{p}')-3(\mathbf{b}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\mathbf{p}')}{[m_Rc^2-(E_{13}+m_{13}c^2)-i\epsilon]}\,. \tag{C1}$$

Here we have written M_R as a 2×2 matrix in nucleon spin space. Further, ζ is an effective coupling constant having the role of the previous product gh, and is essentially constant; m_R is the complex mass of N^*; and finally E_{13} is the total kinetic energy of relative motion of particles 1 (the final nucleon) and 3 (the second emitted pion) which make up the N^*.

In this expression $\mathbf{b}$ is a constant vector, and E_{13} is an invariant. However, $\mathbf{p}'$ must be re-expressed in terms of vectors defined in the $\pi\pi$ center of mass. In this frame the nucleon (particle 1) has momentum $\mathbf{p}$, and the N^* pion (particle 3) has momentum $-\mathbf{q}$ (cf. Fig. 6). One readily finds

$$\mathbf{p}'=\mathbf{p}-\frac{m_1(\mathbf{p}-\mathbf{q})}{m_{13}}=\frac{m_1\mathbf{q}+m_3\mathbf{p}}{m_{13}}\,,$$

where we have ignored the difference between m_{13} and m_R. This leads to

$$M_R=-\frac{\zeta}{2m_Rc^2}\frac{[b^2\boldsymbol{\sigma}-3(\mathbf{b}\cdot\boldsymbol{\sigma})\mathbf{b}]\cdot[m_1\mathbf{q}+m_3\mathbf{p}](1/m_{13})}{[p^2/(2m_{123})-q^2/(2m_2)+Q_{W\rightarrow R2}-(\mathbf{p}-\mathbf{q})^2/(2m_{13})+i\epsilon]}\,. \tag{C2}$$

This can be compared with (3.3), and we see that only the numerator of M_R has been affected. (Recall that $m_W\rightarrow m_{123}$, $m_R\rightarrow m_{13}$ is accurate to our approximation.)

From this we immediately obtain [cf., Eq. (3.1)] (recall that the pions are spinless, and f is S wave)

$$M_\Delta=\frac{(-i)(-2\pi i)}{(2\pi)^4 2m_{13}2m_2 2m_3c^3}\int d^3q' f\zeta\frac{[b^2\boldsymbol{\sigma}-3(\mathbf{b}\cdot\boldsymbol{\sigma})\mathbf{b}]\cdot[m_1\mathbf{q}'+m_3\mathbf{p}](1/m_{13})}{[q^2/2\mu_{23}-q'^2/2\mu_{23}+i\epsilon]}\times\frac{1}{p^2/2m_{123}-q'^2/2m_2+Q_{W\rightarrow R2}-(\mathbf{p}\cdot\mathbf{q}')^2/2m_R+i\epsilon}\,. \tag{C3}$$

We next perform the angular integration over $d\Omega_{q'}$. Consider that part of the numerator which depends on $\mathbf{q}'$, and write it as $\mathbf{v}\cdot\mathbf{q}'$, where $\mathbf{v}$ is some fixed vector. Then taking axes along $\mathbf{p}$, and performing the azimuthal angular integration, only the $v_z'q_z'$ part is preserved, so that $\mathbf{v}\cdot\mathbf{q}'\rightarrow(\mathbf{v}\cdot\mathbf{p})(\mathbf{p}\cdot\mathbf{q}')$. Hence Eqs. (3.4) and (3.5) become modified to

$$M_\Delta=\frac{(-i)(-2\pi i)f}{(2\pi)^4 2m_2 2m_3 c}\int_0^\infty\frac{\frac{1}{2}q'd(q'^2)2\mu_{23}}{q'^2-q^2-i\epsilon}P(p,q')A(\hat{p})\,, \tag{C4}$$

with

$$P(p,q')=4\pi\frac{1}{2}\int_{-1}^{1}\frac{d(\cos\theta)(-\zeta)[(m_1/m_{13})(\hat{p}\cdot\mathbf{q}')+(m_3/m_{13})p]}{2m_{13}c^2[p^2/2m_{123}-q'^2/2m_2+Q_{W\rightarrow R2}-(\mathbf{p}-\mathbf{q}')^2/2m_{13}+i\epsilon]}, \tag{C5}$$

and

$$A(\hat{p})=b^2(\boldsymbol{\sigma}\cdot\hat{p})-3(\mathbf{b}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\hat{p})\,. \tag{C6}$$

We notice that $P(p,q')$ depends only on the magnitudes of $\mathbf{p}$ and $\mathbf{q}'$, and hence the full angular dependence of M_Δ is given by the factor $A(\hat{p})$. This factor is immediately recognizable as corresponding to a final state in which the nucleon recoils from the (23) system in a relative P wave, the (23) system having $J=0$. The coupling (C6) is the only possible coupling from the initial $D_{3/2}$ state to a final state in which the two pions are in a relative S state. (Notice that p is parallel to the momentum of the nucleon in the over-all c.m. system; in fact $\mathbf{p}_{\text{c.m.}}=m_{23}\mathbf{p}/m_{123}$.)

One readily obtains

$$P(p,q')=(4\pi)(-\zeta)\frac{m_1}{2m_{13}pc^2}+\frac{\pi\zeta}{p^2q'c^2}\left[\left(\frac{m_3}{m_{13}}+\frac{m_1m_2}{2m_{13}m_{123}}\right)p^2+\frac{m_1m_{123}}{2m_2m_{13}}q'^2-m_1Q_{W\to R2}\right]$$
$$\times\ln\left[\frac{(q'+m_2p/m_{123})^2-2Q_{W\to R2}m_2m_{13}/m_{123}-i\epsilon}{(q'-m_2p/m_{123})^2-2Q_{W\to R2}m_2m_{13}/m_{123}-i\epsilon}\right].\quad\text{(C7)}$$

Asymptotically the integrand in (C4) goes as $P(p,q')$. This in turn apparently goes to 1, from both the algebraic and the logarithmic term, since the $\ln\sim1/q'$. But in fact a cancellation occurs, and since $P(p,q')$ is even in q', it actually goes as $(q')^{-2}$. Hence the integral in (C4) is convergent. This must be so, since while (C3) is apparently divergent, it can actually be seen to be convergent by using Feynman symmetric integration methods. [In the relativistic case, the integrand in (C3) is clearly convergent, going as $q_\mu d^4q/q^6$.] Because of this cancellation, however, the procedure analogous to that used in going from (3.4) to (3.7) must be done with care. By careful inspection, one sees that the proper grouping of terms is given by

$$M_\Delta=\frac{A(\hat{p})(-i)(-2\pi i)f}{(2\pi)^4 2m_{23}c^3}\frac{1}{4}\int_{-\infty}^{\infty}\left(\frac{dq'}{q'-q-i\epsilon}+\frac{dq'}{q'+q+i\epsilon}\right)$$
$$\times\left\{\left[(2\pi)(-\zeta)\frac{m_1}{2m_{13}p}\left[q'-\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}\right]+\frac{\zeta\pi}{p^2}\left[\left(\frac{m_3}{m_{13}}+\frac{m_1m_2}{2m_{13}m_{123}}\right)p^2-\frac{m_1m_{123}}{2m_2m_{13}}q'^2-m_1Q_{W\to R2}\right]\right.\right.$$
$$\left.\times\ln\left(\left[q'+\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\Big/\left[q'-\frac{pm_2}{m_{123}}+\left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}+i\epsilon\right]\right)\right]$$
$$\left.+\left[\text{``same'' except for opposite sign in front of } each\ \left(\frac{2Q_{W\to R2}m_2m_{13}}{m_{123}}\right)^{1/2}\right]\right\}.$$

By this grouping each [ ] term *separately* behaves as $1/q'$ as $q'\to\infty$. We can hence close the contours for the two [ ] terms *separately* in opposite half planes.[24a] One finally obtains

$$M_\Delta=A(\hat{p})\left\{\frac{i\pi^3f\zeta m_1[q-(2Q_{W\to R2}m_2m_{13}/m_{123})^{1/2}]}{(2\pi)^4m_{23}m_{13}pc^3}-\frac{i\pi^3f\zeta}{(2\pi)^4m_{23}p^2c^3}\left[\left(\frac{m_3}{m_{13}}+\frac{m_1m_2}{2m_{13}m_{123}}\right)p^2+\frac{m_1m_{123}}{2m_2m_{13}}p^2-m_1Q_{W\to R2}\right]\right.$$
$$\left.\times\ln\left[\frac{q+(pm_2/m_{123})+(2Q_{W\to R2}m_2m_{13}/m_{123})^{1/2}}{q-(pm_2/m_{123})+(2Q_{W\to R2}m_2m_{13}/m_{123})^{1/2}}\right]\right\}.\quad\text{(C8)}$$

One sees that M_Δ is regular at $p=0$, as it must be. We do not discuss the relativistic ansatz to be associated with (C8) in detail, since we have used the Galilean transformation and did not consider the spinor transformations. If one writes the ln in the form given in Eq. (3.15), one should use the same ansatz in *all* terms, to preserve the regular behavior at $p=0$. Hence we suggest

$$M=A(\hat{p})\frac{i\pi^3f\zeta}{(2\pi)^4m_{23}pc^3}\left\{\frac{m_1}{m_{13}}\left(q+q_S-\frac{m_2}{m_{123}}p_S\right)-\frac{1}{p}\left[\left(\frac{m_3}{m_{13}}+\frac{m_1m_2}{2m_{13}m_{123}}\right)p^2+\frac{m_1m_{123}}{2m_2m_{13}}\left(q^2-\left[q_S-\frac{m_2}{m_{123}}p_S\right]^2\right)\right]\right.$$
$$\left.\times\ln\left[\frac{q+(m_2/m_{123})p-q_S+(m_2/m_{123})p_S}{q-(m_2/m_{123})p-q_S+(m_2/m_{123})p_S}\right]\right\}.\quad\text{(C9)}$$