Conformal Group in Space-Time*

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Representations of the conformal group and their physical interpretations are discussed in order to provide a basis for interesting applications of this group in particle physics. The conformal group seems to be of particular interest in connection with the ultraviolet singularities in field theory, for it is associated with test functions which vanish on the light cone and may therefore provide an appropriate regularization of Green's functions in coordinate space. In momentum space this property corresponds to an indefinite metric in Hilbert space with the transformation by reciprocal radii as the metric operator. Two examples of representations are investigated in detail; the first one belongs to the case of spin zero and mass zero, and the second one describes a system with spin zero but nonvanishing mass. Possible physical applications of these representations are illustrated by a simple measuring process which determines the value of the electron mass.

I. INTRODUCTION

IN a recent paper[1] a new quantization procedure for relativistic field theories at very high energies was proposed, based on a new type of representation of the 15-parameter conformal group. It is the purpose of this paper to discuss in more detail some of the underlying physical and mathematical concepts.

The main idea is that we assume the conformal group to become physically important at very high energies. Since this group contains the dilatations, it implies continuous mass values. Thus it cannot be a good symmetry group in the usual sense for particle physics at low energies, for the atomic mass spectrum is discontinuous. But at very high energies, when the rest masses are negligible, conformal invariance may become important.

The hypothesis that the 15-parameter conformal group is the physically interesting generalization of the Poincaré group at very high energies is to a certain extent the counterpart to the well-known nonrelativistic limit of the Poincaré group, where the Lorentz group is replaced by the Galilei group.[2] Both limits can be characterized by an approximation of the basic relation[3]

$$E^2=\mathbf{p}^2+m^2,$$

between energy E, momentum $\mathbf{p}$, and mass m of a free particle. In the nonrelativistic limit we have

$$E=m(1+\mathbf{p}^2/m^2)^{1/2}\approx m+\mathbf{p}^2/2m,$$

and in the extreme relativistic limit we get

$$E=|\mathbf{p}|(1+m^2/\mathbf{p}^2)^{1/2}\approx|\mathbf{p}|.$$

Since m is a constant we have to consider the second term in the nonrelativistic expansion in order to obtain a nontrivial function $E(\mathbf{p})$.

The 15-parameter conformal group contains in addition to the Poincaré group the 1-parameter group of the dilatations

$$x^\mu\to x^{\mu\prime}=\rho x^\mu,\quad \rho>0,\quad \mu=0,1,2,3, \tag{1}$$

and the 4-parameter special conformal group

$$x^\mu\to x^{\mu\prime}=RT(c)Rx^\mu=(x^\mu-c^\mu x^2)/(1-2c\cdot x+c^2x^2), \tag{2}$$

where

$$Rx^\mu=-x^\mu/x^2 \quad\text{and}\quad T(c)x^\mu=x^\mu+c^\mu.$$

In a previous paper[4] we discussed as possible consequences of the dilatations the asymptotic behavior of scattering quantities at very high energies. This approach is mainly phenomenological, but has nevertheless interesting experimental implications. We hope to give a detailed comparison of those consequences with experimental data in the near future.

Another important question is, how the conformal group affects the structure of current quantum field theory, based on the Poincaré group. This question is rather involved and we are far from having a satisfactory answer, but we can mention some interesting features and we shall analyze a few of them in this paper.

The mathematical structure of current quantum field theory can be considered determined basically by the concept of operator-valued distributions combined with unitary representations of the Poincaré group.[5,6] The spectral representations involved are usually closely related to the spectrum of the generators P_μ of the translation group. Since these operators are unbounded, their eigenfunctions generally do not belong to the Hilbert space, but one can consider these eigenfunctions and their derivatives as linear functionals on the space S of test functions in the theory of distributions.[7,8]

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[1] H. A. Kastrup, Phys. Rev. **140**, B183 (1965).
[2] H. A. Kastrup, Lawrence Radiation Laboratory Report UCLR-11998, 1965 (unpublished).
[3] We use $\hbar=c=1$ and the metric $x^2=(x^0)^2-\mathbf{x}^2$.
[4] H. A. Kastrup, Nucl. Phys. **58**, 561 (1964).
[5] R. Haag and B. Schroer, J. Math. Phys. **3**, 248 (1962).
[6] A. S. Wightman, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963).
[7] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. I.
[8] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. IV, Chap. I.

The only known version of this concept of quantum field theory which leads to numerical results, is the canonical Lagrange formalism, combined with perturbation theory. But this formalism has the well-known fundamental ultraviolet divergencies.[9] These divergencies are immediately connected with the singularities on the light cone of the propagator Δ_F of, for instance, a scalar field with mass m. Near the light cone this propagator has the form[9]

$$\Delta_F(x)\approx\frac{1}{4\pi}\delta(x^2)+\frac{1}{4\pi^2 ix^2}+\frac{im^2}{8\pi^2}\ln\frac{m|x^2|^{1/2}}{2}-\frac{m^2}{16\pi}\theta(x^2).$$

In order to get finite results in perturbation theory one has to subtract the leading singularities out, for example by the method of Pauli and Villars.[10]

Since the leading singularities of the propagator are on the light cone and mass independent, they belong to the realm of our high-energy approximation $E\approx|\mathbf{p}|$, and we have therefore the interesting question, in which way these singularities are affected by the structure of the conformal group and whether the generalization of the Poincaré group to the conformal group will even bring a solution of the ultraviolet difficulties. We shall not solve the ultraviolet problem in this paper, but we shall give indications that the structure of the conformal group strongly affects the canonical quantization of fields, and that this group might be the key to a solution of those difficulties.

The subgroup (2) of the 15-parameter conformal group is Abelian and isomorphic to the group of translations.[11] Its eigenfunctions which correspond to the plane waves, the eigenfunctions of the translations in coordinate space, are the functions

$$e^{-ih\cdot x/x^2},$$

where the four-vectors $h=(h_0,\mathbf{h})$ are the eigenvalues of the generators of the special conformal group (2). The derivatives of these eigenfunctions with respect to h have singularities on the light cone which become higher with every higher order of the derivatives. These functionals are therefore in general not defined on the class S of test functions. The class S was introduced[7] in order to have a class of test functions, for which all derivatives of the plane waves $e^{ip\cdot x}$ are defined. The corresponding class S_R of test functions $\psi(x)$ for the functionals $e^{-ih\cdot x/x^2}$ can be obtained by a mapping of the testfunctions $\varphi(x)$ of S. We define the $\psi(x)\in S_R$ by

$$\psi(x)=(x^2)^{-4}\varphi(Rx)\,,\quad \varphi\in S\,, \tag{3}$$

where R is the transformation by reciprocal radii given in Eq. (2). The factor $(x^2)^{-4}$ is convenient because the volume element d^4x is transformed into $(x^2)^{-4}d^4x$ under the transformation R.

Since the functions $\varphi(x)$ vanish at infinity faster than any power of $1/x^\mu$, the testfunctions $\psi(x)$ vanish faster than any power of x^2 on the light cone. Thus all derivatives of $e^{-ih\cdot x/x^2}$ with respect to h are defined on S_R and this class of test functions plays the same role for the functionals $e^{-ih\cdot x/x^2}$ as the class S for the plane waves.

From this example we see that the conformal group essentially affects the behavior of field quantities on the light cone. For instance, a delta function $\delta(x^2)$ on the light cone is equivalent to zero with respect to the class S_R, for the test functions $\psi(x)$ vanish on its support. We shall give a detailed discussion of the functions $e^{-ih\cdot x/x^2}$ and their relation to field theory elsewhere and shall deal in this paper mainly with the structure of representations of the conformal group in momentum space.

In renormalization theory the Pauli-Villars regularization, for instance, is tantamount to an indefinite metric in Hilbert space. In the case of the conformal group such an indefinite metric appears in the following way[1,11]:

If P_1 is the usual self-adjoint generator of the translations in the x^1 direction and $e^{i\alpha D}$, $\rho=e^\alpha$ the operator which represents the dilatations (1) in the same linear space, then we have the commutation relation[11]

$$e^{i\alpha D}P_1e^{-i\alpha D}=e^{-\alpha}P_1\,. \tag{4}$$

It follows from Eq. (4) that $e^{-i\alpha D}$ transforms an eigenvector $|p_1\rangle$ of P_1 into an eigenvector $|e^{-\alpha}p_1\rangle$. This is the physical meaning of a dilatation. If we start from a single eigenvalue $p_1^{(0)}\neq 0$ and its eigenvector $|p_1^{(0)}\rangle$, we can construct the whole set of eigenvectors $|p_1\rangle$, where p_1 has the same sign as $p_1^{(0)}$, by applying all possible transformations $e^{-i\alpha D}$ to $|p_1^{(0)}\rangle$.

Since the dilatations form a noncompact group, the operator D is unbounded, and its eigenfunctions in general do not belong to the Hilbert space and the spectrum is continuous when we are dealing with a unitary representation of the conformal group. This question is analyzed in detail in Sec. IV. At first glance the situation seems to be completely analogous to the case of the translations, namely, that D has a continuous real eigenvalue spectrum. But the crucial difference can be seen from the following: If $|s\rangle$ is an (improper!) eigenvector of $e^{i\alpha D}$ with the property

$$e^{i\alpha D}|s\rangle=e^{i\alpha s}|s\rangle\,,\quad |e^{i\alpha s}|=1\,,$$

then it follows from the commutation relation (3) that

$$e^{i\alpha D}P_1|s\rangle=e^{\alpha(is-1)}P_1|s\rangle\,.$$

This means that $P_1|s\rangle$ is also an eigenvector of $e^{i\alpha D}$, but the absolute value of the eigenvalue is no longer one.

The conventional framework of unitary representations obviously does not cope with this new situation, and it was pointed out in Ref. 1 that this peculiar

[9] See, for instance, N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience Publishers, Inc., New York, 1959).

[10] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949); see also Ref. 9.

[11] H. A. Kastrup, Ann. Physik **9**, 388 (1962). This paper contains references to earlier work on the conformal group.

feature can be understood immediately, if we consider the indefinite metric $\langle\phi_1|\phi_2\rangle=(\phi_1,R\phi_2)$ instead of (ϕ_1,ϕ_2), where R is the operator which represents the transformation by reciprocal radii.

We want to emphasize that we do not maintain that the unitary representations of the conformal group are inconsistent, but rather that the properties of the eigenvalues of D can be better understood within this new framework, if the corresponding eigenfunctions are generated by the elements of the Lie algebra itself.

We shall see in Sec. IV that the application of the operators P_μ to the improper eigenfunctions $|s\rangle$ leads to mathematically reasonable results. The point is that this procedure goes beyond the usual notion of a Hilbert space.

The appearance of the transformation R as metric operator in Hilbert space apparently corresponds to the mapping (3) of the test function class S. We do not elaborate on the details of this correspondence in this paper.

Several years ago,[12] Heisenberg supposed that some kind of oscillations of the field operators on the light cone might lead to a regularizing indefinite metric in Hilbert space. Now the functions $e^{-ih\cdot x/x^2}$ have such oscillations on the light cone and it was shown in Ref. 1 how this behavior implies an indefinite metric in Hilbert space. This result confirms Heisenberg's conjecture.

In Ref. 1 it was also shown how the metric R can lead to a less singular quantization, at least in the limit $E=|\mathbf{p}|$.

The main problem of this new formalism is, of course, its physical interpretation. One can distinguish two crucial questions in this context which are closely related. The first one is the question of how the conformal group can be of significance in the description of physical systems and for the structure of their dynamical laws. This question is tantamount to the problem of determining where continuous eigenvalues of physical quantities occur, since the dilatations become significant in the case of such continuous eigenvalues.[4] We examine this question in detail in Sec. II, with the measurement of the electron mass as an example. A careful analysis of the measuring process seems to be crucial for the physical interpretation of the conformal group.

The second problem is the probability interpretation of the indefinite metric and its physically consistent incorporation into field theory. We do not deal with this question here since we think it has to be considered in the context of a special physical example. In order to prepare the ground for such an example we investigate some interesting properties of the conformal group in Secs. III–V. In Sec. III we consider some general features of the representations of the conformal group and their possible physical application, in Sec. IV we investigate the special example of a system with both mass and spin equal to zero, and in Sec. V we consider a system with spin zero but nonvanishing mass.

[12] W. Heisenberg, in *Proceedings of the International Conference on High-Energy Physics* (CERN, Geneva, 1958), p. 119.

II. DILATATION INVARIANCE IN THE CONTEXT OF AN ACTUAL MEASUREMENT

Several aspects of the physical significance of the dilatations have been discussed in Refs. 4 and 10, and independently by Maris in a different context.[13] Numerous divergent remarks on the physical interpretation of the conformal group are contained in the earlier literature, an extensive list of which is given in Ref. 11. Instead of invoking or repeating general arguments let us consider a special example: What does it mean if we say the electron has a rest mass of about 9.1×10^{-28} g, or equivalent, a Compton wavelength of about 2.4×10^{-10} cm?

First we observe that it is not possible to give the value of the electron mass without mentioning the standard unit to which we refer. In the first case the standard unit was the gram, in the second case, the centimeter. Both kinds of units are physically equivalent since we know by actual measurements how they are related. If we change the standard unit e by a numerical factor f, $e\rightarrow e'=fe$, for instance, 1 cm $\rightarrow$ 1m $=100$ cm, then the numerical value of the electron mass with respect to the new unit e' is obtained by multiplying the old numerical by a factor f^n, where n is a real number called the dimension of the mass with respect to the considered kind of units.

It is generally assumed that there are no principally privileged standard units as far as macroscopic objects are concerned. One can measure in m, cm, in., mile, and so forth. This means there is a continuous manifold of physically realizable lengths; in other words, the factor f can be an arbitrary positive number as far as macroscopic physics is concerned.

Mathematically, the set of all multiplications by a factor f forms a 1-parameter Abelian group, the dilatations. If a certain quantity is multiplied by f^n then this is a representation of the dilatations. The representation is characterized by the dimension n.

A further question is whether it is significant which kind of quantities we use as reference system in order to express mass values. In other words, can we use g as well as cm^{-1} or m^{-1}? There is a definite answer to this question.[11] If we identify the generators P_μ of the translations with the energy momentum operators of a physical system, then energy, momentum, and mass must have the dimension of length $n=-1$. This follows from the commutation relations between the translations and the dilatations, a special example of which is given in Eq. (4). This is tantamount to using velocity and action as the two other independent kinds of units. In order not to complicate our following considerations we chose as their units the velocity of light and Planck's constant divided by 2π.

[13] Th. A. J. Maris, Nuovo Cimento **30**, 378 (1963).

Now let us consider how the value of the electron mass is determined. Since the measurement of an atomic quantity always implies some sort of interaction with other objects and since the electron can interact via weak, electromagnetic, and gravitational forces, one has to take all these interactions into account if one wants to have a complete discussion. Since the weak and gravitational forces are comparatively small, we confine ourselves to the electromagnetic interactions. Here we can conceive of two possibilities of measuring the electron mass.

The first one is the measurement of the electron mass by means of its interaction with other *atomic* particles only. This problem is closely related to the question of measuring field quantities in the framework of quantum field theory. In their fundamental papers on this subject Bohr and Rosenfeld[14] concluded that one has to employ *macroscopic* test bodies in order to measure microscopic field quantities. Although their results depend on the canonical field quantization, they seem to be quite plausible independent of this special formalism, for in one way or another an atomic particle has to interact with our macroscopic world, in order to make itself visible. At the moment we do not know to what extent the answer to this question depends on the use of a particular quantization procedure.

Finally we arrive at the most common way of measuring the mass of charged atomic particles by their interaction with macroscopic electric and magnetic fields. In order to analyze such a measuring process for the electron, we consider a simple apparatus, which demonstrates the essential features.

An electron beam of constant velocity $\beta_x=\beta$ moves in the x direction through a condenser of length l (Fig. 1). In the condenser we have a constant homogeneous field F in the z direction. The deflection a of the beam at the right end of the condenser is determined by the Lorentz force

$$(d/dt)(m\beta_z)=qF\,,$$

where m is the mass of the electron, q is its charge and β_z is its velocity in the z direction imparted by the electric field F. Since $\beta_z=tg\phi$, $tg\phi\approx a/l$, we get in the nonrelativistic case

$$m=(ql^2F/\beta^2)a^{-1}. \tag{5}$$

We assume that the charge q was determined independently, for instance by Millikan's experiment (which is a nice example of measuring atomic quantities by means of gravitational interactions). The velocity β can be measured by applying a homogeneous magnetic field B in the y direction of such a magnitude that the beam deflection a in the combined electric and magnetic fields vanishes. The Lorentz force implies that this is the case if

$$B=-\beta E\,.$$

[14] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **12**, No. 8 (1933); Phys. Rev. **78**, 794 (1950).

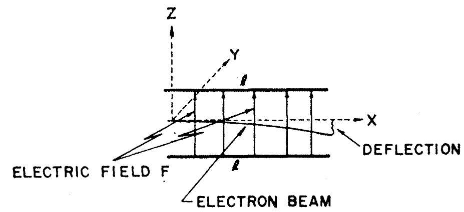

FIG. 1. Measuring of the electron mass by macroscopic electromagnetic fields.

So far we have not fixed the units of the electric and magnetic fields. For this purpose we assume the length l of the condenser to be our unit of length. To make this clear we write for all concerned quantities A

$$A=N(A)\hat{l}^n\,,$$

where n characterizes the dimension of length of the quantity A and $N(A)$ its numerical value expressed in terms of l. We therefore have

$$l=1\times\hat{l}\,,\quad N(l)=1\,,$$

and because of our system of units

$$F=N(F)\hat{l}^{-2}\,,\quad B=N(B)\hat{l}^{-2}.$$

We therefore have to calibrate the electric instruments in units $\hat{l}^{-2}$. In this system q and β are represented by some numbers

$$N(q)\,,\quad N(\beta)\,,$$

so that we finally get the equation

$$m=N(m)\hat{l}^{-1}=[N(q)N(F)/N(\beta^2)]N^{-1}(a)\hat{l}^{-1}.$$

For given $N(q)$, $N(\beta^2)$, and $N(F)$ the numerical value $N(m)$ of the mass m is given by the numerical value $N(a)$ of the deflection a.

If we now consider a beam of all possible atomic particles with charge q—not only electrons—which all have the same velocity but different masses, then we can distinguish the different masses by their different deflections $N(a_i)$. What we are particularly interested in here is the result that the set of all $N(a_i)$ is discontinuous. This is exactly what is meant if one says that the atomic mass spectrum is discontinuous.

The relation of this situation to the dilatations is the following: let us replace our measuring device by a similar one in such a way that the new condenser has the length

$$l'=\rho l=N(l')\hat{l}\,,$$

where ρ is an arbitrary number >0. The other macroscopic quantities are changed according to their dimension of length:

$$F'=\rho^{-2}F=N(F')\hat{l}^{-2}\,,\quad B'=\rho^{-2}B=N(B')\hat{l}^{-2}.$$

We still keep $\hat{l}$ as our standard length. The construction of such a similar apparatus implies that our macroscopic world is in principle invariant under dilatations.

Since the macroscopic bodies consist of atomic particles with discontinuous masses, this assumption is only approximately true. But because of the smallness of the atomic masses in comparison to the macroscopic ones, this approximation is extremely good.

Although we can construct an apparatus which is similar to the original one, we do not have an electron with mass m', the value $N(m')$ of which differs from the original one by the relation

$$N(m')=\rho^{-1}N(m).$$

In this sense atomic systems are not invariant under dilatations. Nevertheless, we can measure the original electron mass by our new apparatus, now calibrated in units of $\hat{l}'$. We then get

$$m=N'(m)(\hat{l}')^{-1}=N(m)\hat{l}^{-1}.$$

In this sense the numerical values of the electron mass can be subject to a scale transformation, but one has to keep in mind that this is a consequence of the *macroscopic* invariance of the measuring apparatus under dilatations.

These last considerations show that the numerical value of the electron mass is completely arbitrary as long as we have not fixed a system of reference numerically. This means that in such a situation the electron has to be described in a dilatation invariant way if we want to describe it adequately. A description of that kind has to allow for all possible numerical mass values, a specific one of which is picked out if we perform a measurement, which introduces a certain unit of length. In mathematical terms this dilatational invariant description means that the linear space of the states which describe such a system contains states with all positive mass values. If such a space is associated with unitary representations of the Poincaré group, then these representations are reducible, for the irreducible representations of this group are characterized by one single mass value.[15] The measuring process reduces this manifold of representations and picks out one irreducible representation.

This situation bears some similarity to the reduction of a wave packet in the interpretation of quantum mechanics,[16] for in that case the measuring process also selects a certain state out of a linear manifold of possible states. The essential difference is that in the latter case different states describe different microscopic physical situations, whereas in our example the scale of the macroscopic apparatus is the varying quantity. We give a mathematical example of such a set of representations in Sec. V.

We have already mentioned in the introduction that even atomic systems seem to become invariant under dilatations if we go to extremely high energies,[4] for the discontinuous masses are negligible in that case. We would like to illustrate this a bit more by means of Eq. (5). In the relativistic case we have to replace it by

$$\mathbf{p}^2/E=qFl^2a^{-1},$$

and for $E\gg m$ we have

$$|\mathbf{p}|=qFl^2a^{-1}.$$

This last equation is quite different from Eq. (5) as far as the dilatations are concerned; for we can physically realize, at least in principle, every high momentum of the electron and the deflections $N(a)$ now form a continuous set. In this approximation not only the macroscopic but also the atomic world is dilatation invariant.

III. REPRESENTATIONS OF THE CONFORMAL GROUP AND THEIR PHYSICAL APPLICATIONS

The unitary representations of the conformal group are already described in the literature.[17–19] In order to get more physical insight into their structure we shall start from the unitary representations of the Poincaré group. The irreducible unitary representations of this group are characterized by the values of the spin S and the squared mass P^2 of the system considered.[15] Essential for the representations of the conformal group is the commutation relation of P^2 with the dilatations:

$$e^{i\alpha D}P^2e^{-i\alpha D}=e^{-2\alpha}P^2. \tag{6}$$

This relation has the following consequences:

If $P^2=0$, then we can extend the irreducible representations of the Poincaré group to irreducible representations of the conformal group.[19] This is usually meant if one says, for instance, that Maxwell's equations or the neutrino equations are conformal invariant. We shall discuss some mathematical details of the example $P^2=0$, $S=0$ in the next section, particularly the features, which are connected with the indefinite metric of Ref. 1.

These representations with $P^2=0$ are of physical interest in several aspects: First, they describe physical particles with zero rest mass. This has been the main subject explored in the literature concerning the conformal group.

Second, one can consider these representations as approximate descriptions for systems of particles with such extremely high energies that their rest masses are negligible. From this one can derive some new results from the asymptotic behavior of physical quantities such as cross sections at very high energies,[4] and one can also analyze the consequences for the high-energy behavior of the Green's functions in field theory. In the case of the dilatations some interesting applications are

[15] E. P. Wigner, Ann. Math. 40, 149 (1939).
[16] See, for instance, W. Pauli, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, Göttingen, Heidelberg, 1958), Vol. 5/1, p. 69.
[17] Y. Murai, Progr. Theoret. Phys. (Kyoto) 9, 147 (1953).
[18] A. Esteve and P. G. Sona, Nuovo Cimento 32, 473 (1964).
[19] L. Gross, J. Math. Phys. 5, 687 (1964).

already discussed in the literature.[20–22] But, as already pointed out in the Introduction, in field theory the essential new properties seem to be connected with the special conformal transformations (2).

Third, if we assume the mass of the atomic particles to be a consequence of their interactions, described, for instance, through coupled fields which do not contain a kinematical mass term, contrary to the conventional Lagrange formalism, then we can couple fields which belong to the above massless representations and which are quantized in the canonical way, etc., or by the method of Ref. 1. An example is the nonlinear equation

$$\Box A(x)+gA^3(x)=0.$$

Without further assumptions the mass spectrum of such a theory would be continuous since the interaction term is dilatational invariant, too. One, therefore, has to introduce some symmetry-breaking mechanism as, for instance, an unsymmetric ground state, an external field or some other kind of perturbation. Such field theoretical models have been discussed by Maris,[13] Maris and Haag,[23] Johnson *et al.*,[24] and Dürr *et al.*[25] Since the generator of the dilatations is given in spin space by[11] γ_5, even such dynamical approaches as the work[26] of Nambu and Jona-Lasinio seem to make implicit use of the dilatations.

Conformal symmetry can perhaps provide also a deeper understanding of those theories, which introduce some features of theories for low-temperature physics such as superconductivity[26] and ferromagnetism[27] into elementary particle physics. The reason is that the asymptotic invariance under the conformal group at very high energies corresponds to the same invariance for nonrelativistic energies in the limiting case of vanishing kinetic energies. Indeed, the equation

$$\Delta\phi(\mathbf{x})+\mathbf{k}^2\phi(\mathbf{x})=0$$

becomes conformal invariant[28] for $\mathbf{k}\to 0$. In a very rough sense this vanishing energy is tantamount to the vanishing of the absolute temperature. Since this is often accompanied by some new phenomena such as superconductivity, superfluidity, etc., it may be that the limiting conformal invariance can shed some new light on these properties.

If $P^2>0$, then Eq. (6) implies that all possible eigenvalues $p^2>0$ occur in the corresponding irreducible representation of the conformal group. The Hilbert space, for instance, of an irreducible representation of the conformal group for a particle with nonvanishing mass and spin zero is, therefore, a direct sum of all irreducible representations of the Poincaré group with spin zero and $m^2>0$.

Since the generators K_μ of the special conformal group are given by[1]

$$K_\mu=RP_\mu R,$$

where R represents the transformation by reciprocal radii, defined in Eq. (2), the spectrum of these operators is identical with that of the generators P_μ; they require nothing new. The mathematical details of this example are given in Sec. V.

IV. THE CASE $m=0$, $S=0$

This example was already discussed in Ref. 1. We would like to give some more details here.

We start from the assumption that the representation of each 1-parameter subgroup of that part of the conformal group which is continuously connected with the unity can be written in the form

$$e^{i\beta A},$$

where β is the group parameter. If one considers unitary representations, then according to Stone's theorem[29] the generators A are self-adjoint. If $M_{\mu\nu}$, P_μ, K_μ, and D are the generators of the orthochronous Lorentz group, the translations, the special conformal group (2), and the dilatations (1), respectively, they form the Lie algebra[11]

$$[M_{\kappa\lambda},M_{\mu\nu}]=i(g_{\lambda\mu}M_{\kappa\nu}-g_{\kappa\mu}M_{\lambda\nu}+g_{\kappa\nu}M_{\lambda\mu}-g_{\lambda\nu}M_{\kappa\mu}), \quad (7a)$$

$$[P_\lambda,M_{\mu\nu}]=i(g_{\lambda\mu}P_\nu-g_{\lambda\nu}P_\mu), \quad (7b)$$

$$[K_\lambda,M_{\mu\nu}]=i(g_{\lambda\mu}K_\nu-g_{\lambda\mu}K_\mu), \quad (7c)$$

$$[K_\mu,P_\nu]=2i(g_{\mu\nu}D-M_{\mu\nu}), \quad (7d)$$

$$[M_{\mu\nu},D]=0, \quad (7e)$$

$$[D,P_\mu]=iP_\mu, \quad (7f)$$

$$[D,K_\mu]=-iK_\mu, \quad (7g)$$

$$[P_\mu,P_\nu]=0, \quad (7h)$$

$$[K_\mu,K_\nu]=0. \quad (7i)$$

Under the transformation by reciprocal radii R the generators are transformed this way[1]:

$$RM_{\mu\nu}R=M_{\mu\nu}, \quad (8a)$$

$$RDR=-D, \quad (8b)$$

$$RP_\mu R=K_\mu, \quad (8c)$$

$$RK_\mu R=P_\mu, \quad (8d)$$

[20] D. Amati, S. Fubini, and A. Stanghellini, Nuovo Cimento 26, 896 (1962).
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[27] H. P. Dürr and W. Heisenberg, Z. Naturforsch. 16a, 726 (1961).
[28] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949), p. 140.
[29] F. Riesz and B. Sz.-Nagy, *Vorlesungen über Funktionalanalysis*, (VEB Deutscher Verlag der Wissenschafter, Berlin, 1956), p. 363.

Because of relations (7d) and (8c) one can generate the complete Lie algebra, if the operators P_μ and R are given.

In the momentum space of the Klein-Gordon equation without rest mass the above generators have the form:

$$M_{jk}=i(p_j\partial_k-p_k\partial_j)\,,\quad \partial_j=\frac{\partial}{\partial p_j}\,,\quad j,k=1,2,3\,, \tag{9a}$$

$$M_{0j}=ip_0\partial_j\,,\quad j=1,2,3\,, \tag{9b}$$

$$P_\mu=p_\mu\,,\quad p_0=(\mathbf{p}^2)^{1/2}\,, \tag{9c}$$

$$K_0=-p_0\Delta\,,\quad \Delta=\partial_j\partial_j\,,\quad K_j=-2\partial_j-2p^k\partial_k\partial_j-p_j\Delta\,,\quad j=1,2,3\,, \tag{9d}$$

$$D=i(p^j\partial_j+1)\,, \tag{9e}$$

In the case of the operators which belong to the Poincaré group this is well known.[30] For the rest it can be seen from the transformation properties of the solution

$$f(x)=1/(2\pi)^{3/2}\int\frac{d^3p}{2p_0}\phi(p)e^{ip\cdot x}\,,\quad p_0=(\mathbf{p}^2)^{1/2}\,,$$

under the generators D and K_μ in coordinate space, where they have the form[11]

$$D=(1/i)(x^\mu\partial_\mu+1)\,,\quad K_\mu=(1/i)(2x_\mu+2x_\mu x^\nu\partial_\nu-x^2\partial_\mu)\,.$$

R is an integral operator in momentum space and in Ref. 1 we obtained as its matrix elements

$$R(p,h)=(e_p,Re_h)=(1/2\pi)J_0[(2p\cdot h)^{1/2}]\,, \tag{10}$$

where e_p, e_h are eigenfunctions of the momentum operators.

From Eq. (8c) it follows that Re_h is an eigenfunction of the operators K_μ with eigenvalues h_μ. That means that the quantities $R(p,h)$ are the eigenfunctions of the operators K_μ in momentum representation. This can be verified in our special example by applying the operators (9d) to the function (10). This general feature provides an important practical method of determining the matrix elements of R in any representation of the conformal group in momentum space.

The above operators are selfadjoint with respect to the scalar product

$$(\phi_1,\phi_2)=\int\frac{d^3p}{2p_0}\phi_1{}^*(p)\phi_2(p)\,. \tag{11}$$

Next we want to determine the eigenvalues and eigenfunctions of the dilatations. For the special conformal group we know them already. In order to determine the eigenfunctions of the operator (9e) we first assume that we have a unitary representation of the conformal group with the scalar product (11). Since the dilatations form a noncompact group the operator D is always an unbounded operator in Hilbert space and its spectrum is continuous. This means that its eigenfunctions do not belong to the Hilbert space. In this way the situation is analogous to that of the translation group.

[30] H. Joos, Fortschr. Physik **10**, 65 (1962).

By introducing spherical coordinates in momentum space we find as eigenfunctions $u(p)$ of

$$D=i(p^j\partial_j+1)=i(p_0(\partial/\partial p_0)+1)\,,$$

with real eigenvalues s, the homogeneous functions

$$u_{slm}(\mathbf{p})=p_0{}^{-(is+1)}Y_{lm}(\mathbf{p}/p_0)\,. \tag{12}$$

An expansion

$$f(\mathbf{p})=\sum_{l,m}\int_{-\infty}^{+\infty}ds\,g_{lm}(s)p_0{}^{-(is+1)}Y_{lm}(\mathbf{p}/p_0)$$

of a function $f(\mathbf{p})$ in terms of these eigenfunctions is essentially the Mellin transformation.[31] With the scalar product (11) the eigenfunctions $u(p)$ have the improper norm

$$(u_1,u_2)=\pi\delta(s_1-s_2)\delta_{l_1l_2}\delta_{m_1,m_2}\,.$$

In our deliberations above we have assumed that the eigenvalues of the operator D are real, in analogy to the situation in the case of translations and according to the theory of Gel'fand,[8] which proves such reality properties even for continuous eigenvalues of self-adjoint operators.

But, as we have already pointed out in the Introduction, the new problem in this context is given by the commutation relations (7g) and (7f): If $|s\rangle$ is an eigenvector of D with the real eigenvalue s, then it follows from these relations that

$$DP_\mu|s\rangle=(s+i)P_\mu|s\rangle\,,\quad DK_\mu|s\rangle=(s-i)K_\mu|s\rangle\,.$$

The operators P_μ and K_μ therefore generate new eigenfunctions of D, the corresponding eigenvalues of which are no longer real.

This can be seen explicitly in our example: If we multiply the eigenfunctions $u(p)$ in Eq. (12) by $p_0{}^n$, $n=1,2,\cdots$, then we get eigenfunctions of D with eigenvalues $s+in$. If we start from $s=0$, then we have the new set

$$v_{nlm}(\mathbf{p})=p_0{}^{n-1}Y_{lm}(\mathbf{p}/p_0) \tag{13}$$

of eigenfunctions with pure imaginary and discontinuous eigenvalues. An expansion of a function $F(\mathbf{p})$ in terms of these eigenfunctions is the usual Taylor series. The same arguments apply to the relation (7g) with the only difference being that the operators K_μ lower the eigenvalues by i.

A well-known expansion in terms of the eigenfunctions (13) is that of the plane wave

$$e^{i\mathbf{p}\cdot\mathbf{x}}=\sum_{l=0}(2l+1)i^l(\pi/2p_0r)^{1/2}J_{l+1/2}(p_0r)P_l(\cos\gamma)\,, \tag{14}$$

[31] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I, p. 103.

where the Bessel function $J_{l+1/2}$ is given by a power series in $p_0 r$.

The above considerations indicate a surprising situation: We have a self-adjoint operator whose commutation relations with other self-adjoint operators lead to a pure imaginary point spectrum.

This is not a contradiction. Self-adjointness is usually defined with respect to a positive definite Hilbert space, whereas our above mathematical procedure goes beyond this framework.

It was pointed out in Ref. 1 that the puzzle can be solved if one introduces the indefinite metric

$$\langle\phi_1,\phi_2\rangle=(\phi_1,R\phi_2)\,, \tag{15}$$

for the operator D is skew Hermitian with respect to the scalar product (15) if it is Hermitian with respect to the scalar product (11). This is a consequence of relation (8b). Since skew Hermitian operators generally have imaginary eigenvalues, the above set of imaginary eigenvalues is no longer paradoxical.

Finally, we wish to discuss explicitly the transformation properties of the eigenfunctions (13), $n=0,\ \pm1,\ \pm2,\cdots$, under the transformation of reciprocal radii.

In Ref. 1, we obtained for $R(p,h)$ the following expansion in terms of spherical harmonics

$$R(p\cdot h)=(1/2\pi)(p_0h_0)^{-1/2}\times\sum_{l=0}^{\infty}(2l+1)J_{2l+1}[2(p_0h_0)^{1/2}]P_l(z)\,. \tag{16}$$

The basic formula for our purpose here is[32]

$$\int_0^\infty x^\mu J_{2l+1}(xy)dx=2^\mu y^{-\mu-1}\frac{\Gamma(l+1+\frac{1}{2}\mu)}{\Gamma(l+1-\frac{1}{2}\mu)}\,. \tag{17}$$

It holds for $-(2l+1)<\mathrm{Re}\,\mu<\frac{1}{2}$. But since the gamma function is a meromorphic function in the whole complex plane with simple poles on the negative real axes, we can define the left-hand side of Eq. (17) by analytic continuation of the right-hand side. The right-hand side is a meromorphic function of μ with simple poles at the points

$$\mu=-2(l+1+k)\,,\quad k=0,1,2,\cdots.$$

It follows from Eqs. (16) and (17) that

$$1/\Gamma(l+1+\nu)\int\frac{d^3p}{2p_0}R(p,h)p_0^{\nu-1}Y_{lm}(\mathbf{p}/p_0)=1/\Gamma(l+1-\nu)h_0^{-\nu-1}Y_{lm}(\mathbf{h}/h_0)\,, \tag{18}$$

where ν is a complex number. Equation (18) shows explicitly how the operator R transforms an eigenfunction of D with the eigenvalue $i\nu$ into an eigenfunction with the eigenvalue $-i\nu$. Since Eq. (17) is symmetric in ν, it is convenient to define the eigenfunctions of D as

$$v_{\nu lm}(\mathbf{p})=1/\Gamma(l+1+\nu)p_0^{\nu-1}Y_{lm}(\mathbf{p}/p_0)\,. \tag{19}$$

This definition comprises the functions (12) as well as (13). They are given by the special values $\nu=-is$ and $\nu=n$, respectively. Because of the poles of the gamma function the functions (19) are zero if ν is a negative integer $\leqslant -l-1$.

If we want to avoid these zeros, we can, for instance, start with the function

$$v_{3/2lm}=1/\Gamma(l+1+\tfrac{3}{2})p_0^{1/2}Y_{lm}(\mathbf{p}/p_0)$$

and then apply the operators P_0, K_0 to this function in order to obtain a set of eigenfunctions. Such a set is contained in the above expansion of the plane waves, for the Bessel function is given by

$$J_{l+1/2}(x)=(\tfrac{1}{2}x)^{l+1/2}\sum_{k=0}^{\infty}(-1)^k[k!\Gamma(l+k+\tfrac{3}{2})]^{-1}(\tfrac{1}{2}x)^{2k}.$$

The scalar product

$$\langle v_1|v_2\rangle=(v_{\nu_1l_1m_1},Rv_{\nu_2l_2m_2})$$

of two eigenfunctions of the form (19) has the value

$$\langle v_1|v_2\rangle=\delta_{l_1l_2}\delta_{m_1m_2}\times[\Gamma(l+1+\nu_1)\Gamma(l+1-\nu_2)]^{-1}\int_0^\infty dp_0p_0^{\nu_1-\nu_2-1}\,,$$

where the integral diverges. This has to be expected since the dilatations form a noncompact group. But in analogy to the eigenfunctions of the translations we can consider the functions

$$p_0^\lambda,\quad \lambda=\nu_1-\nu_2-1\,,$$

as linear functionals on the space S of test functions ϕ in the theory of distributions. These functionals have been studied in detail by Gel'fand and Shapiro.[33] Their main result is the following: The functional $p_0^\lambda(\phi)$, $\phi\in S$, is a meromorphic function in the whole λ plane with simple poles at the points $\lambda=-k$, $k=1,2,\cdots$. The poles have the residuum

$$\phi^{(k-1)}(0)/\Gamma(k)\,,$$

where $\phi^{(k-1)}(0)$ is the $(k-1)$th derivative of the testfunction $\phi(p_0)$ at the point $p_0=0$. We, therefore have the following properties of the functional

$$[\Gamma(l+1+\nu_1)\Gamma(l+1-\nu_2)]^{-1}p_0^{\nu_1-\nu_2-1}(\phi),$$

if ν_1 and ν_2 are real integers: It has simple poles for $\nu_1-\nu_2=-k$, $k=0,1,2,\cdots$, if $l+1+\nu_1$ and $l+1-\nu_2$ are not equal to $0,-1,-2,\cdots$. If the last is the case for

[32] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 49.

[33] I. M. Gel'fand and Z. Ya. Shapiro, Uspekhi Mat. Nauk **10**, 3 (1955); Am. Math. Soc. Translations, Ser. 2, **8**, 21 (1958); see also Ref. 7.

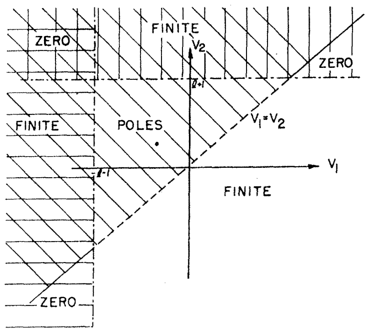

FIG. 2. Properties of the functional $[1/\Gamma(l+1+\nu_1)] \times[1/\Gamma(l+1-\nu_2)] \times p_0^{\nu_1-\nu_2-1}(\phi)$ for integer ν_1 and ν_2.

$l+1+\nu_1$ or $l+1-\nu_2$, then the functional is finite since the simple pole of the corresponding gamma function in the denominator cancels the pole in the numerator. If both gamma functions in the denominator have poles for certain integers ν_1 and ν_2, then the value of the functional is zero. Figure 2 shows the different areas of poles, zero, etc., in the (ν_1,ν_2) plane, where the pairs of integer ν_1 and ν_2 form a lattice.

The above results provide also an inversion of the Taylor series in the following sense: In a Hilbert space with a positive definite metric (ϕ_1,ϕ_2) the coefficients c_n of the expansion

$$\phi=\sum_n c_n\phi_n$$

are given by (ϕ_n,ϕ) if the ϕ_n form an orthogonal normalized set.

If $\phi(p_0)$ is a test function with the expansion

$$\phi(p_0)=\sum_{n=0}^{\infty} c_n p_0{}^n,$$

then, according to the already quoted results of Gel'fand and Shapiro, c_n is given by

$$c_n=\operatorname*{Res}_{\lambda=-(n+1)} p_0{}^{\lambda}(\phi)=\phi^{(n)}(0)/\Gamma(n+1).$$

Since quantities like $p_0{}^{\nu}$ etc. occur in physics far more frequently for real ν than for imaginary ones, the above results indicate the possible importance of indefinite metrics in infinite dimensional linear spaces for physica theories.

V. AN EXAMPLE FOR $m>0$, $S=0$

We have already mentioned in Sec. III that for $m^2>0$ all values $m^2>0$ occur in an irreducible representation of the conformal group and that such a representation is, therefore, a superposition of irreducible representations of the Poincaré group. In the case of spin zero we get the scalar product of the corresponding Hilbert space by taking the direct integral of all scalar products:

$$(\phi_1,\phi_2;m^2)=\int\frac{d^3p}{2p_0}\phi_1{}^*(\mathbf{p})\phi_2(\mathbf{p}),\quad p_0=(\mathbf{p}^2+m^2)^{1/2}.$$

Thus we have

$$(\phi_1,\phi_2)=\int_0^\infty dm^2\rho(m^2)(\phi_1,\phi_2;m^2),$$

where $\rho(m^2)$ is an appropriate weight function.

As a special example we chose the 4-dimensional generalizations of the operators (9a)–(9e):

$$P_\mu=p_\mu,\quad \mu=0,1,2,3;\quad M_{\mu\nu}=i(p_\mu\partial_\nu-p_\nu\partial_\mu);$$
$$K_\mu=-2\partial_\mu-2p^\nu\partial_\nu\partial_\mu+p_\mu\Box;\quad D=i(p^\mu\partial_\mu+1).$$

These operators are Hermitian with respect to the scalar product

$$\int_{p^2>0,\,p_0>0}\frac{d^4p}{p^2}\phi_1{}^*(p)\phi_2(p),$$

where we have replaced the variable m by the variable p_0.

We shall first consider the operators K_μ. Since they commute with each other, they have a common set of eigenfunctions $R(p,h)$ with eigenvalues h_μ:

$$K_\mu R(p,h)=h_\mu R(p,h).$$

The ansatz $R(p,h)=R(p\cdot h)$ leads to the equation

$$(p_\mu h^2-2h_\mu p\cdot h)R''-2h_\mu R'=h_\mu R,$$

where

$$R'=dR(y)/dy,\quad y=p\cdot h.$$

Multiplying both sides by h^μ and summing over μ yields

$$yR''+2R'+R=0.$$

The solution of this equation which is regular for $y=0$ is, except for a constant factor,

$$R(p,h)=1/\pi^3(p\cdot h)^{-1/2}J_1[2(p\cdot h)^{1/2}],$$

where J_1 is the Bessel function of order one. The quantity $R(p,h)$ therefore has a similar structure here to that in the case $m=0$ of Sec. IV.

Since $R(p,h)$ is the momentum representation of the transformation by reciprocal radii, it is interesting to determine its eigenfunctions $e(p)$. In Ref. 1 it was convenient to do this in the case of mass zero by expanding $R(p,h)$ in terms of spherical harmonics. In our 4-dimensional case we have to expand in terms of the corresponding functions of the homogeneous Lorentz group.[34] We show in the Appendix that the following relation holds:

$$R(p,h)=(1/4\pi^3)(\mu\lambda)^{-1/2}(\sinh a\,\sinh b)^{-1/2}$$
$$\times\sum_l(2l+1)\int_0^\infty dq\,|\Gamma(iq+l+1)|^2|\Gamma(iq/2)|^2$$
$$\times|\Gamma(iq)|^{-2}J_{\nu(q)}[2(\mu\lambda)^{1/2}]P_{iq-1/2}{}^{-l-1/2}(\cosh a)$$
$$\times P_{iq-1/2}{}^{-l-1/2}(\cosh b)P_l(z),$$

[34] V. Bargmann, Ann. Math. 48, 568 (1947); see also Ref. 30.

where $P_\nu{}^\mu(x)$ means a Legendre function, $J_\nu(x)$ a Bessel function, and where

$$\nu(q)=(3-4q^2)^{1/2};\quad z=\cos(\mathbf{p},\mathbf{h});$$
$$p_0=\mu\cosh a,\quad h_0=\lambda\cosh b,$$
$$p_1=\mu\sinh a,\quad h_1=\cdots;$$
$$p_2=\mu\sinh a\sin\vartheta_1\cos\phi_1,$$
$$p_3=\mu\sinh a\sin\vartheta_1\sin\phi_1;$$

Because of the relation[30]

$$\int_0^\infty da\,\sinh a P_{iq-1/2}{}^{-l-1/2}(\cosh a)P_{iq-1/2}{}^{-l-1/2}(\cosh b)$$
$$=|\Gamma(iq)|^2|\Gamma(iq+l+1)|^2\delta(q-q'),\quad(20)$$

and the addition theorem for the spherical harmonics, the ansatz

$$e(p)=f(\mu;\nu(r))P_{ir-1/2}{}^{-l-1/2}(\cosh a)Y_{lm}(\mathbf{p}/|\mathbf{p}|)$$

leads to the integral equation

$$f(\mu;\nu(r))=\frac{1}{\pi^2\mu^{1/2}}|\Gamma\tfrac{1}{2}ir|^2$$
$$\times\epsilon\int_0^\infty d\lambda\,\lambda^{1/2}J_{\nu(r)}[2(\mu\lambda)^{1/2}]f(\lambda;\nu(r)),\quad(21)$$

where ϵ is the eigenvalue ±1. This integral equation is the same as in Ref. 1 if

$$|\Gamma\tfrac{1}{2}ir|^2=\pi^2.$$

Because of[35]

$$|\Gamma\tfrac{1}{2}ir|^2=2\pi/r\sinh(\tfrac{1}{2}\pi r)$$

this means that r has to be a solution of the equation

$$\tfrac{1}{2}\pi r\sinh(\tfrac{1}{2}\pi r)=1.\quad(22)$$

It is easy to see[36] that there is only one real solution $r_0>0$. It lies between 0 and 1. This feature, that there is only one isolated value $r=r_0$, for which the function

$$P_{ir_0-1/2}{}^{-l-1/2}(\cosh a)$$

is a part of the eigenfunctions $e(p)$, is essential, for it means that $e(p)$ is normalizable with a finite norm. This would not be the case for a continuous set [see Eq. (20)]. The eigenfunctions of Eq. (21) are given in Ref. 1. They are

$$f(\mu;\nu(r_0))=L_n{}^{\nu(r_0)}(2\mu)e^{-\mu}(2\mu)^{\nu(r_0)/2-1},\quad\epsilon=(-1)^n,$$

where the $L_n{}^\alpha$ are Laguerre's polynominals.

[35] See Ref. 32, Vol. 1, p. 3.
[36] *Hyperbolic Functions*, Smithsonian Mathematical Tables (Smithsonian Institution, Washington, D. C., 1909).

The above considerations have the following interesting application: The function

$$F(x)=1/x^2\int_{V_+}d^4h\,\phi(h)e^{-ih\cdot x/x^2},$$

where V_+ means the cone $p^2>0$, $p_0>0$, is a solution of the Klein-Gordon equation

$$(\Box_x+m^2)F(x)=0,$$

if $\phi(h)$ is a solution of

$$h^2\Box_h{}^2\phi(h)=m^2\phi(h).$$

This follows from

$$\Box_x(1/x^2e^{-ih\cdot x/x^2})=-h^2/(x^2)^3e^{-ih\cdot x/x^2}$$
$$=-h^2\Box_h{}^2(1/x^2e^{-ih\cdot x/x^2})$$

and a partial integration. Because of $K^2=-p^2\Box_p{}^2$ this means that the operator P^2 has the same form in h space as the operator K^2 in p space. This has to be the case since the two spaces and their operators are connected by the unitary transformation $R(p,h)$.

This simple example shows how one can utilize the representations of the conformal group in the case of nonvanishing rest masses.

An analysis of the operator $D=i(p^\mu\partial_\mu+1)$ brings nothing essentially new in comparison to the corresponding one in Sec. IV. We merely have to replace Y_{lm} by

$$P_{iq-1/2}{}^{-l-1/2}(\cosh a)Y_{lm}$$

and p_0 by μ.

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APPENDIX: THE EXPANSION OF $R(p,h)$ IN TERMS OF SPHERICAL HARMONICS

The following procedure is analogous to the expansion (14) of the 3-dimensional plane waves in terms of spherical harmonics and the result is similar to that given by Joos for the 4-dimensional plane waves.[30] We recall that the Bessel functions $J_{l+1/2}$ in the expansion (14) result from a separation of the spherical variables in the equation

$$\Delta\phi(\mathbf{x})+\mathbf{k}^2\phi(\mathbf{x})=0.$$

The corresponding operator in our case is

$$K^2=p^2\Box^2.$$

With the "spherical" coordinates of the Minkowsky space,

$$p_0=m\cosh a\,,\quad p_2=m\sinh a\sin\vartheta\cos\phi\,,$$
$$p_1=m\sinh a\,,\quad p_3=m\sinh a\sin\vartheta\sin\phi\,,$$

we get for the Klein-Gordon operator[37]

$$\Box=m^{-3}\frac{\partial}{\partial m}m^3\frac{\partial}{\partial m}-m^{-2}\Delta_L\,,$$

$$\Delta_L=(\sinh a)^{-2}\frac{\partial}{\partial a}\sinh^2 a\frac{\partial}{\partial a}$$
$$+(\sinh a)^{-2}\left(1/\sin\vartheta\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta}+1/\sin^2\vartheta\frac{\partial^2}{\partial\phi^2}\right).$$

The eigenvalue equation

$$\Delta_L w_q=-(q^2+1)w_q$$

has the solutions[38]

$$w_q=(\sinh a)^{-1/2}P_{iq-1/2}{}^{-l-1/2}(\cosh a)Y_{lm}(\vartheta,\phi)\,,$$

where q is real and positive in our case.[30]

The eigenvalue equation

$$K^2f=p^2\Box^2f(p)=h^2f(p)$$

has on the one hand the particular solution $R(p\cdot h)$ and yields, on the other hand, the differential equation

$$m^4g^{(4)}(m)+6m^3g^{(3)}(m)+(2q^2+5)m^2g''(m)$$
$$+(2q^2-1)mg'(m)+[(q^2+1)^2-m^2h^2]g(m)=0\,,$$

if we make the ansatz

$$f(p)=g(m)w_q(a,\vartheta,\phi)\,.$$

Four independent solutions of this equation are[39] the cylinder functions

$$m^{-1/2}J_\nu[2(m\lambda)^{1/2}],\quad m^{-1/2}I_\nu[2(m\lambda)^{1/2}],\quad m^{-1/2}Y_\nu[2(m\lambda)^{1/2}],$$
$$m^{-1/2}K_\nu[2(m\lambda)^{1/2}],\quad\text{where}\quad \nu=(3-4q^2)^{1/2},\quad \lambda=(h^2)^{1/2}.$$

In analogy to the results of Ref. 1 we expand $R(p,h)$ in terms of the first one of these four functions:

$$(p\cdot h)^{-1/2}J_1[2(p\cdot h)^{1/2}]=(\mu\lambda)^{-1/2}(\sinh a\sinh b)^{-1/2}\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dq\,A_l(q)J_{\nu(q)}[2(\mu\lambda)^{1/2}]$$
$$\times P_{iq-1/2}{}^{-l-1/2}(\cosh a)P_{iq-1/2}{}^{-l-1/2}(\cosh b)P_l(z)\,,$$

where

$$p\cdot h=\mu\lambda(\cosh a\cosh b-\sinh a\sinh b\,z)\,,\quad z=\cos(\mathbf{p},\mathbf{h})\,.$$

The quantity $A_l(q)$ has to be determined. The following way of doing this is a special case of a general method given by Vilenkin and Smorodinsky.[38]

First we integrate both sides of Eq. (A1) over λ, and because of the relation[40]

$$\int_0^\infty dx\,J_\nu(xy)=y^{-1}\,,$$

we get

$$(\cosh a\cosh b-\sinh a\sinh b\,z)^{-1}=\sum_{l=0}^{\infty}(2l+1)\int_0^\infty dq\,A_l(q)(\sinh a\sinh b)^{-1/2}$$
$$\times P_{iq-1/2}{}^{-l-1/2}(\cosh a)P_{iq-1/2}{}^{-l-1/2}(\cosh b)P_l(z)\,.$$

The orthogonality of the Legendre polynominals leads to

$$Q_l\left(\frac{\cosh a\cosh b}{\sinh a\sinh b}\right)=(\sinh a\sinh b)^{1/2}\int_0^\infty dq\,A_l(q)P_{iq-1/2}{}^{-l-1/2}(\cosh a)P_{iq-1/2}{}^{-l-1/2}(\cosh b)\,,$$

where

$$Q_l(y)=\frac{1}{2}\int_{-1}^{+1}dt\frac{P_l(t)}{t-y}$$

is the Legendre function of the second kind.

[37] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 113.
[38] N. Ya. Vilenkin and Ya. A. Smorodinsky, Zh. Eksperim. i Teor. Fiz. **46**, 1793 (1964) [English transl.: Soviet Phys.—JETP **19**, 1209 (1964)].
[39] E. Kamke, *Differentialgleichungen* (Akademische Verlagsgesellschaft, Leipzig, 1953), 2nd ed., Vol. I, p. 534.
[40] *Tables of Integral Transforms*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1954), Vol. II, p. 22.

Applying the relation (20) yields

$$A_l(q)|\Gamma(iq)|^2|\Gamma(iq+l+1)|^{-2}(\sinh a)^{1/2}P_{iq-1/2}{}^{-l-1/2}(\cosh a)$$
$$=\int_1^\infty d\cosh b\, Q_l\left(\frac{\cosh a \cosh b}{\sinh a \sinh b}\right)P_{iq-1/2}{}^{-l-1/2}(\cosh b)(\sinh b)^{-1/2}.$$

In order to determine $A_l(q)$ from this equation it is convenient to consider small a. For in that case we can replace $Q_l(y)$ under the integral by its asymptotic expansion for large

$$y=\frac{\cosh a \cosh b}{\sinh a \sinh b}$$

and compare the coefficients of the first terms in the power series of $\cosh a/\sinh a$ on both sides of Eq. (A1). Since[41]

$$Q_l(y)=\pi^{1/2}2^{-l-1}\Gamma(l+1)\Gamma(l+\tfrac{3}{2})^{-1}y^{-l-1}[1+O(y^{-2})]$$

for $y\gg1$ and because of the relation[42]

$$P_{iq-1/2}{}^{-l-1/2}(\cosh a)=e^{\pi q}(2/\pi)^{1/2}\Gamma(iq+l+1)^{-1}(\sinh a)^{-1/2}Q_l{}^{iq}\left(\frac{\cosh a}{\sinh a}\right)$$
$$=\sqrt{2}\; 2^{-l-1}(\sinh a)^{-1/2}\Gamma(l+\tfrac{3}{2})\left(\frac{\cosh a}{\sinh a}\right)^{-l-1}\left[1+O\left(\left\{\frac{\cosh a}{\sinh a}\right\}^{-2}\right)\right] \text{ for } \cosh a\gg\sinh a,$$

we get

$$A_l(q)=(\tfrac{1}{2}\pi)^{1/2}|\Gamma(iq+l+1)|^2|\Gamma(iq)|^{-2}\Gamma(l+1)\int_1^\infty x^{-l-1}(x^2-1)^{\frac{1}{2}(l+\frac{1}{2})}P_{iq-1/2}{}^{-l-1/2}(x)dx.$$

The value of the integral is[43]

$$\int_1^\infty x^{-l-1}(x^2-1)^{\frac{1}{2}(l+\frac{1}{2})}P_{iq-1/2}{}^{-l-1/2}(x)dx=2^{-1}(2\pi)^{-1/2}\Gamma(l+1)^{-1}|\Gamma(\tfrac{1}{2}iq)|^2,$$

so that we finally have

$$A_l(q)=\tfrac{1}{4}|\Gamma(iq+l+1)|^2|\Gamma(\tfrac{1}{2}iq)|^2|\Gamma(iq)|^{-2}.$$

It should be noted that our parameters in the above integral do not fulfill the conditions given in Ref. 43. But we can define the left-hand side by a unique analytic continuation of the gamma functions on the right-hand side if we avoid the poles of those functions on the negative real axis.

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[41] See Ref. 32, Vol. 1, p. 122.
[42] See Ref. 32, Vol. 1, p. 141.
[43] See Ref. 40, Vol. 2, p. 320.