Nucleon as a Composite Particle State*

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A general scheme is presented for approximately determining the mass M and coupling with pions $g^2/4\pi$ of the nucleon, and the position and width of the πN 33 resonance. No cutoffs are required. Contributions from distant left-hand singularities to the $I=J=\frac{1}{2}$, $\frac{3}{2}$ p-wave amplitudes are approximated by Balázs-type pole terms, and those from nearby left-hand singularities, by crossing symmetry. Balázs-pole residues are determined by requiring crossing symmetry in a form relating the πN partial-wave amplitude on the nearby left-hand cut to physical πN scattering. As a first step in the investigation of our general scheme, we use experimental data on πN scattering in the $I=J=\frac{3}{2}$ p-wave state to determine the nucleon mass and pion-nucleon coupling constant. We find $M=1.03$ in units of the physical-nucleon mass and $g^2/4\pi=18.6$. The latter appears to be weakly dependent on the choice of Balázs-pole positions. The calculated nucleon mass, on the other hand, appears to be practically independent of the choice of pole positions.

I. INTRODUCTION

IN 1962, Chew[1] showed, on the basis of the static model, that the exchange of the N^* (3-3 resonance) can give an attractive force strong enough to produce the nucleon as a pion-nucleon bound state. Combined with previous indications[2] that the N^* is essentially produced by nucleon exchange, this result strongly suggests the possibility of a theory in which both the nucleon and the N^* arise as composite particle states. Subsequent relativistic dispersion-theory calculations[3,4] have given qualitative support to this possibility.

Essentially two different approaches have been used in these calculations. In some of them,[3] left-hand singularities of partial-wave amplitudes are taken from Born terms for N, N^* and (in some cases) ρ exchange, resonances being treated as single-particle states. The N/D method[5] is used to construct unitary amplitudes which are then examined for appropriate bound states and resonances. In these calculations, the unknown contributions from distant left-hand singularities (corresponding to short-range forces) are neglected. Also a cutoff is required to eliminate the well-known divergence due to the N^*-exchange Born term.

The other approach,[4] which (at least formally) circumvents the difficulties associated with unknown short-range forces and the divergence coming from N^* exchange, is based on a method introduced by Balázs.[6] In this method, contributions from distant left-hand singularities of partial-wave amplitudes are approximated by pole terms whose pole positions can presumably be determined by a definite criterion and whose residues are determined by requiring that a fixed-energy dispersion relation be satisfied at a suitable number of points. The dispersion relation yields an equality between a partial-wave amplitude (which in practice is obtained from the N/D method) at some (matching) point in the low-energy region and an integral covering an infinite-energy range, the main contributions to which are approximately expressed in terms of masses and widths of appropriate resonances. Although this method uses a more realistic description of short-range forces than the first, and does not require a cutoff,[7] it involves the neglect of possibly important high-energy contributions to the fixed-energy dispersion relation. This neglect of high-energy effects probably accounts in part for the sensitivity of results to the choice of matching point.[4] In both approaches, elastic unitarity is assumed in solving N/D equations.

In this paper, we will discuss, using a variant of the Balázs method, the problem of "generating" the nucleon given the N^* and vice versa, and also the much more difficult problem of "generating" both the nucleon and N^* in a single self-consistent calculation. We will follow Balázs in parametrizing effects of short-range forces with pole terms but will *not* use fixed-energy dispersion relations to determine residues at the poles. Instead, the residues will be determined by requiring crossing symmetry in a form relating a pion-nucleon partial-wave amplitude on the crossed pion-nucleon cut to physical pion-nucleon scattering. Our approach here is largely motivated by a previous calculation[8] of two of us (T. K. and A. T.) for pion-pion scattering in which Balázs pole residues were also determined by applying crossing relations in a straightforward manner. In fact, the formulation of crossing symmetry, as used in the present work, is much easier to apply practically than was that used in the π-π calculation.

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[1] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).
[2] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956); S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).
[3] E. Abers and C. Zemach, Phys. Rev. **131**, 2305 (1963); J. S. Ball and D. Wong, Phys. Rev. **133**, B179 (1964); M. Der Sarkissian, Nuovo Cimento **34**, 1314 (1964).
[4] V. Singh and B. M. Udgaonkar, Phys. Rev. **130**, 1177 (1963); P. Narayanaswamy and L. K. Pande, Phys. Rev. **136**, B1760 (1964).
[5] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[6] L. A. P. Balázs, Phys. Rev. **128**, 1935, 1939 (1962).
[7] See, however, discussion by F. Zachariason in *Strong Interactions and High Energy Physics, 1963 Edinburgh Summer School Proceedings*, edited by R. G. Moorhouse (Plenum Press, New York, 1964). See also the discussion at the end of Sec. III of this paper.
[8] T. Kanki and A. Tubis, Phys. Rev. **136**, B723 (1964).

In Sec. II of this paper, we present the general formalism including specific forms of partial-wave amplitudes and crossing relations. A brief outline of a program for simultaneously deducing the N and N^* from unitarity, crossing and analyticity is also given. As a first step in the investigation of such a program, we report in Sec. III a calculation in which experimental data on the 33 resonance is used to determine the nucleon mass and pion-nucleon coupling constant. Section IV contains a "Summary and Conclusions".

II. GENERAL FORMALISM

A. Pion-Nucleon Amplitude and Crossing Relations

The pion-nucleon amplitude (see Fig. 1) can be expressed in terms of the four invariant amplitudes, $A^{\pm}$ and $B^{\pm}$ as[9]

$$T_{\alpha\beta}=-A_{\alpha\beta}+\frac{i}{2}\gamma\cdot(q_1+q_2)B_{\alpha\beta}, \tag{2.1}$$

$$A_{\alpha\beta}=\delta_{\alpha\beta}A^{+}+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]A^{-}, \tag{2.2}$$

$$B_{\alpha\beta}=\delta_{\alpha\beta}B^{+}+\tfrac{1}{2}[\tau_\alpha,\tau_\beta]B^{-}, \quad (\alpha,\beta=1,2,3), \tag{2.3}$$

where (q_1,α) and (q_2,β) are the momenta and isospin-state labels of the incident and outgoing pion, respectively; τ_α is the nucleon isospin operator; and $A^{\pm}$ and $B^{\pm}$ are functions of the three Mandelstam variables $s=(p_1-q_1)^2$, $t=(p_2-p_1)^2$ and $u=(p_1-q_2)^2$. In terms of the center-of-mass scattering-angle θ_s in the s channel, t and u are given by[10]

$$t=-2q_s^2(1-\cos\theta_s), \tag{2.4}$$

$$u=\frac{1-\cos\theta_s}{2}\frac{(1-\mu^2)^2}{s}-\frac{1+\cos\theta_s}{2}[s-2(1+\mu^2)], \tag{2.5}$$

where q_s^2 is the square of the center-of-mass momentum in the s channel.

$$q_s^2=\frac{1}{4s}[s-(1+\mu)^2][s-(1-\mu)^2]. \tag{2.6}$$

The substitution rule, when applied to the amplitudes (2.1)-(2.3), leads to the crossing relations

$$\begin{aligned}A^{\pm}(s,t,u)&=\pm A^{\pm}(u,t,s),\\ B^{\pm}(s,t,u)&=\mp B^{\pm}(u,t,s).\end{aligned} \tag{2.7}$$

In terms of isospin eigenamplitudes A^I defined in the s channel by

$$\begin{aligned}A^{3/2}(s;t,u)&=A^{+}(s,t,u)-A^{-}(s,t,u),\\ A^{1/2}(s;t,u)&=A^{+}(s,t,u)+2A^{-}(s,t,u),\end{aligned}$$

[9] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[10] In the present work, we set the mass of the nucleon equal to unity. Then μ = mass of pion/mass of nucleon = 0.146 and $s+t+u=2(1+\mu^2)$.

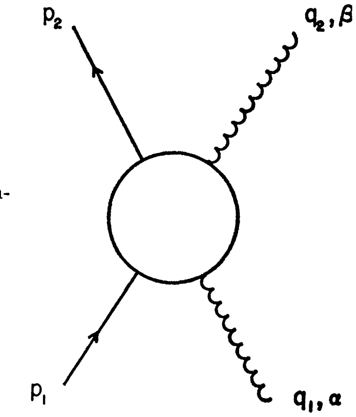

FIG. 1. The pion-nucleon interaction $\pi+N\to\pi+N$.

with similar expressions for the B^I, the crossing relations (2.7) become

$$A^I(s;t,u)=\sum_{I'=\frac{1}{2},\frac{3}{2}}C_{II'}A^{I'}(u;t,s), \tag{2.8}$$

$$B^I(s;t,u)=-\sum_{I'=\frac{1}{2},\frac{3}{2}}C_{II'}B^{I'}(u;t,s), \tag{2.9}$$

where

$$C_{II'}=\begin{pmatrix}-\frac{1}{3} & \frac{4}{3}\\ \frac{2}{3} & \frac{1}{3}\end{pmatrix}; \quad I,I'=\tfrac{1}{2},\tfrac{3}{2}. \tag{2.10}$$

The amplitudes $A^{I'}$ on the right sides of (2.8) and (2.9) are the eigenamplitudes in the u channel.

The partial-wave amplitude

$$h_{l\pm}{}^I(s)=e^{i\delta_{l\pm}{}^I(s)}\frac{\sin\delta_{l\pm}{}^I(s)}{\rho(q)}$$

which corresponds to the state with total isospin I, orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$, is given by the well-known formula[9,11]

$$\begin{aligned}h_{l\pm}{}^I(s)=\frac{q_s}{2\rho(s)}\int_{-1}^{1}d\cos\theta_s[&P_l(\cos\theta_s)f_1{}^I(s;\cos\theta_s)\\ &+P_{l\pm1}(\cos\theta_s)f_2{}^I(s;\cos\theta_s)],\end{aligned} \tag{2.11}$$

where

$$\begin{aligned}f_1{}^I&=\frac{E(s)+1}{8\pi\sqrt{s}}[A^I+(\sqrt{s}-1)B^I],\\ f_2{}^I&=\frac{E(s)-1}{8\pi\sqrt{s}}[-A^I+(\sqrt{s}+1)B^I].\end{aligned} \tag{2.12}$$

$\rho(s)$ is a kinematical factor which will be specified later, $P_l(\cos\theta_s)$ is the Legendre polynomial of order l, and $E(s)\equiv(s+1-\mu^2)/2\sqrt{s}$ is the nucleon energy in the center-of-mass system.

Inserting (2.8) and (2.9) into (2.12), we obtain the

[11] S. C. Frautschi and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

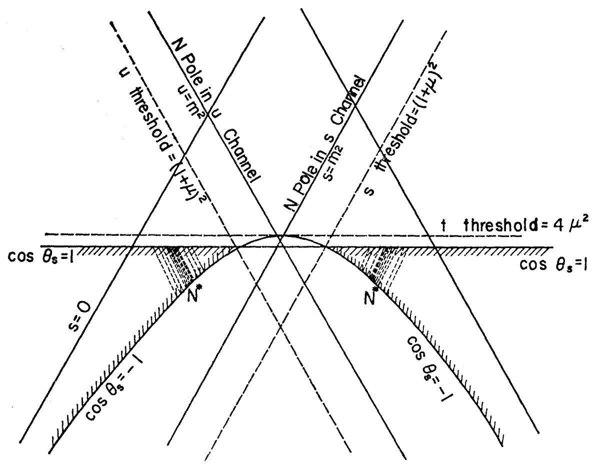

FIG. 2. The Mandelstam plot for the pion-nucleon interaction. The region of the u and s channels where the 33 (N^*) resonance dominates is indicated. The s- and u-channel nucleon poles are associated with a mass m not initially set equal to the physical nucleon mass ($\equiv$unity in this paper) in order to emphasize that this mass should be deduced from a dynamical calculation.

crossing relations in terms of $f_1{}^I$ and $f_2{}^I$.

$$f_1{}^I(s;\cos\theta_s) =\frac{E(s)+1}{2\sqrt{s}}\sum_{I'} C_{II'}\left[\frac{\sqrt{u}-\sqrt{s}+2}{E(u)+1}f_1{}^{I'}(u;\cos\theta_u) -\frac{\sqrt{u}+\sqrt{s}-2}{E(u)-1}f_2{}^{I'}(u;\cos\theta_u)\right], \quad (2.13a)$$

$$f_2{}^I(s;\cos\theta_s) =-\frac{E(s)-1}{2\sqrt{s}}\sum_{I'} C_{II'}\left[\frac{\sqrt{u}+\sqrt{s}+2}{E(u)+1}f_1{}^{I'}(u;\cos\theta_u) -\frac{\sqrt{u}-\sqrt{s}+2}{E(u)-1}f_2{}^{I'}(u;\cos\theta_u)\right]. \quad (2.13b)$$

θ_u, the center-of-mass scattering angle in the u channel, may be obtained by interchanging s and u in (2.5).

We see from (2.4) and (2.5) that $\cos\theta_s=+1$ on the line $t=0$ and $\cos\theta_s=-1$ on the curve $su=(1-\mu^2)^2$. This curve and line are also the boundary curves for $|\cos\theta_u|\leq 1$ since they are invariant under the interchange $s\rightleftarrows u$ (see Fig. 2). Thus, when $|\cos\theta_s|\leq 1$ and s is in the unphysical region $[0<s\leq(1-\mu)^2]$ for the s channel, all $f^{I'}$s on the right-hand sides of (2.13a) and (2.13b) are to be evaluated in the physical region of the u channel. We can thus use the following partial-wave expansions[9,11] for these $f^{I'}$s.

$$f_1{}^I(u;\cos\theta_u)=\frac{\rho(s)}{q_u}\sum_{l=0}^{\infty} h_{l+}{}^I(u)P_l{}'(\cos\theta_u) -\sum_{l=2}^{\infty} h_{l-}{}^I(u)P_{l-1}{}'(\cos\theta_u), \quad (2.14a)$$

$$f_2{}^I(u;\cos\theta_u)=\frac{\rho(s)}{q_u}\sum_{l=0}^{\infty}[h_{l-}{}^I(u)-h_{l+}{}^I(u)]P_l{}'(\cos\theta_u). \quad (2.14b)$$

Equations (2.11), (2.13), and (2.14) thus express by crossing symmetry the $h_{l\pm}{}^I(s)$ in the unphysical region $[0<s\leq(1-\mu^2)^2]$ in terms of their physical values.

Since we only use the Legendre expansions (2.14) for physical amplitudes, they converge rapidly in the low-energy region of the u channel and our crossing relation for $h_{l\pm}{}^I(s)$ will be accurately satisfied for s close to the threshold of the left-hand cut $[s=(1-\mu)^2]$ even if only the first few terms in the Legendre expansion are retained. This fortunate situation is lacking in the case of pion-pion scattering[8] where we had to use Legendre expansions for unphysical amplitudes in order to practically use crossing relations analogous to the one discussed here.

B. Choice of Partial-Wave Amplitudes and Effective-Range Expansions

In Fig. 3 we show the dynamical singularities of $h_{l\pm}{}^I(s)$.[12] The right-hand cut starts from $s=(1+\mu)^2$ and the left-hand cut from $s=(1-\mu)^2$. The nucleon pole is at $s=m^2$ and the nucleon short cut is between $s=(1-\mu^2)^2/m^2$ and $s=2(1+\mu^2)-m^2$. Note that we have used m for the mass of the *internal nucleon*, which corresponds to a pole of the $l=I$, $I=J=\frac{1}{2}$ amplitude. We do not assume initially that m is equal to the *external nucleon* mass because we assume that the mass of the *internal nucleon* should be calculated dynamically as the total energy of the bound system of the *external nucleon* and a pion. In Fig. 3, we have neglected the circular cut which comes from the pion-pion interaction in the t channel. In Appendix A, we discuss the effect of this circular cut and show that it gives only a small correction to the analysis of the $I=J=\frac{1}{2}$ p-wave amplitude in Sec. III.

We now discuss the choice of kinematic factor $\rho(s)$ in (2.11). If $\rho(s)=q_s$, then

$$h_{l\pm}{}^I(s)=(e^{i\delta_{l\pm}{}^I}/q_s)\sin\delta_{l\pm}{}^I(s).$$

This amplitude has the following behaviors[13] near the

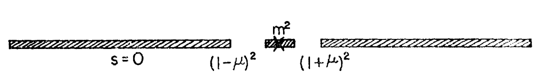

FIG. 3. Singularities of $h_{l\pm}{}^I(s)$. The short crossed nucleon cut extends from $s=(1-\mu^2)^2/m^2$ to $s=2(1+\mu^2)-m^2$. The circular t-channel cut is not shown here but its effect on our calculations is discussed in Appendix A.

[12] Because we work in the complex s plane, it is evident from (2.12) that there is a *kinematic* branch cut starting at $s=0$. This is of no consequence in the present calculations since the Balázs effective range expansion to be employed (see Sec. IIB) in principle accounts for the effect of singularities in the region of this cut.

[13] G. Fry and R. L. Warnock, Phys. Rev. **130**, 478 (1963).

thresholds of the right- and left-hand cuts, respectively:

$$h_{l\pm}{}^{I}(s)\sim q_s{}^{2l}\sim[s-(1+\mu)^2]^l, \qquad s\to(1+\mu)^2; \quad (2.15)$$

$$\sim[s-(1-\mu)^2]^{1/2}+\mathrm{const}\,\delta_{l0}, \quad s\to(1-\mu)^2. \quad (2.16)$$

When we apply the N/D method to this amplitude, the discontinuity of the left-hand cut of the N function should therefore satisfy certain moment-conditions. Now we will shortly introduce a Balázs effective-range expansion[6] for the distant singularities of the N function. From a practical point of view, it is desirable to free this expansion from these moment-restrictions and to build in the threshold behaviors (2.15) and (2.16). Thus, for p-wave amplitudes, we choose $\rho(s)$ so that

$$h_{2I2J}(s)=\left(\frac{s}{s-(1-\mu)^2}\right)^{1/2}\frac{s}{s-(1+\mu)^2}\times\frac{e^{i\delta_{2I2J}}\sin\delta_{2I2J}}{q_s}, \quad (2.17)$$

where the conventional notation ($2I2J$) for partial-wave amplitudes is introduced and will be used from now on. The branch of the square root on the right side of (2.17) is chosen so that it has a cut from $s=0$ to $s=(1-\mu)^2$. The function h_{2I2J} is finite at both the right- and left-hand cut thresholds. Near $s=0$, the behavior of h_{2I2J} is controlled by the high-energy amplitude for backward π-N scattering. In Appendix B, it is shown that h_{2I2J} does not vanish at $s=0$ provided that the backward-scattering amplitudes $A(u;\cos\theta_u=-1)$ and $B(u;\cos\theta_u=-1)$ behave as $\sim\mathrm{const}\times u^{\alpha}$ ($\alpha>\frac{1}{2}$). Unfortunately, we have no available data for backward π-N scattering. In this paper, we will for simplicity assume that h_{2I2J} does *not* vanish at $s=0$.

If we invoke two-particle unitarity over the whole right-hand cut and introduce Balázs effective-range poles for contributions from the distant left-hand cut, the usual N/D formalism[5,6] leads to

$$h_{2I2J}(s)=N_{2I2J}(s)/D_{2I2J}(s), \quad (2.18)$$

$$N_{2I2J}(s)=\frac{1}{\pi}\int_{(1-\mu^2)^2/m^2}^{2(1+\mu^2)-m^2}ds'\frac{\mathrm{Im}h_{2I2J}(s')}{s'-s}D_{2I2J}(s')+\frac{1}{\pi}\int_{L}^{(1-\mu)^2}\frac{\mathrm{Im}h_{2I2J}(s')}{s'-s}D_{2I2J}(s')+\frac{\alpha_{2I2J}}{s}+\frac{\beta_{2I2J}}{s+50}, \quad (2.19)$$

$$D_{2I2J}(s)=1-\frac{s-s_0}{\pi}\int_{(1+\mu)^2}^{\infty}ds'\frac{\rho(s')N_{2I2J}(s')}{(s'-s_0)(s'-s)}. \quad (2.20)$$

The first term on the right side of (2.19) is the contribution from the crossed-nucleon cut. The second term is the contribution of left-hand cut singularities from $s=L$ to $s=(1-\mu)^2$. We will specify L later. The contribution from unknown "distant" left-hand singularities ($s<L$) is accounted for by the Balázs pole terms.

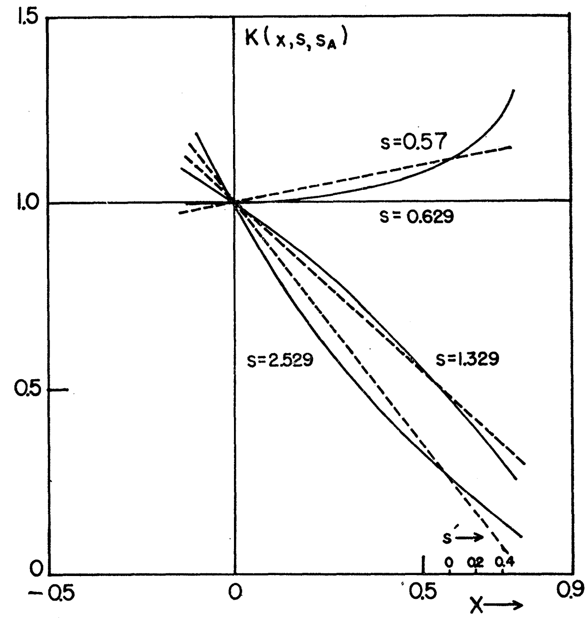

FIG. 4. Plots of $K(x,s,s_A)\equiv[1+x(s_A-(1-\mu)^2-1)]/[1+x(s-(1-\mu)^2-1)]$, where $s_A=0.629$, for several values of s. Some relevant values of $s'\equiv1+(1-\mu)^2-1/x$ are indicated along the x axis.

The positions of the poles have been chosen in accordance with Balázs' criterion. If we set $s'=[1+(1-\mu)^2-1/x]$, the contribution to N_{2I2J} from "distant" left-hand singularities may be written as

$$\frac{-1}{\pi}\int_0^{x_L}\frac{dx\,K(x;s;s_A)}{x\{1+x[s_A-(1-\mu)^2-1]\}}\times\mathrm{Im}h_{2I2J}[s'(x)]D_{2I2J}[s'(x)], \quad (2.21)$$

where

$$1/x_L=1+(1-\mu)^2-L,$$

$$K(x,s,s_A)=\frac{1+x[s_A-(1-\mu)^2-1]}{1+x[s-(1-\mu)^2-1]},$$

$$s_A=0.629.$$

The behavior of K as a function of x for several s values is shown in Fig. 4. The straight-line approximation shown there corresponds to the pole terms of (2.19). We see that reasonable accuracy is obtained for $0.5\gtrsim s\gtrsim2.5$ (the region of interest in this work) and $-0.1\gtrsim x\gtrsim0.8$ (corresponding to $-\infty<s'\gtrsim0.45$ and $12\gtrsim s'<+\infty$). The validity of the approximation for $K(x;s;s_A)$ in the region corresponding to $12\gtrsim s'<\infty$ means that some effects of inelastic unitarity and the breakdown of our representation for $N_{2I2J}(s')$, appearing in the integrand of $D_{2I2J}(s)$ for $s'>12$, are taken into account.[6]

C. General Scheme of Self-Consistent Calculation

We shall briefly discuss here a general program for self-consistently determining the mass and coupling with pions of the nucleon, and the mass and width of the 33 resonance (N^*). We have previously shown in the case of pion-pion scattering[8] that the existence and approximate mass and width of the $I=J=1(\rho)$ resonance follow as simple consequences of analyticity, unitarity and crossing symmetry. In this paper, we essentially adapt the approach used in pion-pion scattering to the problem of determining pion-nucleon p-wave amplitudes for $I=J=\frac{1}{2}$ and $I=J=\frac{3}{2}$.

Approximate unitarity and analyticity requirements are built into the N/D expressions (2.18)–(2.20). We now impose crossing symmetry by requiring that each partial-wave amplitude h_{2I2J} given by (2.18)–(2.20) satisfy the crossing relation following from (2.11)–(2.14), on the nearby left-hand cut $[L\leq s\leq(1-\mu)^2]$. This crossing relation gives both the real and imaginary part of the h_{2I2J} on this cut in terms of their physical values. The relation for the imaginary part may be used to calculate the integral over the nearby left-hand cut of the N function [the second integral of (2.19)]. The relation for the real part may then be used to determine the pole residues α and β of (2.19) in a way similar to the procedure used in the pion-pion calculation.[8] Actually, it will be more convenient in this work to use these "real-part" crossing relations in a somewhat different manner which will be discussed further on.

In the second integral of (2.19), the maximum value of u in the Legendre expansions (2.14) appearing in the integrand is $(1-\mu^2)^2/L$. For $L=0.4$, which is roughly the value we use in this work, the corresponding laboratory pion kinetic energy is about 560 MeV. Therefore, a reasonable approximation for the Legendre expansions (2.14) is to retain only the h_{33} term on the right-hand sides. We will discuss this approximation in detail in Secs. 3 and 4.

The compositeness assumption for the nucleon implies that in N/D partial-wave calculations, the nucleon should appear dynamically as a zero of D_{11} as well as a short (crossed-nucleon) cut in all the h_{2I2J}. Thus, assuming that the 33 contribution to the crossing relation just discussed dominates the other contributions, we may, in terms of the coupled h_{11} and h_{33} problem, set up an approximate self-consistent procedure for simultaneously deducing the mass and width of the 33 resonance and the mass and coupling with pions of the nucleon.[14]

For example, we might make an initial guess for these parameters, use them to determine the first two integrals on the right side of (2.19), find values for α_{11}, β_{11}, α_{33}, and β_{33} so that the calculated h_{11} and h_{33} have poles and residues at these poles corresponding to the input parameters, investigate the self-consistency of the input parameters by seeing how well the "real-part" crossing relations are satisfied on the nearby left-hand cut, and finally repeat this procedure for various sets of input parameters until the "real-part" crossing relations are satisfied to an accuracy compatible with the approximate nature of the calculation. In the next section, we will discuss a calculation based on a more limited program in which the N^* parameters are assumed and only the nucleon parameters are determined. Calculations based on the full program are now in progress and will be reported at a later time.

D. Internal- and External-Nucleon Masses

A few words should be said here concerning the relationship between the *external-* and *internal-*nucleon masses. The mass of the *internal nucleon*, according to the composite-nucleon picture, is associated with the position of a (calculated) zero of D_{11}. This might not, in general, be equal to the mass of the *external nucleon*. In other words, we have assumed that the *internal nucleon* is a dynamical bound state of a pion and the *external nucleon* so that we cannot *a priori* set these masses equal. Instead, this equality of internal and external masses should be derived from dynamical principles. Actually, this equality can be understood as a consequence of Chew's bootstrap principle,[15] namely, that every one particle state be a compound of particles which are able to couple with this one particle state.

To show this, we first note that if these masses are not equal, there exist two different nucleon states, N and N', similar with respect to all quantum numbers and differing only in mass with the basic vertex now being $\pi NN'$. We must now consider the $\pi+N\rightarrow\pi+N$ as

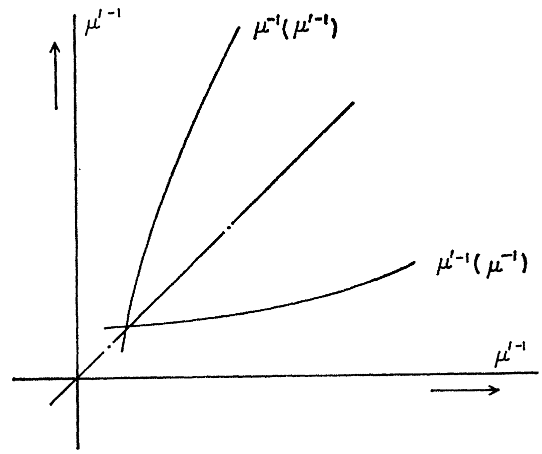

FIG. 5. Plot illustrating the equality of *internal* and *external* nucleon masses according to Chew's hypothesis. The actual nucleon mass corresponds to the intersection of the two curves. The curve $\mu^{-1}(\mu'^{-1})$ gives the N-pion mass-ratio in terms of the N'-pion mass ratio and similarly for the curve $\mu'^{-1}(\mu^{-1})$. The two curves differ only by the interchange $\mu\rightleftarrows\mu'$.

[14] To the extent that we work with the coupled equations for h_{11} and h_{33}, the program to be described here is similar in spirit to the "reciprocal bootstrap" calculations (Refs. 1, 3, 4) previously mentioned but quite different in methodology.

[15] G. F. Chew, Rev. Mod. Phys. 34, 394 (1962).

well as the $\pi+N'\rightarrow\pi+N'$ amplitude. N' should appear as a pole of the amplitude for $\pi+N\rightarrow\pi+N$ scattering and N as a pole of the amplitude for $\pi+N'\rightarrow\pi+N'$ scattering. Now our theory has only one parameter for each scattering, namely the mass ratios $\mu=$ (pion mass)/(N mass) and $\mu'=$ (pion mass)/(N' mass), respectively. First consider π-N scattering. According to Chew's hypothesis, the mass of the N' which is essentially given by μ'^{-1} is dynamically calculable. Thus μ'^{-1} is a function of μ. The functional dependence may be represented by a curve in the $\mu^{-1}-\mu'^{-1}$ plane (see Fig. 5). Now consider π-N' scattering. Since N and N' are identical except for mass, the dynamical equations for π-N' scattering should be the same as those for π-N scattering except for the fact that μ and μ' are interchanged. Thus we obtain another functional relation between μ^{-1} and μ'^{-1} which is obtained from that of π-N scattering by the interchange $\mu\leftrightarrow\mu'$. Chew's principle shows that the solution of our problem should correspond to the intersection of these two curves, thus leading to the equality of both masses.

In the calculation of h_{11} in Sec. III of this paper, we fix the external-nucleon mass at its physical value and only determine the internal-nucleon mass. This is because we are using experimental data on one side of the crossing relations discussed in Sec. IIA, thus making it awkward to vary the external mass. Variation of both internal and external masses is now being carried out in connection with the full self-consistency program outlined in Sec. IIC.

III. DETERMINATION OF NUCLEON MASS AND COUPLING WITH PIONS FROM THE N^* PARAMETERS

We now discuss the details of a calculation of the nucleon parameters from the assumed (experimental) scattering in the 33 state. The general procedure has already been briefly described in Sec. IIC.

The 11 amplitude is given by (2.18)–(2.20) with $I=J=\frac{1}{2}$. For the crossed-nucleon-cut contribution [first term on right side of (2.19)], we use[9,11]

$$\mathrm{Im}h_{11}^*(s+i\epsilon)=\frac{g^2}{4\pi}\left(\frac{1}{s-(1-\mu)^2}\right)^{1/2}\frac{s}{s-(1+\mu)^2}\frac{\pi}{8q_s^2}\times\left[[E(s)+1](\sqrt{s}-1)\left(1-\frac{s+m^2-2-2\mu^2}{2q_s^2}\right)+[E(s)-1](\sqrt{s}+1)\right],\quad(3.1)$$

which is written in terms of the conventional coupling-constant $g^2/4\pi$ (≈ 15 experimentally). For the nearby left-hand-cut contribution [second term on the right side of (2.19)], we use the crossing relation obtained by substituting (2.13) and (2.17) into (2.11)[16]

$$h_{2I2l\pm}(s+i\epsilon)=-i\left(\frac{s}{|s-(1-\mu)^2|}\right)^{1/2}\frac{s}{s-(1+\mu)^2}\int_{u_+(s)}^{u_-(s)}du\frac{s}{(s-[1-\mu^2])^2-4\mu s}\Bigg[P_l(\cos\theta_s)\frac{E(s)+1}{2\sqrt{s}}$$
$$\times\sum_{I'}C_{II'}\left(\frac{\sqrt{u}-\sqrt{s}+2}{E(u)+1}f_1^{I'}(u-i\epsilon;\cos\theta_u)-\frac{\sqrt{u}+\sqrt{s}-2}{E(u)-1}f_2^{I'}(u-i\epsilon;\cos\theta_u)\right)-P_{l\pm1}(\cos\theta_s)$$
$$\times\frac{E(s)-1}{2\sqrt{s}}\sum_{I'}C_{II'}\left(\frac{\sqrt{u}+\sqrt{s}+2}{E(u)+1}f_1^{I'}(u-i\epsilon;\cos\theta_u)-\frac{\sqrt{u}-\sqrt{s}+2}{E(u)-1}f_2^{I'}(u-i\epsilon,\cos\theta_u)\right)\Bigg],\quad(3.2)$$

$$L\leq s\leq(1-\mu)^2;\quad\epsilon\rightarrow0_+,\qquad(3.3)$$
$$u_-(s)=(1-\mu^2)^2/s,$$
$$u_+(s)=-(s-2-2\mu^2),\qquad(3.3\text{a})$$
$$\cos\theta_s=\frac{u_+(s)+u_-(s)-2u}{u_-(s)-u_+(s)},\qquad(3.4)$$
$$\cos\theta_u=-1+\frac{2[(1-\mu^2)^2-su]}{[u-(1+\mu^2)]^2-4\mu^2}.\qquad(3.4\text{a})$$

The $f_{1,2}^I(u,\cos\theta_u)$ are given by (2.14).

[16] We have included the $i\epsilon$ explicitly to emphasize the fact that if s approaches the real axis from above, and $\cos\theta_s$ is physical, then u approaches the real axis from below.

As was previously mentioned, we will only retain the 33 contribution to the right-hand side of (3.2). The quantitative reliability of this procedure is illustrated in Table I where contributions to the crossing relations from various states are given.[17] The real and imaginary parts of h_{11} along the nearby left-hand cut are plotted in Fig. 6.

L in (2.19) was chosen as 0.49. This value corresponds to a maximum pion lab kinetic energy of 340 MeV in the u integration of (3.2), an energy low enough so that the assumption of (33) dominance of crossing relations is valid. Also, the imaginary part of $h_{11}(s)$ as given by crossing relations vanishes at this value. By starting the

[17] L. D. Roper, R. M. Wright, and B. T. Feld, Phys. Rev. **138**, B190 (1965).

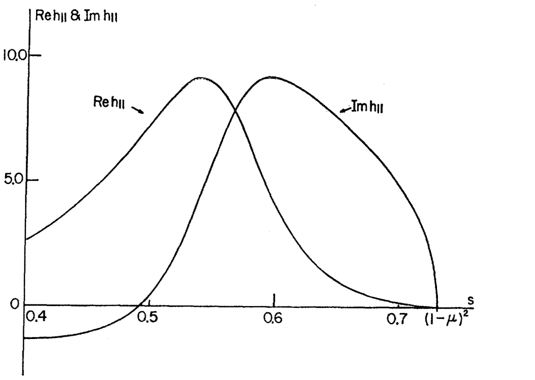

FIG. 6. h_{11} along the nearby left-hand cut calculated from the crossing relation (3.2) with experimental 33 data (Ref. 17).

integral of the nearby left-hand cut contribution to N_{11} at this value of L, we thus avoid a spurious end-point singularity, which would be hard to compensate for with Balázs poles.

The determination of m^2 and g^2 from crossing, unitarity, and analyticity assumptions is most conveniently accomplished by an iterative procedure based on the general program of Sec. IIC.

We make an initial choice of g^2 and m^2 and use experimental (33) information to evaluate the crossing relation for $\mathrm{Im}h_{11}(s)$, $L\leq s\leq (1-\mu)^2$. We then approximate D_{11} in the integrals of (2.19) by

$$D_{11}(s)=(s-m^2)/[(1+\mu)^2-m^2], \tag{3.5}$$

which corresponds to the linear approximation for $D_{11}(s)$ normalized to unity at $s=s_0=(1+\mu)^2$ and leading to a bound state at $s=m^2$. D_{11}, given by (2.20), may now be calculated and will be a linear function of α_{11} and β_{11}.

The conditions for a bound-state pole at m^2 and residue at the pole corresponding to the conventional pion-nucleon coupling-constant $g^2/4\pi$ are

$$D_{11}(m^2)=0, \tag{3.6}$$

$$\frac{N_{11}}{D_{11}'(m^2)}=-3\frac{g^2}{4\pi}\left(\frac{m^2}{m^2-(1-\mu)^2}\right)^{1/2} \times\frac{m^2}{m^2-(1+\mu)^2}\,\frac{E(m^2)-1}{2m}(m+1), \tag{3.7}$$

$$D_{11}'=dD_{11}/ds;$$

Eqs. (3.6) and (3.7) are two linear inhomogeneous equations for α_{11} and β_{11} which may be readily solved. We may then iterate this procedure using instead of (3.5) the actual D_{11} calculated the first time with α_{11} and β_{11} determined from (3.6) and (3.7). The iterations are continued until the changes in α_{11} and β_{11} for successive

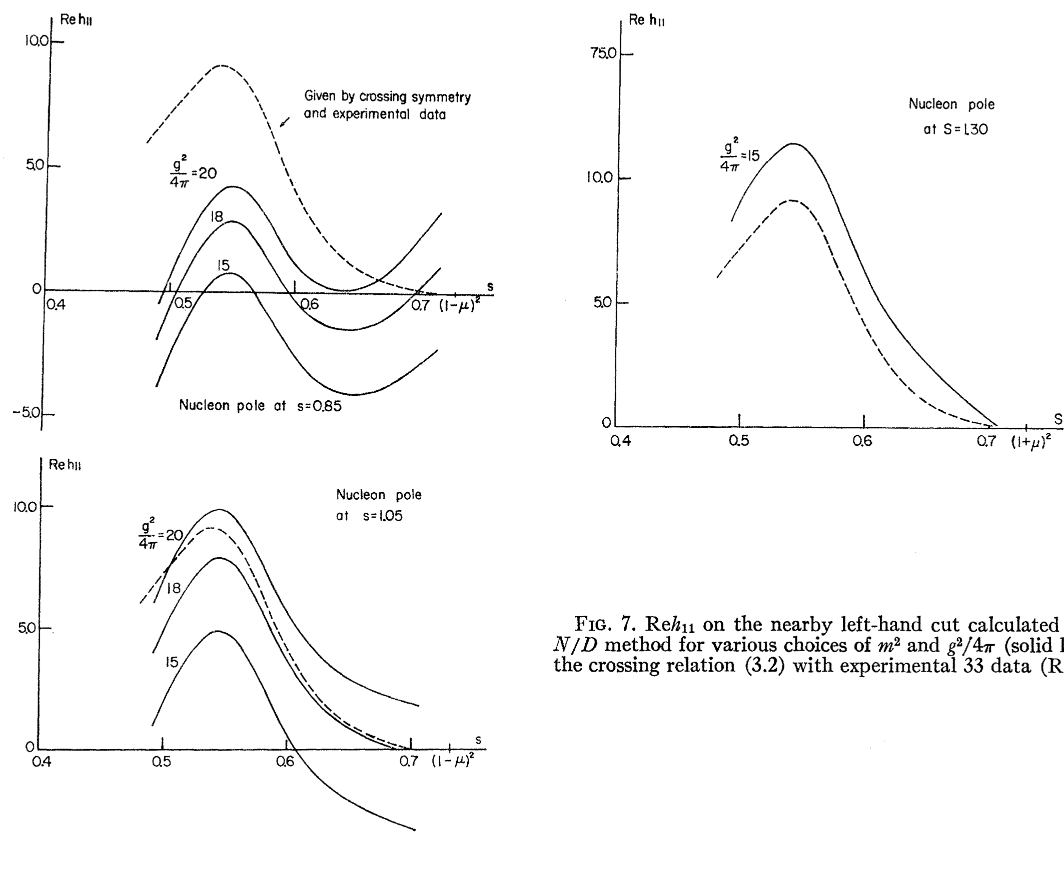

FIG. 7. Reh_{11} on the nearby left-hand cut calculated from the N/D method for various choices of m^2 and $g^2/4\pi$ (solid lines) and the crossing relation (3.2) with experimental 33 data (Ref. 17).

iterations are negligible. It was found in practice that only three iterations were usually needed.

This whole procedure is now repeated for various values of $g^2/4\pi$ and m^2. The final choice of values is that leading to an amplitude $h_{11}(s)$ which most accurately satisfies the crossing relation for $\mathrm{Re}h_{11}(s)$, $L<s\leq(1-\mu)^2$, as determined from experiment. In Fig. 7, we show how accurately this crossing relation is satisfied for various choices of $g^2/4\pi$ and m^2. A general feature of the results is that for fixed m^2, the calculated $\mathrm{Re}h_{11}(s)$ tends to *move vertically* without much change in shape for changes in g^2, and for fixed g^2, the slope of the calculated $\mathrm{Re}h_{11}(s)$ near the left-hand cut threshold is a very sensitive function of m^2. This feature allows one to make a fairly unique choice for m^2 and $g^2/4\pi$, our final values being

$$\begin{aligned} m^2&=1.05\,, \\ g^2/4\pi&=18.6\,. \end{aligned} \tag{3.8}$$

The experimental values are $m^2=1$, $g^2/4\pi\approx 15$.

It is interesting to compare the present calculation with that of Narayanaswamy and Pande[4] who also introduced Balázs poles. Instead of determining the pole residues as done here, they matched the value and derivative of the 11 amplitude at a point between the left-hand and crossed-nucleon cuts with that given by fixed-energy dispersion-relations. The low-energy parts of the fixed-energy dispersion integrals were approximated by the N^* and ρ contributions while the high-energy parts were evaluated according to the strip approximation of Singh and Udgaonkar.[4] It was found that results were very sensitive to the choice of matching point. As was mentioned in the Introduction, this probably indicates an inadequate description of high-energy effects in the fixed energy dispersion relations. Such "matching-point" sensitivity of results is absent in the present calculation since crossing relations are required to be satisfied over a *finite* region where we are fairly certain that all important contributions to the relations are taken into account.

Although we do not have "matching-point" sensitivity in our calculation, the results are slightly dependent on the choice of Balázs-pole positions. In Fig. 8,

TABLE I. Non-negligible contributions to h_{11} along the nearby left-hand cut calculated from the crossing relation (3.2) and experimental data (Ref. 17).

		$\mathrm{Re}h_{11}$		
s	P_{33}	S_{31}	S_{11}	Total
0.70	0.12	−0.02	0.01	0.11
0.62	2.47	−0.09	0.02	2.40
0.54	9.18	−0.12	0.03	9.09
0.46	4.99	−0.12	0.03	4.90
		$\mathrm{Im}h_{11}$		
S	P_{33}	S_{31}	S_{11}	Total
0.70	4.66	0.19	−0.05	4.80
0.62	8.61	0.20	−0.05	8.76
0.54	4.35	0.18	−0.04	4.49
0.46	−1.05	0.12	−0.03	−0.96

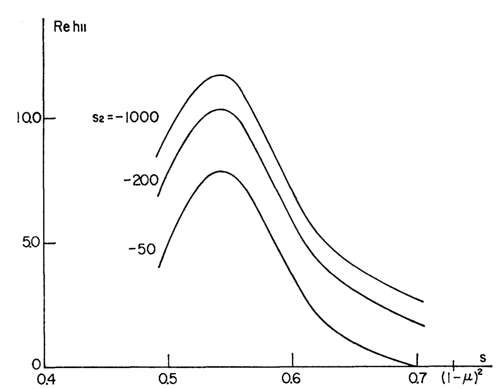

FIG. 8. $\mathrm{Re}h_{11}$ on the nearby left-hand cut calculated from the N/D method for $m^2=1.05$, $g^2=18$ and Balázs pole positions $s_1=-50$, $s_2=-50$, -200, -1000. The curve for $s_2=-50$ is in fairly good agreement with $\mathrm{Re}h_{11}$ calculated according to the crossing relation (3.2) with experimental 33 data (Ref. 17).

we give the real part of h_{11} on the nearby left-hand cut for Balázs-pole positions $s_1=0$ and $s_2=-50$, -200, and -1000.[18] The curves are all calculated under the assumption that the nucleon pole is at $s=1.05$ and the pion-nucleon coupling is 18. These nucleon parameters correspond roughly to the best fit to the real part crossing relations for $s_1=0$, $s_2=-50$. We see from the figure that the nucleon mass determined from our analysis is essentially independent of the assumed Balázs-pole position but that the "calculated" pion-nucleon coupling constant seems to decrease slowly as the distant Balázs pole moves toward $-\infty$. For $s_2=-1000$, the coupling constant $g^2/4\pi$ as determined by our analysis is about 14 compared to 18.6 for $s_2=-50$.

In Fig. 9, we give the P_{11} phase shift associated with the parameter choice (3.8). This behavior for δ_{11} is very similar to that obtained by Balázs[6] and Narayanaswamy and Pande[4] and is in rather strong disagreement with the phase shift implied by Roper *et al.*[17] However, there is some conflict among several current phase-shift analyses[19] and it is consequently difficult to draw any strong conclusions from the disagreement in phase shifts.

IV. SUMMARY AND CONCLUSIONS

Reasonable values for the nucleon mass and pion-nucleon coupling constant have been obtained by using a variation of the "standard" Balázs technique.[4,6] A Balázs parametrization is made for the contribution from distant left-hand singularities to the p wave $I=J=\frac{1}{2}$ pion-nucleon amplitude. The Balázs-pole residues are determined not by means of a fixed-energy dispersion relation[4,6] but by requiring crossing symmetry in the form relating a partial-wave amplitude on the nearby left-hand cut to physical πN scattering (assumed to be essentially given by experimental πN-

[18] These choices of pole positions are all compatible with the accuracy stated in Sec. IIC.

[19] See, e.g., R. J. Cence and M. Y. Cha, Bull. Am. Phys. Soc. **10**, 528 (1965).

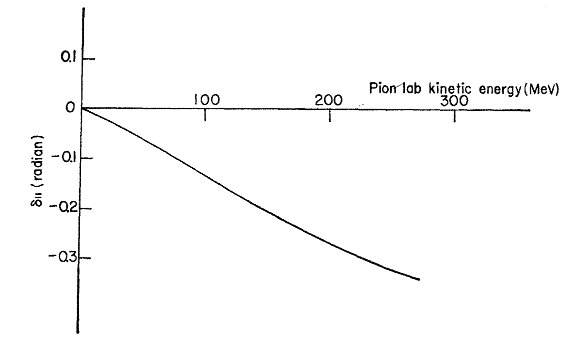

FIG. 9. P_{11} phase shift predicted by N/D solution with parameter choice (3.8).

scattering data for the p wave $I=J=\frac{3}{2}$ state). By using crossing relations in this manner, one does not meet with difficulties such as those associated with unknown high-energy contributions to the fixed-energy dispersion relations.[4,6] The matching-point sensitivity of the "standard" Balázs calculation is, of course, not present in our approach although there is a weak dependence of the calculated coupling constant (but not of the nucleon mass) on the Balázs pole positions.

In conclusion, the results of this paper are consistent with the picture of the nucleon as a composite particle state whose approximate mass and coupling with pions are deducible from standard assumptions concerning charge independence, crossing symmetry, analyticity, and unitarity.

ACKNOWLEDGMENTS

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APPENDIX A: ρ-MESON CONTRIBUTIONS TO N_{11} AND N_{33}

In the main part of this paper, we have neglected terms in h_{11} and h_{33} coming from the $I=J=1$ pion-pion (ρ)-resonance contribution to the $N\bar{N}\rightarrow\pi\pi$ amplitude of the t channel. If the extremely narrow resonance approximation is made, these terms are[11,20,21]

$$\begin{bmatrix} h_{11}^{\pi\pi}(s) \\ h_{33}^{\pi\pi}(s)\end{bmatrix} = \begin{bmatrix} 2 \\ -1\end{bmatrix}\frac{3}{\pi}\left(\frac{s}{s-(1-\mu)^2}\right)^{1/2}\frac{s}{[s-(1+\mu)^2]^2[s-(1-\mu)^2]}$$
$$\times[(\sqrt{s}+1)^2-\mu^2]L_1(s)Q_1[1+(t_R/2q_s^2)]+\{[\sqrt{s}-1]^2-\mu^2\}L_2(s)\begin{Bmatrix} Q_0\left(1+\dfrac{t_R}{2q_s^2}\right) \\ Q_2\left(1+\dfrac{t_R}{2q_s^2}\right)\end{Bmatrix}, \quad \text{(A1)}$$

$$L_1(s)=C_2\left(s+\frac{t_R}{2}-1-\mu^2\right)-(\sqrt{s}-1)(C_1+2C_2)\mu^2, \quad \text{(A2)}$$

$$L_2(s)=-C_2\left(s+\frac{t_R}{2}-1-\mu^2\right)-(\sqrt{s}+1)(C_1+2C_2)\mu^2, \quad \text{(A3)}$$

$$Q_l(z)=-\frac{1}{2}\int_{-1}^{1}\frac{P_l(y)}{y-z}dy, \quad \text{(A4)}$$

$[C_1=-4.14,\quad C_2=-1.12,\quad t_R=30\mu^2$.

The singularities of $h_{11}^{\pi\pi}(s)$ and $h_{33}^{\pi\pi}(s)$ are indicated in Fig. 10. It is well known that the discontinuities across the cut along the real axis ($-\infty<s\leqslant 0$) and all but a small segment of the circular cut (labeled ρ in the figure) cannot be calculated from (A1). This is due to the failure of the Legendre expansion of the t-channel amplitude to converge in the region of the singularities. These cuts associated with noncalculable discontinuities are labeled C in the figure and it is assumed that Balázs-pole terms in (2.18) will account for their effect.

The discontinuity across the cut $[0<s<(1-\mu)^2]$ is due to our choice of kinematic factor in the definition of h_{2I2J}. Note, however, that this discontinuity is only part of the total discontinuity of h_{2I2J} given *exactly* by (3.2).

Thus, the effect, on our analysis of the π-π interaction in the t channel, is to give an extra term to the N function.

$$\Delta N_{2I2J}^{\pi\pi}(s)=\frac{1}{\pi}\int_{\rho\ \text{cut}}ds'\frac{\text{disc}[D_{2I2J}(s')h_{2I2J}^{\pi\pi}(s')]}{s'-s}, \quad \text{(A5)}$$

where disc stands for discontinuity. It is easily seen from (A1) that, for the D functions of this paper, $D_{2I2J}h_{2I2J}^{\pi\pi}$ satisfies a no-subtraction dispersion rela-

[20] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603, 1609 (1960).

[21] The values for C_1 and C_2 are derived from a fit to the nucleon electromagnetic form factors [R. Hofstadter and R. Herman, Phys. Rev. Letters **6**, 293 (1961)] using forms (79) and (80) of J. Hamilton *et al.*, Phys. Rev. **128**, 1881 (1962). A full width at half-maximum of 112 MeV was assumed for the ρ resonance.

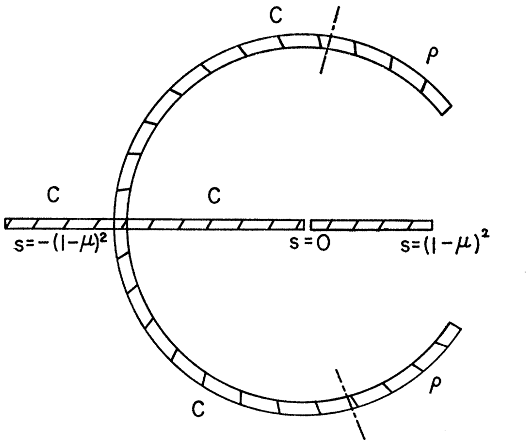

FIG. 10. Singularities of $h_{2I2J}(s)$ due to ρ-resonance contribution to the t-channel amplitude.

tion. Thus,

$$\Delta N_{2I2J}{}^{\pi\pi}(s) = D_{2I2J}(s)h_{2I2J}{}^{\pi\pi}(s) - \frac{1}{\pi} \times \int_C ds' \frac{\mathrm{disc}[D_{2I2J}(s')h_{2I2J}{}^{\pi\pi}(s')]}{s'-s} - \frac{1}{\pi}\int_0^{(1-\mu)^2} ds' \frac{D_{2I2J}(s')\,\mathrm{Im}h_{2I2J}{}^{\pi\pi}(s')}{s'-s}. \quad (A6)$$

The integral over C in (A6) may, as previously discussed, be thought of as already contained in the Balázs pole contribution to N_{2I2J} and consequently dropped.

TABLE II. $\mathrm{Im}h_{11}{}^{\pi\pi}(s)$ and $\mathrm{Im}h_{11}(s)$ as given by the crossing relation (3.2) and experimental data (Ref. 17).

s	$\mathrm{Im}h_{11}{}^{\pi\pi}(s)$	$\mathrm{Im}h_{11}(s)$
0.70	0.01	0.11
0.62	0.04	2.40
0.54	0.07	9.09
0.46	0.09	4.90

We now take, as a measure of the importance of $\Delta N_{2I2J}{}^{\pi\pi}$ to our analysis, its value along the cut $[0<s\leq(1-\mu)^2]$. In this region, D_{2I2J} is real and $\mathrm{Re}h_{2I2J}{}^{\pi\pi}=0$. Since $\Delta N_{2I2J}{}^{\pi\pi}$ must be real here, we have the "effective" relation

$$\Delta N_{2I2J}{}^{\pi\pi}(s) = -\frac{1}{\pi}P\int_0^{(1-\mu)^2} ds' \frac{D_{2I2J}(s')\,\mathrm{Im}h_{2I2J}{}^{\pi\pi}(s')}{s'-s}. \quad (A7)$$

In Table II, we compare $\mathrm{Im}h_{11}{}^{\pi\pi}(s)$ on the nearby left-hand cut with $\mathrm{Im}h_{11}(s)$ as given by the crossing relation (3.2). We see that it is a rather good approximation to drop the ρ-meson contribution.

APPENDIX B: BEHAVIOR OF $h_{l\pm}{}^I(s)$ IN THE NEIGHBORHOOD OF $s=0$

The behavior of $h_{l\pm}{}^I(s)$ in the neighborhood of $s=0$ may be investigated by means of the defining relations (2.11), (2.12), and (2.17).

$$h_{l\pm}{}^I(s) = \left(\frac{s}{s-(1-\mu)^2}\right)^{1/2} \frac{s}{s-(1+\mu)^2}\frac{1}{16\pi\sqrt{s}} \times\{(E(s)+1)[A_l{}^I(s)+(\sqrt{s}-1)B_{l\pm 1}{}^I(s)] + (E(s)-1)[-A_l{}^I(s) + (\sqrt{s}+1)B_{l\pm 1}{}^I(s)]\}, \quad (B1)$$

$$\begin{bmatrix} A_l{}^I(s) \\ B_l{}^I(s) \end{bmatrix} = \frac{1}{2}\int_{-1}^{1} d\cos\theta_s P_l(\cos\theta_s) \begin{bmatrix} A^I(s,u) \\ B^I(s,u) \end{bmatrix}. \quad (B2)$$

If we change the integration variable to u according to

$$\cos\theta_s = \frac{(1-\mu^2)^2 - s(s+2u-2-2\mu^2)}{[(1-\mu^2)^2+s(s-2-2\mu^2)]} \quad (B3)$$

we have

$$A^I(s) = \frac{-s}{(1-\mu^2)^2+s(s-2-2\mu^2)} \times \int_{(1-\mu^2)^2/s}^{2(1+\mu^2)-s} du\, P_l(\cos\theta_s[s,u])A^I(s,u) \quad (B4)$$

and a similar expression for $B_l(s)$.

Now if we assume that

$$|A(s,u)| \sim \beta(s)u^\alpha \qquad \beta(0)\neq 0, \quad (B5)$$

and similarly for $B_l(s)$, we find

$$A^I(s),\ B^I(s) \sim \cdots (1/s)^\alpha \qquad s\to 0, \quad (B6)$$

and consequently

$$h_{l\pm}{}^I(s) \sim \cdots s^{1/2-\alpha} \qquad s=0, \quad (B7)$$

so that $h_{l\pm}{}^I(s)$ will not vanish at $s=0$ provided $\alpha>\frac{1}{2}$, as stated in Sec. IIB.