Here we see that this assumption might be suspicious. The $V\theta$ bound-state pole at ω_B has residue proportional to Z_1^2/Z_U in the limit of Eq. (36), so we add the requirement

$$\lim_{Z_1\to 0;\, Z_U\to 0} Z_1^2/Z_U=0. \tag{36c}$$

This condition has a natural interpretation in terms of the unrenormalized $UV\theta$ coupling constant which is given by[5]

$$\lambda_0^2=(Z_1^2/Z_U)(\lambda^2/Z_V). \tag{37}$$

Since we have assumed[11] $Z_V\neq 0$ and, from Eq. (18) have finite λ^2, Eq. (37) yields

$$\lambda_0^2=0. \tag{38}$$

We expect something like Eq. (38) to hold when we discuss bootstraps and bound states, and here we see explicitly how it arises and what it means.

Finally, from Eqs. (32) and (24) we see that if the strong condition

$$\lim_{Z_1\to 0;\, Z_U\to 0} Z_1/Z_U=0 \tag{39}$$

is satisfied then we have, in this limit

$$\omega_B=\omega_U. \tag{40}$$

Equation (40) was the starting assumption of Kaus and Zachariasen[8] and they arrived at Eq. (39). Of course Eq. (36c) is satisfied *a fortiori* so our entire discussion still holds. The point here is that it does not seem to be necessary to require the $V\theta$ bound state to move down to the U-particle mass, the bootstrap really depends on the point $\bar{\omega}_0$ doing this. Since $\bar{\omega}_0$ always lies between ω_U and ω_B we can force

$$\bar{\omega}_0=\omega_U \tag{41}$$

by requiring (40) to hold, but in fact we can also obtain Eq. (41) with less stringent requirements.

It is interesting that if Eq. (39) holds then we have vanishing self-mass in the limit, while finite self-mass implies Eq. (36c). Therefore we can phrase the bootstrap conditions in the simple form,

$$Z_U=0,\quad \delta m_U \text{ finite}$$

for the theory of Ref. 1 while

$$Z_U=0,\quad \delta m_U=0$$

for the theory of Ref. 8.

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Linear Integral Equations for Relativistic Multichannel Scattering*

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A set of three-dimensional coupled linear integral equations is presented for relativistic scattering and production processes. The new equations seem more amenable to numerical analysis than the standard Bethe-Salpeter equations. In addition one may discuss multiparticle scattering and hence inelastic processes. As a numerical example we discuss low-energy pion-pion scattering in the simplest approximation. We find that there are no self-consistent values for the mass and width of the ρ meson if one takes into account only the elastic and π-ω channels.

I. INTRODUCTION

Most of the difficulties in performing dynamical calculations with on-the-mass-shell amplitudes using the Mandelstam representation are now evident. The nonlinearity of the equations makes it very difficult to make rigorous statements about the solutions. One particularly important unanswered question concerns the stability of the solution as higher mass states are added. It is difficult in any case to include inelastic states correctly because of their extremely complicated analytic structure.

We will explore in this paper a set of linear, off-the-energy-shell equations.[1,2] The most popular linear

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[1] A. A. Langunov and A. N. Tavkhelidze, Nuovo Cimento **29**, 380 (1963); G. Tiktopoulos (unpublished); B. Lee and R. Sawyer (unpublished). We wish to thank Professor Sawyer for conversations on the subject of this paper and for pointing out an important error in an early version of this work.

[2] R. Blankenbecler and R. Sugar, in *Proceedings of the 12th Annual International Conference on High-Energy Physics at Dubna, 1964* (Atomizdat, Moscow, 1965).

equation is the classic Bethe-Salpeter approach. However, this is a very complicated equation to solve because of its four-dimensional character and the Lorentz metric. (See, however, the most recent work of Zemach and Schwartz.[3]) Furthermore, the Bethe-Salpteter equation is not an expansion in the number of particles in intermediate states since even in the ladder approximation inelastic states which contain an arbitrary number of "exchange" particles are present. The purely technical problems associated with solving the Bethe-Salpeter equation in physically interesting cases makes it quite difficult to apply in strong-interaction problems.

One would like to be able to treat models of strong interactions which go beyond the usual restriction to two-particle intermediate states. Therefore, one would like to have equations that look like potential scattering since there are convenient mathematical tools for discussing multiparticle potential scattering.[4] However, how does one then handle particle production? It does not seem possible or convenient to generally assume that this situation can be treated in analogy with potential scattering; that is, to consider a multiparticle scattering situation and then force the bound states among these particles to be the lower particle channels. The gymnastics required to apply this method to the treatment of the coupled $\pi-\pi$ and $\bar{N}-N$ systems are obvious.

We will present an approach to this problem which contains certain "good" features of potential scattering; among these are the three-dimensional nature of the equations and the separation into channels. This approach will suffer from many of the same faults which plague all other strong-interaction schemes; crossing symmetry is particularly badly handled.

For the two-particle sector the method will be motivated by discussing the Bethe-Salpeter equation and reducing it to a three-dimensional equation. The techniques developed for two-particle channels can be extended to multiparticle channels. The procedure is illustrated by considering a concrete model which has both a two- and a three-particle channel. Some general properties of the simplest approximation to these equations have already been presented at the Dubna Conference.[2] However, since the proceedings of this conference are still on the unphysical sheet, we will repeat some of our results here. We will also present some numerical work on the physically interesting case of pion-pion scattering.

We have recently received a report by Alessandrini and Omnes in which our method for constructing elastic Green's functions is applied to the three-particle Faddeev equations.[5] We will also present relativistic three-body equations but in a form which we feel has some advantages in cases which involve the coupling between the two- and three-particle channels and which involve a direct three-body force.

II. DYNAMICAL EQUATIONS

A. Two-Particle Channels

We will motivate our equations in the two-particle sector by considering the Bethe-Salpeter (B-S) equation in the ladder approximation and rewriting it to bring out certain analytic properties. However, once the equations are written down, we will give arguments that their form is valid quite generally. We will write the B-S equation for the scattering of particles a and b each with mass 1 by exchange of particle c with mass m in the form

$$M(p,q)=V(p,q)-i(2\pi)^{-4}\int d^4k\, V(p,k)\times G_a(\tfrac{1}{2}P+k)G_b(\tfrac{1}{2}P-k)M(k,q)\,, \quad (1)$$

where

$$V(p,q)=g^2[m^2+(p-q)^2]^{-1}\,,$$
$$G_{a,b}(k)=[1+k^2]^{-1}.$$

The center-of-mass momentum has been denoted by P and we define $s=-P^2$. In Eq. (1) we may put q on the energy shell, that is $q_0=0$, $q^2=s/4-1$.

Let us now introduce a two-particle Green's function E_2 which can only produce two-particle cuts in the physical region. This will be accomplished by writing a function which is singular only when both legs of the ladder are on the mass shell.

$$E_2(k)=2\pi\int ds'(s'-s)^{-1}\delta[1+(\tfrac{1}{2}P'+k)^2]\times\delta[1+(\tfrac{1}{2}P'-k)^2]\,, \quad (2)$$

where $P'^2=-s'$ and P' has only a fourth component. The fact that E_2 has the proper imaginary part for a two-particle discontinuity can easily be checked. The integration can be carried out in the center-of-mass frame and the result is

$$E_2(k)=\tfrac{1}{2}\pi\delta(k_0)[(k^2+1)^{1/2}(k^2-q^2)]^{-1}. \quad (3)$$

We now write

$$-iG_aG_b=E_2+R_2 \quad (4)$$

and note that R_2 cannot produce two-particle singularities. It should be noted that E_2 cannot be unique since one may add any function which is not singular along the positive cut of s. However, the choice (2) seems to be simplest and leads to a potential with simple analytic properties.

The B-S equation can now be written as

$$M=W+WE_2M\,, \quad (5)$$

where

$$W=V[1-R_2V]^{-1}.$$

This integral equation for $M(p,q)$ only requires knowl-

[3] Bull. Am. Phys. Soc. **10**, 464 (1965), Paper DC1.
[4] R. Sugar and R. Blankenbecler, Phys. Rev. **136**, B472 (1964).
[5] V. A. Alessandrini and R. L. Omnes, Phys. Rev. **139**, B288 (1965).

edge of M for $p_0=q_0=0$, because of the delta function present in E_2. Therefore, we only need the effective potential $W(p,q)$ for $p_0=q_0=0$. This fact eliminates the necessity of performing a Wick rotation.

The simplest approximation is to neglect all higher corrections and set $R_2=0$. We then have

$$W(p,q)=V(p,q)=g^2[(p-q)^2+m^2]^{-1}. \tag{6}$$

In other words, W is exactly the Yukawa potential. Equation (5) then becomes,[1]

$$M(p,q)=V(p,q)+\tfrac{1}{4}\int \frac{d^3k}{(2\pi)^3}\frac{V(p,k)M(k,q)}{[k^2+1]^{1/2}[k^2-q^2]}. \tag{7}$$

The only difference between Eq. (7) and the usual Lippmann-Schwinger equation in potential scattering is the factor of $(k^2+1)^{-1/2}$ in the Green's function. Equation (7) is, therefore, slightly more convergent than the standard potential scattering equation, and it satisfies a relativistic (elastic) unitarity condition. However, the presence of this square-root factor severely complicates the analytic structure and this amplitude does not satisfy the Mandelstam representation.[6]

In fact, it is easy to see that the second Born approximation is given incorrectly by Eq. (7). In order to estimate the error introduced by this approximation for W, we have evaluated explicitly the second iteration of (7) and the correct fourth-order graph in the forward direction. The interation of (7) contains a left-hand cut starting at $s=4-4(1+m)^2$ which is *not* present in the correct second Born term. However, the difference between them is just the second-order term in R_2, which is less than 10% of the correct second-order term at threshold ($s=4$) and goes to zero as s increases.

Therefore, one is led to expect that the partial-wave amplitudes coming from such an equation will have left-hand cuts representing multiparticle exchange which are given accurately for $s\geq 4-4(1+m)^2$ by the approximation $W=V$.

Of course, by adding the appropriate correction terms to W, one can get an exact left-hand cut for any finite number of exchanged particles. We expect this approach to be most useful in including multiparticle states into dynamical calculations and, therefore, consider this accuracy on the left as a bonus.

The left-hand cut due to the exchange of N particles of mass m extends up to $s=4-N^2m^2$. Thus, we see that the approximation $W=V$ should give a reasonable approximation to the left-hand cut out to $N^2\approx 4$ for $m\gg 1$ and $N^2\approx 4/m^2$ for $m\ll 1$.

[6] M. K. Polivanor and S. S. Khoruzhi, Zh. Eksperim. i Teor. Fiz. **46**, 339 (1963) [English transl.: Soviet Phys.—JETP **19**, 232 (1964)]. The effective potentials considered in this work have only right-hand cuts. Of course, the exact W for the ladder graphs also has a left-hand cut and the ladder graphs do satisfy the Mandelstam representation. It is erroneously stated in this reference that the factor of $(k_2+1)^{\frac{1}{2}}$ in E_2 gives rise to a square-root branch in M at $s=0$. The difficulty is that the bound on E_2 used in this work blows up at $s=0$ whereas E_2 is actually finite at this point.

The effective potential W has no two-particle singularities but it can have poles corresponding to bound states. This is certainly the case in pi-nucleon scattering, and it may be worthwhile to consider such a situation in some detail. We take the two-body potential [see Fig. 1(d)] in our equal-mass model to be

$$\bar{V}_{22}=g^2{}_0[(p_1+p_2)^2+M_0^2]^{-1}.$$

If $\bar{V}_{22}$ is the only contribution to the potential, then we are merely summing bubble diagrams, and it is clear that it will be necessary to renormalize the mass and coupling constant. In this case, the scattering amplitude is given formally by

$$\begin{aligned}M_{22}(p,q)&=g_0^2(M_0^2-s)^{-1}+g_0^2(M_0^2-s)^{-1}\\&\quad\times\int\frac{d^3k}{(2\pi)^3}\frac{M_{22}(k,q)}{(k^2+1)^{1/2}[4(q^2+1)-s]}\\&=g_0^2[M_0^2-s-g_0^2I(s)]^{-1},\end{aligned}$$

where

$$\begin{aligned}I(s)&=\int\frac{d^3k}{(2\pi)^3}\{(k^2+1)^{1/2}[4(k^2+1)-s]\}^{-1}\\&=I(M^2)+(s-M^2)I'(M^2)+(s-M^2)^2I_R(s),\end{aligned}$$

$$I_R(s)=\int\frac{d^3k}{(2\pi)^3}\{(k^2+1)^{1/2}[4(k^2+1)-M^2]^2\times[4(k^2+1)-s]\}^{-1}.$$

In order that the scattering amplitude have a pole at the physical mass M and that the residue be the square of the physical coupling constant g we must have

$$g_0^2I(M^2)=M_0^2-M^2,\quad g^2=g_0^2[1+g_0^2I'(M^2)]^{-1}.$$

Then,

$$M_{22}=g^2\{(M^2-s)[1-(M^2-s)g^2I_R(s)]\}^{-1}.$$

If one now takes other forces into account, the position and residue of the pole will be shifted by a finite amount, and it will be necessary to perform a second, finite, renormalization.

B. Three-Particle Channels

Now that we have extracted the two-particle states by introducing the propagator E_2, we can proceed in the same fashion on the three-particle singularities. In order to ensure that the scattering amplitude satisfies unitarity in the three-particle sector, the three-particle propagator must have a cut starting at the three-particle threshold, and the discontinuity across this cut must be given by

$$\Delta_3=(2\pi)^3 i\prod_{j=1}^{3}\delta(q_j^2+m_j^2), \tag{8}$$

where

$$\textstyle\sum_j q_j=P.$$

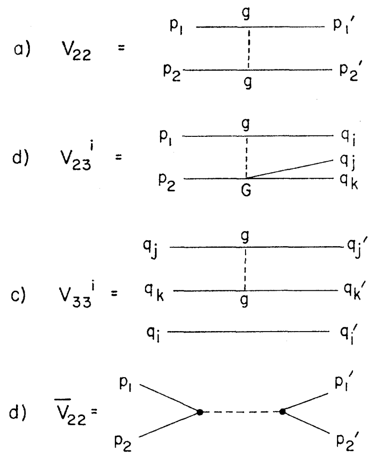

FIG. 1. Potential graphs.

The two-particle propagator was written as a dispersion integral in the energy variable $s=-P^2$. The three-particle propagator can be written in the same form; however, one can also disperse in the energy of any pair of particles or even in the mass of a single particle. We will only consider the first two possibilities. To be specific we introduce the Green's functions

$$E_3{}^i \equiv (2\pi)^2 \int_{(m_1+m_2+m_3)^2}^{\infty} ds'(s'-s)^{-1}\delta(q_i{}^2+m_i{}^2) \times\delta[(\tfrac{1}{2}(P'-q_i)+q_{jk})^2+m_j{}^2] \times\delta[(\tfrac{1}{2}(P'-q_i)-q_{jk})^2+m_k{}^2], \quad (9)$$

where

$$P'=(s'/s)^{1/2}P \quad \text{and} \quad q_{jk}=\tfrac{1}{2}(q_j-g_k), \quad i\neq j\neq k.$$

We also introduce

$$F^i \equiv (2\pi)^2 \int_{(m_j+m_k)^2}^{\infty} d\sigma_i'(\sigma_i'-\sigma_i)^{-1}\delta(q_i{}^2+m_i{}^2) \times\delta[(\tfrac{1}{2}P_{jk}'+q_{jk})^2+m_j{}^2]\delta[(\tfrac{1}{2}P_{jk}'-q_{jk})^2+m_k{}^2],$$

where

$$\sigma_i=-(P-q_i)^2=-P_{jk}{}^2, \quad P_{jk}'=(\sigma_i'/\sigma_i)^{1/2}P_{jk}.$$

One can introduce propagators for more than three particles in precisely the same manner. We note that the N-particle Green's function is to be integrated over $N-1$ four-dimensional coordinates, and there are just enough delta functions to reduce these to $N-1$ three-dimensional coordinates. As a result, the equations will look like potential scattering except that the Green's functions will be integrated over a relativistic phase space.

In order to obtain "potentials" for the three-body channel, one must go beyond the ladder approximation to the B-S equation which is inconsistent in the multiparticle sector. We will illustrate our approach by considering a simple model with coupled two- and three-particle channels. The "potentials" are given by the diagrams shown in Fig. 1. We have

$$V_{22}=g^2[(p_1-p_1')^2+M^2]^{-1}(2\pi)^4\delta^4(p_1+p_2-p_1'-p_2'),$$
$$V_{23}{}^i=gG[(p_1-q_i)^2+M^2]^{-1}(2\pi)^4\delta^4(p_1+p_2-q_1-q_2-q_3),$$
$$V_{33}{}^i=g^2[(q_i-q_j')^2+M^2]^{-1}(2\pi)^4 \times\delta^4(q_1+q_2+q_3-q_1'-q_2'-q_3') \times 2\omega_i(2\pi)^3\delta^3(\mathbf{q}_i-\mathbf{q}_i'), \quad (10)$$

where

$$\omega_i=(q_i{}^2+m_i{}^2)^{1/2}, \quad V_{23}=\sum V_{23}{}^i, \quad V_{33}=\sum V_{33}{}^i.$$

It is convenient in the three-particle case to keep the over-all momentum-conservation delta function.

One-particle exchange forces have been used for definiteness only. The equations that we shall obtain are independent of the functional form of the potentials. The only real requirement on them is that they do not contain any two- or three-particle singularities. Direct three-body forces have been omitted, but their inclusion would not cause any additional difficulty.

In order to clarify the meaning of the potentials and Green's functions, let us write down a few examples from second-order perturbation theory. The diagrams that we will consider are shown in Fig. 2.

$$\text{(a)} \quad V_{22}E_2V_{22}=\int \frac{d^4q_1d^4q_2}{(2\pi)^8} g^2[(p_1-q_1)^2+M^2]^{-1}(2\pi)^4\delta^4(P-Q)g^2[(p_1'-q_1)^2+M^2]^{-1}(2\pi)^4\delta^4(P'-Q) \times 2\pi\int ds'(s'-s)^{-1}\delta[(\tfrac{1}{2}P'+q)^2+m_1{}^2]\delta[(\tfrac{1}{2}P'-q)^2+m_2{}^2],$$

where

$$P=p_1+p_2, \quad P'=p_1'+p_2', \quad Q=q_1+q_2, \quad q=\tfrac{1}{2}(q_1-q_2).$$

Performing the integration over the delta function in the center-of-mass system, we find[7]

$$V_{22}E_2V_{22}=(2\pi)^4\delta^4(P-P')\int \frac{d^3q}{(2\pi)^3} g^2[(p-q)^2+m^2]^{-1} \frac{2(\omega_1+\omega_2)}{2\omega_1 2\omega_2[(\omega_1+\omega_2)^2-s]} g^2[(p'-q)^2+m^2]^{-1},$$

[7] It should be noted that in the unequal-mass case the potential depends on the fourth component of the relative momentum which, is however, determined by its spatial components. One can always choose the variable so that the relative momentum has a zero fourth component [R. Stora (private communication)].

where

$$\omega_i=(q^2+m_i^2)^{1/2}, \quad p=\tfrac12(p_1-p_2), \quad p'=\tfrac12(p_1'-p_2').$$

(b) $$V_{33}{}^1F^1V_{33}{}^1=\int\frac{d^4q_1d^4q_2d^4q_3{}^2}{(2\pi)^{12}}g^2[(p_2-q_2)^2+m^2]^{-1}g^2[(p_2'-q_2)^2+m^2]^{-1}(2\pi)^3 2(p_1{}^2+m_1{}^2)^{1/2}\delta^3(p_2-q_1)$$
$$\times(2\pi)^4\delta^4(P-Q)(2\pi)^3 2(p_1'+m_1{}^2)^{1/2}\delta^3(p_1'-q_1)(2\pi)^4\delta^4(P'-Q)$$
$$\times(2\pi)^2\delta(q_1{}^2+m_1{}^2)\int d\sigma'[\sigma'-\sigma]^{-1}\delta[(\tfrac12P_{23}'+q_{23})^2+m_2{}^2]\delta[(\tfrac12P_{23}'-q_{23})^2+m_3{}^2],$$

$$P=p_1+p_2+p_3,\quad \sigma=-P_{23}{}^2,\quad Q=q_1+q_2+q_3,\quad P_{23}'=(\sigma'/\sigma)^{1/2}P_{23},\quad P'=p_1'+p_2'+p_3',\quad q_{23}=\tfrac12(q_2-q_3),\quad P_{23}=P-p_1.$$

Going to the center of mass of the 2–3 system we have

$$V_{33}{}^1F^1V_{33}{}^1=2\omega_1(2\pi)^3\delta^3(p_1-p_1')(2\pi)^4\delta^4(P-P')$$
$$\times\int\frac{d^3q_{23}}{(2\pi)^3}g^2[(P_{23}-q_{23})^2+m^2]^{-1}\frac{2(\omega_2+\omega_3)}{2\omega_2 2\omega_3[(\omega_2+\omega_3)^2-\sigma]}g^2[(P_{23}'-q_{23})^2+m^2]^{-1},$$
$$\omega_2=[q_{23}{}^2+m_2{}^2]^{1/2},\quad \omega_3=[q_{23}{}^2+m_3{}^2]^{1/2}.$$

(c) $$V_{23}'E_3'V_{32}'=\int\frac{d^4q_1d^4q_2d^4q_3}{(2\pi)^{12}}g^2G^2[(p_1-q_1)^2+M^2]^{-1}[(p_1'-q_1)^2+M^2]^{-1}(2\pi)^4\delta^4(P-Q)(2\pi)^4\delta(P'-Q)$$
$$\times(2\pi)^2\delta(q_1{}^2+M_1{}^2)\int ds'(s'-s)^{-1}\delta\{(\tfrac12(P'-q_1)+q_{23})^2+M_2{}^2\}\delta\{(\tfrac12(P'-q_1)-q_{23})^2+M_3{}^2\}.$$
$$P=p_1+p_2,\quad P'=p_1'+p_2',\quad Q+q_1+q_2+q_3,\quad q_{23}=\tfrac12(q_2-q_3).$$

In the center-of-mass system of the three particles, we have

$$V_{23}'E_3'V_{32}'=(2\pi)^4\delta^4(P-P')$$
$$\times\int\frac{d^3q_1d^3q_{23}}{(2\pi)^6}g^2G^2[(p_1-q_1)^2+M^2]^{-1}[(q_1-p_1')^2+M^2]^{-1}\frac{2(\omega_1+\omega_2+\omega_3)}{2\omega_1 2\omega_2 2\omega_3}[(\omega_1+\omega_2+\omega_3)^2-s]^{-1}.$$

We now return to the problem of formulating integral equations whose structure will guarantee that the scattering amplitude obeys unitarity in the two- and three-particle sectors. Let us start by neglecting the coupling between the channels and considering the pure three-body amplitude. The only difficulty is the

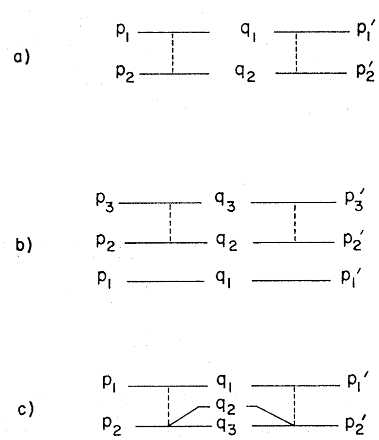

FIG. 2. Typical second-order graphs.

choice of Green's functions. In analogy with the two body case one might write

$$M_{33}=V_{33}+\sum_{i=1}^{3}V_{33}{}^iE_3{}^iM_{33}. \tag{11}$$

Equation (11) can be brought to Faddeev form by writing

$$M_{33}=\sum_{i=1}^{3}M_{33}{}^i, \tag{12}$$

where $M_{33}{}^i$ is that part of M_{33} in which the ith potential has interacted last. We then have[8]

$$M_{33}{}^i=V_{33}{}^i+V_{33}{}^iE_3{}^iM_{33}=T^i+T^iE_3{}^i(M_{33}{}^j+M_{33}{}^k),\quad j,k\neq i \tag{13a}$$

where

$$T_i=[1-V_{33}{}^iE_3{}^i]^{-1}V_{33}{}^i=t^i2\omega_i\delta^3(\mathbf{q}_i-\mathbf{q}_i').$$

[8] Equation (13) is essentially the same as Eq. (11.1) of Ref. 5. These authors are also aware of the difficulties with Eq. (13); V. Alessandrini (private communication).

The difficulty with Eq. (13a) is that because of the form of $E_3{}^i$ the two-particle amplitude t^i is a function of the total energy of the three particles instead of the energy of the jk pair. Although t^i does satisfy the two-particle unitarity condition, it is clearly not the amplitude one would obtain using E_2's and the cluster property. As can be seen from example (c), these difficulties can be overcome by replacing (13a) by

$$\begin{aligned} M_{33}{}^i &= V_{33}{}^i + V_{33}{}^i F^i M_{33}{}^i + V_{33}{}^i E_3{}^i (M_{33}{}^j + M_{33}{}^k) \\ &= T^i + T^i E_3{}^i (M_{33}{}^j + M_{33}{}^k), \quad \text{(13b)} \end{aligned}$$

where T_i is now given by

$$T^i = [1 - V_{33}{}^i F^i]^{-1} V_{33}{}^i = t^i 2\omega_i \delta^3(\mathbf{q}_i - \mathbf{q}_i).$$

Alternatively, one can replace all $E_3{}^i$'s in Eq. (13a) by F^i's. In either case the resulting M_{33} will automatically satisfy three-particle unitarity because of the form of the E_3's and F's.

There are now two alternative forms for the coupled equations. One can write

$$\begin{aligned} M_{22} &+ V_{22} + V_{22} E_2 M_{22} \\ &+ \sum_{i=1}^{3} V_{23}{}^i (F^i M_{32}{}^i + E_3{}^j M_{32}{}^j + E_3{}^k M_{32}{}^k), \\ M_{32}{}^i &= V_{32}{}^i (1 + E_2 M_{22}) + V_{33}{}^i F^i M_{32}{}^i \quad \text{(14a)} \\ &\quad + V_{33}{}^i E_3{}^i (M_{32}{}^j + M_{32}{}^k), \\ M_{33}{}^i &= V_{33}{}^i + V_{32}{}^i E_2 M_{23} + V_{33}{}^i F^i M_{33}{}^i \\ &\quad + V_{33}{}^i E_3{}^i (M_{33}{}^j + M_{33}{}^k), \end{aligned}$$

where

$$M_{32} = \sum_{i=1}^{3} M_{32}{}^i, \quad M_{33} = \sum_{i=1}^{3} M_{33}{}^i, \quad \text{and} \quad i \neq j \neq k.$$

Alternatively one can write

$$M_{22} = V_{22} + V_{22} E_2 M_{22} + \sum_{i=1}^{3} V_{23} F^i M_{32}{}^i,$$

$$M_{32} = V_{32}(1 + E_2 M_{22}) + \sum_{i=1}^{3} V_{33}{}^i F^i M^{32}, \quad \text{(14b)}$$

$$M_{33} = V_{33} + V_{32} E_2 M_{23} + \sum_{i=1}^{3} V_{33}{}^i F^i M_{33}.$$

In both cases unitarity is automatically satisfied. There is no obvious reason for choosing between Eqs. (14a) and (14b); however, in the course of a practical calculation, where the V's are given explicitly, a particular choice of variables may turn out to be more natural or more convenient.

Let us briefly consider an example in which Eq. (14b) is clearly the more convenient choice. Suppose that the potential $V_{33}{}^1$ is strong enough to produce a narrow resonance between particles 2 and 3. Neglecting $V_{33}{}^2$ and $V_{33}{}^3$ for the moment, the effective potential in the two-body channel is given by

$$W_{22}{}^1 = V_{22} + V_{23} F^1 [1 - V_{33}{}^1 F^1]^{-1} V_{32} = V_{22} + V_{23} I V_{32}.$$

Taking the discontinuity of I across the three-particle cut gives

$$\Delta I = [1 - F^{1*} V_{33}{}^1]^{-1} \Delta F^1 [1 - V_{33}{}^1 F^1]^{-1},$$

where

$$\Delta F^1 = (2\pi)^3 i \delta(q_1{}^2 + m_1{}^2) \delta(q_2{}^2 + m_2{}^2) \delta(q_3{}^2 + m_3{}^2).$$

If there is a sharp resonance in the 2-3 system then

$$\begin{aligned} [1 - F^{1*} V_{33}{}^2]^{-1} &\cong (2\pi)^4 \delta^4(P - P')(2\pi)^3 2\omega_1 \delta^3(p_1 - p_1') \\ &\quad \times \psi(P_{23}) \chi^*(P_{23}')[\sigma - \sigma_r - i |g(\sigma)|^2]^{-1}, \end{aligned}$$

where

$$p_{23} = \tfrac{1}{2}(p_2 - p_3), \quad \sigma = -(P - p_1)^2,$$

$$V_{33}{}^1 = 2\omega_1 (2\pi)^3 \delta^3(p_1 - p_1') \nu(p_{23}, p_{23}{}^1),$$

$$\begin{aligned} \chi(p_{23}) &= \int \frac{d^3 p_{23}'}{(2\pi)^3} \nu(p_{23}, p_{23}') \psi(p_{23}') g^2(\sigma) \int \frac{d\Omega}{32\pi^2} |\chi(\sigma, \Omega)|^2 \\ &\quad \times \frac{[\sigma - (m_2 + m_3)^2]^{1/2} [\sigma - (m_2 - m_3)^2]^{1/2}}{2\sigma} \end{aligned}$$

and $\psi(p_{23})$ is the wave function for the resonant state. We then have

$$\begin{aligned} \Delta I = (2\pi)^4 \delta^4(P - P') &\int \frac{d^4 q_1 d^4 q_{23} d^4 Q_{23}}{(2\pi)^{12}} |\psi) \\ &\times (2\pi)_i{}^3 \delta(q_1{}^2 + m_1{}^2) \delta[(\tfrac{1}{2}(P - q_1) + q_{23})^2 + m_2{}^2] \\ &\times \delta[(\tfrac{1}{2}(P - q_1) - q_{23})^2 + m_3{}^2](2\pi)^4 \delta^4(q_1 + Q_{23} - P) \langle \psi |. \end{aligned}$$

In the limit of an infinitely narrow resonance, $|g|^2 \times [(\sigma - \sigma_r)^2 + |g|^2]^{-1} \to \pi \delta(\sigma - \sigma_r)$, and we find

$$\begin{aligned} \Delta I = (2\pi)^4 \delta^4(P - P') &\int \frac{d^4 q_1 d^4 Q_{23}}{(2\pi)^8} |\psi\rangle (2\pi)^2 i \delta(q_1{}^2 + m_1{}^2) \\ &\times \delta(Q_{23}{}^2 + \sigma_r)(2\pi)^4 \delta^4(q_1 + Q_{23} - P) \langle \psi |, \end{aligned}$$

which is exactly what one would expect if there were a two-particle inelastic channel with particles of mass m_1 and $\sqrt{\sigma_r}$. This result suggests that even if one does not wish to use the zero-width approximation it will be convenient to introduce a Green's function

$$\epsilon^1 = (2\pi i)^{-1} \int ds' (s' - s)^{-1} \Delta I(s'),$$

so that the effective potential is given by

$$W_{22}{}^1 = V_{22} + V_{23} \epsilon^1 V_{32}.$$

Now that the "most important" singularities have been dealt with, other contributions to the three-body potential can be added in. If $V_{33}{}^2$ and $V_{33}{}^3$ do not give rise to any two-particle bound states or resonances, then one might write

$$W_{22}=V_{22}+V_{23}\epsilon^1[1-(V_{33}^2+V_{33}^3)\epsilon^1]^{-1}V_{32}.$$

However, if there are overlapping resonances, it is probably more convenient to work with the F's and use either Eq. (14a) or (14b) directly.

III. π-π SCATTERING

As an example of the application of the foregoing equations we will consider an extension of the ρ-meson bootstrap model of Zachariasen and Zemach.[9] Let us start by considering only the $\pi-\pi$ channel. We assume that the force is due to ρ exchange and we restrict ourselves to two-particle intermediate states. Following $(Z-Z)$ we take the $\pi\pi\rho$ vertex to be [see Fig. 3(a)]

$$2i\gamma_{\rho\pi\pi}\epsilon_{ijk}(q_1+q_2)_\mu, \tag{15}$$

where i, j, k are isotopic spin indices.

The integral equation for the $T=1$ scattering amplitude then becomes

$$M(p,q)=V(p,q)+\frac{1}{4}\int\frac{d^3k}{(2\pi)^3}\frac{V(p,k)M(k,q)}{[k^2+1]^{1/2}[k^2-q^2-i\epsilon]}, \tag{16}$$

where

$$V(p,q)=4\pi f[s+(p+q)^2][(p-q)^2+m^2]^{-1},\quad f=\gamma_{\delta\pi\pi}{}^2/4\pi,$$

and

$$q^2=\tfrac{1}{4}s-1.$$

The mass of the ρ is m and we are using units in which the pion mass is 1. It should be noted that the second Born term of (16) contains an incorrect cut starting at the point $q^2=-(1+m)^2\approx-42$, whereas the regular left-hand cut from two ρ exchange starts at $q^2=-M^2\cong-30$. Therefore, the left-hand cut is probably well represented by (16).

Since M obeys the elastic unitarity condition, and the left-hand cut is given quite accurately, all of the "near-by" singularities should be well represented. There is, of course, no solution to Eq. (16) because of the vector nature of the ρ. In order to obtain a well-defined model it is necessary to introduce a cutoff. If the ρ were a Regge pole there would be no covergence difficulties; however, in the absence of any knowledge of the trajectory one can really do no better than to introduce a cutoff. Naturally one of the tests of the model will be whether the qualitative features of the solution change as the cutoff is varied.

The next step is to project out the $J=T=1$ amplitude. We find

$$M_{11}(p,q)=V_{11}+V_{11}EM_{11}=V_{11}+\frac{1}{8\pi^2}\int_0^\kappa\frac{k^2dkV_{11}(p,k)M_{11}(k,q)}{[k^2+1]^{1/2}[k^2-q^2-i\epsilon]}, \tag{17}$$

where

$$V_{11}(q,p)=\frac{4\pi f}{qp}[\tfrac{1}{2}(s+m^2)+p^2+q^2]Q_1\left(\frac{p^2+q^2+m^2}{2pq}\right)$$

and

$$\kappa^2=\tfrac{1}{4}\Lambda-1.$$

Following Zachariasen and Zemach we will use a cutoff mass $\Lambda=140$, about two nucleon masses. Equation (17) has been solved numerically with the aid of the Columbia University IBM 7094 computer. In order to compare our results with those of Ref. 9, let us imagine solving Eq. (17) by the Fredholm method. Keeping terms to first order in f, we have the determinantal approximation:

$$N=V_{11}(q,q),\quad D=1-\mathrm{tr}[V_{11}E]. \tag{18}$$

This is just the model of Ref. 9. Physically, the main difference between the two models is that Eq. (17) takes into account multiple ρ exchange whereas Eq. (18) does not. Since f is expected to be of order of magnitude 1, there is no reason to expect the predictions of the models to agree.

Phase shifts for the two models are shown in Fig. 4, for typical values of m and f. The phase shifts given by Eq. (17) are consistently larger than those given by (18) indicating that multiple ρ-exchange diagrams give rise to more attraction.

We are primarily interested in searching for self-consistent parameters for the ρ. For each value of m we fix the input coupling constant f so that there is a resonance at $s_r=m^2$. The output coupling constant is related to the width of the resonance by[9]

$$\Gamma=8f[(m^2/4-1)^{3/2}/3m^2]. \tag{19}$$

In Fig. 5 we have plotted f_{in} and f_{out} versus m^2 for a cutoff of $\Lambda=140$. There are *no* self-consistent parameters. It will be recalled that the (Z-Z) model gives the self-consistent parameters $m=350$ MeV, $f=2.4$. If one increases the cutoff both the input and output coupling constants decrease, and the curves move further apart. By decreasing the cutoff to $\Lambda=70$, one can produce a self-consistent ρ of mass 2. The mass will increase as the cutoff is decreased further, but such small cutoffs do not make sense physically. What has happened here is that for any m the input coupling constant needed to give a self-consistent mass is smaller in our model than in one of (Z-Z) because of the attractive forces arising from multiple ρ exchange. On the other hand, the

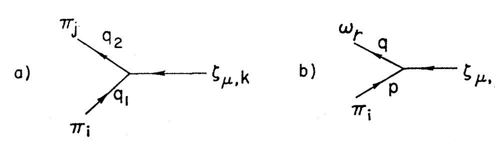

FIG. 3. Vertex diagrams for pion-pion calculation.

[9] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962); to be called (Z-Z).

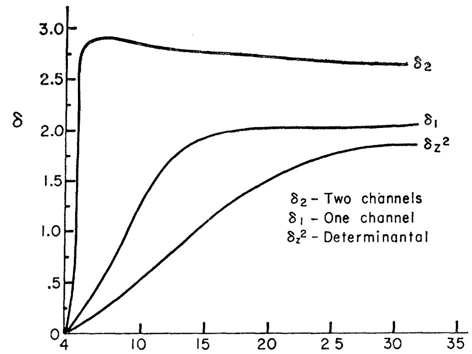

FIG. 4. Phase-shift comparison.

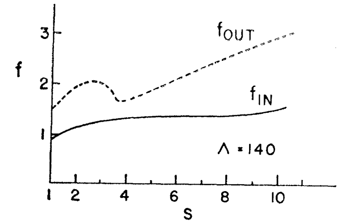

FIG. 5. Input and output coupling constants versus self-consistent mass for one-channel case.

widths of the resonances in our model have not decreased appreciably from the determinantal approximation, so self-consistency is no longer possible.[10,11]

It has been pointed out by several authors[9,12] that the $\pi\omega$ channel should be extremely important for the formation of the ρ since its threshold is only slightly above the physical ρ mass. One of the principal difficulties with all of the bootstrap calculations is that the coupling constants are always too large. The introduction of a closed channel will decrease the input coupling constant since such a channel always gives rise to an attractive force.[4] The introduction of a closed channel will also decrease the output coupling constant if the resonance is near the threshold of the closed channel. To see this we recall that any two-channel problem is equivalent to a one-channel problem with the effective potential

$$W=V_{11}+V_{12}g_2V_{21}, \tag{20}$$

where g_2 is the full Green's function in the second channel and V_{ij} is the 2×2 potential matrix. We now introduce the effective potential

$$W(X)=V_{11}+XV_{12}g_2V_{21} \tag{21}$$

which interpolates between the one- and two-channel problems as X varies from 0 to 1. Now

$$d\delta(X)/dX=-\langle\psi(X)|V_{12}g_2V_{21}|\psi(X)\rangle, \tag{22}$$

where $\delta(X)$ is the phase shift for the potential $W(X)$ and $\psi(X)$ is the wave function.[4] Differentiating Eq. (22) with respect to energy we find *near* the threshold of channel two, where we can neglect the energy dependence of the ψ's,

$$d^2\delta(X)/dEdX\approx\langle\psi(X)|V_{12}g_2g_2V_{21}|\psi(X)\rangle \tag{23}$$

so that

$$d\delta(1)/dE-d\delta(0)/dE\gtrsim0. \tag{24}$$

Since the width of a resonance is inversely proportional to $d\delta/dE$, we see that the introduction of a closed channel will decrease the width of a resonance which lies close to its threshold if the other parameters are held fixed.

Following (Z-Z) we take the $\pi\rho\omega$ vertex to be [see Fig. 3(b)]

$$\gamma_{\pi\rho\omega}\delta_{ij}\epsilon_{\mu\nu\gamma\sigma}q_\sigma p_\lambda. \tag{25}$$

Restricting ourselves to two-particle intermediate states, we have a two-channel problem with the potential matrix given by the ρ-exchange diagrams shown in Fig. 6. It is again necessary to introduce a cutoff in order to have a well-defined model. The projection of the potential matrix onto the $J=T=1$ state is cumbersome and we will not display the result here. The resulting integral equations have been solved numerically.

In searching for a self-consistent ρ we have taken the ω mass from experiment. We then ask if the input and output values of the ρ mass, f and $g=\gamma_{\delta\pi\omega}/4\pi$ can be equal.

For cutoffs $\Lambda=140$ and $\Lambda=280$ it is impossible to satisfy all three conditions simultaneously. The difficulty is that the ρ width is still too large. In order to have $f_{\mathrm{in}}=f_{\mathrm{out}}$, it is necessary for the ρ to be very close to the $\pi-\omega$ threshold. However, in this region it is impossible to have $g_{\mathrm{in}}=g_{\mathrm{out}}$.

If the force in the $\pi-\pi$ channel is really well approximated by ρ exchange then it seems necessary to treat the inelastic channels in a more sophisticated manner than was done here in order to achieve self-consistency. Another way to look at the problem is to fix the ρ mass and coupling constant at their physical values ($m=760$ MeV, $f_{\mathrm{in}}=0.45$) and adjust the $\pi\rho\omega$ coupling constant to give a resonance at the ρ mass. For a cutoff of $\Lambda=140$ this requires $g_{\mathrm{in}}=7.5$ whereas experimentally $g<1$. Thus, the effect of the $\pi-\omega$ channel must be greatly exaggerated, or other inelastic channels must be taken into account in order to produce a resonance at the ρ mass. The cross section for this set of input parameters is plotted as a function of energy in Fig. 7. The ρ width is 147 MeV, which is much closer to the experimental

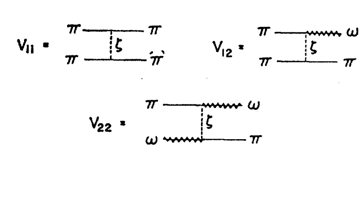

FIG. 6. Potential graphs for two-channel model.

[10] A similar result has been obtained by J. L. Gervais, Paris report (unpublished), who has modified the model of Ref. 5 to take into account contributions of the two-pion exchange diagrams to the left-hand cut. We would like to thank Dr. T. N. Truong for bringing this work to our attention.

[11] J. R. Fulco, G. L. Shaw, and D. Y. Wong, Phys. Rev. 137, B1242 (1965).

[12] R. Blankenbecler, Phys. Rev. 125, 755 (1962).

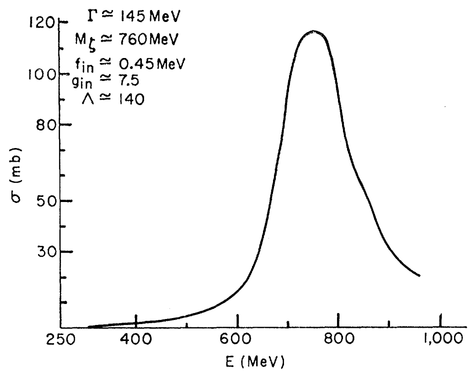

FIG. 7. Resonant pion-pion scattering cross section.

width than previous calculations. Of course, this favorable result may be partly due to the fact that we have adjusted the position of the resonance by overestimating the inelastic coupling constant instead of overestimating the cutoff as was done in Ref. 11. We have thus emphasized the low-energy part of the inelastic channel which, as was explained above, is precisely the part that is most effective in reducing the width of the resonance. However, when the cutoff is increased to $\Lambda=280$, the value of the $\pi\rho\omega$ coupling constant needed to give the experimental ρ mass is reduced to $g_{\text{in}}=3.2$. On the other hand, the ρ mass is only increased to 157 MeV.

A physical picture of this model of the ρ is that the large value of the $\pi\rho\omega$ coupling constant gives rise to a very strong attractive force between the π and the ω due to ρ exchange. This force leads to a bound (or virtual) state in this inelastic channel. If this bound state could be coupled to the $\pi\pi$ channel with a small coupling constant then it would give rise to a very narrow resonance. However, since the coupling between the channels is via ρ exchange, the same large coupling constant is involved and this broadens the resulting resonance as well as shifting its position.

DISCUSSION

The type of equation presented here seems to have several advantages which may prove useful in selected applications. The major disadvantages are the lack of crossing symmetry (a common disease) and the fact that the equations do not form a practical way of summing selected Feymann graphs exactly. The advantages are that the equations are three-dimensional and the solutions satisfy relativistic unitarity even in the multiparticle sector.

They give a practical method for evaluating multiexchange contributions to the left-hand cut (in dispersion language) and also for evaluating the multiparticle contribution to the right-hand cut, or to the optical potential if one prefers to use this language. The structure of the equations allows one to construct the effective two-body potential W_{22} by including multiparticle states in a nonperturbative manner. This is certainly true if one uses the methods discussed in Ref. 4. These equations can be used to discuss bound states and resonant states in many-particle systems.

There are several obvious problems to which one could apply these methods. One is an estimate of the four-pion continuum contributions to the ρ meson. Another problem of theoretical interest is to examine the properties of Regge poles and residues on the basis of rather realistic relativistic models with inelasticity present. Another possible application is to high-energy diffraction scattering and the effect of unitarity on the low partial waves. One could use the equations presented here in a phenomenological-optical-model analysis of both elastic and inelastic scattering.

Finally, we remark that these equations can be extended to reactions involving fermions. The main difference is that one must use the natural fermion variable $\sqrt{s}$ instead of s in the Green's functions and include the fermion projection operators.[13]

[13] F. Gilman (private communication).