Radiative Corrections to $K_{l3}^{\pm}$ Decays*

EDWARD S. GINSBERG
Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania
(Received 27 September 1965)

The radiative corrections to the lepton spectrum and decay rate for $K_{l3}^{\pm}$ decays are calculated, assuming a phenomenological weak K-π vertex and using perturbation theory. The answer depends logarithmically on a cutoff. The result applied to pion beta decay yields a fractional change in lifetime of -1.2% and is not sensitive to reasonable variations in the cutoff. The radiative correction to the electron spectrum from $K_{e3}^{\pm}$ decays at rest is substantial, averaging about 5% in absolute magnitude over most of the measurable energy range, and is insensitive to the cutoff. The correction to the $K_{e3}^{\pm}$ lifetime, the $K_{\mu3}^{\pm}$ lifetime and muon spectrum is probably small (a fraction of a percent) although the numerical estimates are sensitive to the cutoff.

THE study of the three-body leptonic decays of K mesons can furnish valuable information about the nature of the strangeness-changing weak interactions. The coarser features of these decays appear to be rather well established experimentally.[1–3] There is agreement with the presence of only $V-A$ interactions, μ-e universality, lepton locality, etc. However, more accurate experimental knowledge of decay spectra and branching ratios than is presently available would be desirable, particularly for the determination of the form factors involved. It is to be expected that sufficiently detailed measurements of these decays will require an estimate of the radiative corrections. In this paper we will describe a calculation of such radiative corrections. Recent experiments[4] have sought an accurate measurement of the lepton momentum spectrum; thus we shall concentrate on the radiative correction to the lepton spectrum and to the decay rate.

We adopt the point of view that a complete phenomenological description of the K-$\pi l\nu$ vertex in $K_{l3}^{\pm}$ decay is given by an appropriate strangeness-changing weak Hamiltonian, or equivalently, by the corresponding transition matrix element. We then calculate the radiative corrections using ordinary perturbation theory. This approach ignores the possible effect of electromagnetic corrections to the strong-interaction renormalization graphs. However, there is no satisfactory technique at present to calculate such effects. An unfortunate feature of the simple perturbation-theory approach is that the radiative corrections turn out to be cutoff-dependent. This fact has its origin in the nonrenormalizability of the conventional theory of weak interactions. Thus, the usefulness of the present calculation is open to question. Nevertheless, we can estimate the numerical value of the radiative corrections using a "reasonable" value of the cutoff, provided the estimate is relatively insensitive to variations in the cutoff. (This is customary in nucleon leptonic decay, where the radiative corrections are also cutoff-dependent.)

Let us begin by writing down the transition matrix element for the zero-order K_{l3}^{+} process (Fig. 1a)

$$\mathfrak{M}_0=-i(2\pi)^{-2}m_l^{1/2}(4E_KE_\pi E_l)^{-1/2}\delta^{(4)}(p_K-p_\pi-p_l-p_\nu)$$
$$\times[(p_K+p_\pi)_\alpha f_+ +(p_K-p_\pi)_\alpha f_-]\bar{u}_\nu\gamma_\alpha\tfrac{1}{2}(1-i\gamma_5)v_l. \quad (1)$$

Here, the mass m, energy E, and four-momentum p of the various particles are denoted by appropriate subscripts, u and v are Dirac spinors. The form factors $f_\pm$ are, for locally coupled leptons, functions of the invariant four-momentum transfer $(p_K-p_\pi)^2$. They are analytic in the cut plane with a cut starting at $(m_K+m_\pi)^2$. There is some justification in assuming that $f_\pm$ are slowly varying functions since the physical region of momentum transfer, $m_l^2\leq(p_K-p_\pi)^2\leq(m_K-m_\pi)^2$ is relatively far removed from the cut.[5,6] Experimental data thus far are consistent with constant form factors. Since the radiative correction is itself small we can safely neglect any small energy dependence of the form factors in calculating it. (These remarks apply *a fortiori* to π_{e3}).

Concerning the normalization of $f_\pm$, in the limit of unitary symmetry, the form of the weak $K-\pi$ vertex is the same as the weak $\pi-\pi$ vertex, the latter being given by the conserved-vector-current hypothesis.[7] Assuming also the octet hypothesis of Cabibbo,[8] one has

$$f_+\rightarrow 2\times\tfrac{1}{2}\times G^V{}_\beta\tan\theta\,,\quad \xi\equiv f_-/f_+\rightarrow 0 \qquad (2)$$

where $G^V{}_\beta$ is the weak coupling constant determined from O^{14} decay, and θ is the Cabibbo angle. The factor of 2 arises from the definition[7] of $\mathfrak{M}_0$, while the $\frac{1}{2}$ is an

* Supported by the National Science Foundation.
[1] G. L. Jensen, F. S. Shaklee, B. P. Roe, and D. Sinclair, Phys. Rev. **136**, B1431 (1964) (Refs. 1 and 2 contain references to other experimental work).
[2] V. Bisi, G. Borreani, A. Marzari-Chiesa, G. Rinaudo, M. Vigone, and A. E. Werbrouck, Phys. Rev. **139**, B1068 (1965).
[3] D. Luers, I. S. Mittra, W. J. Willis, and S. S. Yamamoto, Phys. Rev. **133**, B1276 (1964).
[4] P. T. Eschstruth, R. Cester, E. B. Hughes, G. K. O'Neill, B. Quassiati, D. Yount, J. Dobbs, A. K. Mann, W. K. McFarlane, and D. H. White, Bull. Am. Phys. Soc. **10**, 466 (1965).
[5] P. Dennery and H. Primakoff, Phys. Rev. **131**, 1334 (1963).
[6] Other theoretical discussions of the form factors are contained in the following works. A. Pais and S. B. Treiman, Phys. Rev. **105**, 1616 (1957); S. W. MacDowell, *ibid* **116**, 1047 (1959). N. Brene, L. Egardt and B. Qvist, Nucl. Phys. **22**, 553 (1961). R. H. Dalitz, in *Proceedings of the International School of Physics "Enrico Fermi"* [*Villa Monastero, Varenna, Como, Italy, 1964*] (to be published).
[7] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[8] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).

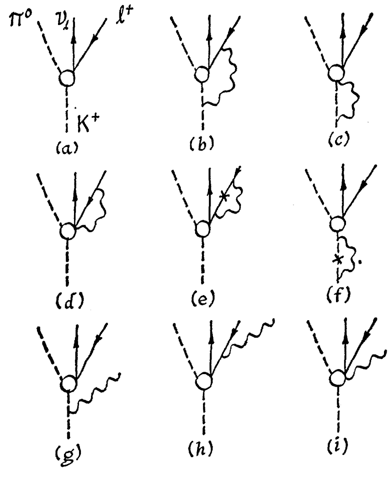


FIG. 1. Feynman diagrams for the lowest order radiative corrections to K_{l3}^+ decay. (a) zero-order diagram; (b)–(f) virtual diagrams; (g)–(i) inner bremsstrahlung.

$SU(3)$ Clebsch-Gordan coefficient. For the actual decay, symmetry-breaking interactions associated with the difference of m_K and m_π can introduce a nonzero f_- term. Thus, for example, consideration of such interactions together with the assumption of K^* dominance of the $K-\pi$ vertex leads to a value for ξ of $-(m_K^2-m_\pi^2)/m_{K^*}^2\approx-0.31$. Experimentally the exact value of ξ is still uncertain, but it is believed to be small.

Finally, a word should be said about time-reversal invariance. If T is conserved f_+ and f_- are relatively real; if not ξ is complex. The decay rate and spectra are relatively insensitive to the imaginary part of ξ (Imξ enters only in $|\xi|^2$); however, Imξ enters directly into the component of the lepton polarization perpendicular to the decay plane.[5,9] As yet, there is no experimental indication of a nonzero Imξ.

We now turn to the calculation of the radiative correction to K_{l3}^+ decay.[10] The lowest order perturbation diagrams are shown in Fig. 1[(b)–(i)]. As mentioned earlier, we treat the weak $K-\pi$ vertex phenomenologically, neglecting possible electromagnetic corrections to the strong-interaction renormalization graphs. The amplitudes for the diagrams in Fig. 1[(c), (d), and (i)] are given by the gauge-invariant substitution $p_\alpha\rightarrow p_\alpha-eA_\alpha$.

The zero-order matrix element, Eq. (1), can be rewritten as

$$\mathfrak{M}_0=-if_+(2\pi)^{-2}m_l^{1/2}(4E_KE_\pi E_l)^{-1/2}\times\delta^{(4)}(p_K-p_\pi-p_l-p_\nu)\bar{u}_\nu M_0\tfrac{1}{2}(1-i\gamma_5)v_l\,,\quad(3)$$

where

$$M_0=2p_\pi\cdot\gamma+(1+\xi)p_l\cdot\gamma\,.\quad(4)$$

The evaluation of the matrix element arising from the virtual diagrams is straightforward and, with a definition similar to Eq. (3), is found to be

$$M_{\text{virtual}}=2p_\pi\cdot\gamma\tfrac{1}{2}A+(1+\xi)p_l\cdot\gamma\tfrac{1}{2}B\,,\quad(5)$$

where

$$A=(\alpha/\pi)[\tfrac{3}{2}\ln(\Lambda/ml)-1-2\ln(\lambda/m_l)+t_1[-\tfrac{1}{2}t_2m_l^2(1+\xi)(p_K-p_l)^2]\,,\quad(6)$$

and

$$B=(\alpha/\pi)[-\tfrac{3}{2}\ln(\Lambda/ml)-(7/4)-2\ln(\lambda/m_l)+t_1-2t_2/(1+\xi)]\,.\quad(7)$$

In Eqs. (6) and (7), Λ is the ultraviolet cutoff, which is introduced as a Feynman regulator for the photon propagator, and λ is the "fictitious" photon mass. The infrared divergence associated with the latter will be exactly cancelled when the contribution of the inner bremsstrahlung diagrams is included. For simplicity, we will only give expressions for t_1 and t_2 in the center-of-mass system $[p_K=(0,m_K)]$ which we will use henceforth. The lepton energy and (three-) momentum will be denoted by E and p.

$$t_1=\frac{E}{p}\left\{\mathrm{Li}_2\left(\frac{2p}{m_K-E+p}\right)-\mathrm{Li}_2\left(\frac{2m_Kp}{m_K(E+p)-m_l^2}\right)-\ln\left(\frac{m_K-E+p}{m_K-E-p}\right)\ln\left(\frac{E+p}{m_K}\right)+\left[2\ln\frac{\lambda}{m_l}+1-\ln\left(\frac{E+p}{m_l}\right)\right]\ln\left(\frac{E+p}{m_l}\right)\right\},\quad(8)$$

$$t_2=[(m_K-E)/p]\ln[(E+p)/m_l]-\ln(m_K/m_l)\,.\quad(9)$$

In Eq. (8) the function $\mathrm{Li}_2(x)$ is the dilogarithm (the negative of the Spence function) defined by[11]

$$\mathrm{Li}_2(x)\equiv-\int_0^x\frac{\ln(1-z)}{z}dz=\sum_{n=1}^{\infty}\frac{x^n}{n^2}\,.\quad(10)$$

[9] N. Cabibbo and A. Maksymowicz, Phys. Letters **9**, 352 (1964).
[10] In this paper we consider the decay of charged kaons only. We present an explicit calculation for positively charged kaons although the results apply also to negative kaons. Radiative corrections to the three-body leptonic decays of neutral kaons will be the subject of a future paper.
[11] L. Lewin, *Dilogarithms and Associated Functions* (MacDonald, London, 1958).

From the above equations we obtain the lepton energy spectrum in the c.m. system (i.e., for decays at rest):

$$\left(\frac{d\Gamma}{dE}\right)_0=\frac{|f_+|^2}{(2\pi)^3}\frac{p}{2m_K}\left(1-\frac{m_\pi^2}{H^2}\right)^2\left\{p_l\cdot H\left(H^2+\frac{m_l^2|1+\xi|^2}{4}\right)+m_l^2H^2\,\mathrm{Re}(1+\xi)\right\}, \tag{11}$$

$$\left(\frac{d\Gamma}{dE}\right)_{\text{virtual}}=\frac{|f_+|^2}{(2\pi)^3}\frac{p}{2m_K}\left(1-\frac{m_\pi^2}{H^2}\right)^2\left\{p_l\cdot H\left(H^2A+\frac{m_l^2|1+\xi|^2B}{4}\right)+\tfrac{1}{2}m_l^2H^2\,\mathrm{Re}[(1+\xi)(A^*+B)]\right\}, \tag{12}$$

where

$$m_\pi^2\leqslant H^2\equiv(p_K-p_l)^2=m_K^2-2m_KE+m_l^2\leqslant(m_K-m_l)^2. \tag{13}$$

It now remains to calculate the contribution from the inner bremsstrahlung (IB), diagrams (g), (h), and (i) in Fig. 1. The IB differential decay rate is found using standard techniques to be (c.m. system)

$$d\Gamma_{\text{IB}}=\frac{\alpha|f_+|^2}{(2\pi)^6}\frac{d\mathbf{p}d\mathbf{q}}{8m_KEq_0}\left(1-\frac{m_\pi^2}{G^2}\right)^2\left\{\left(2G^2+\frac{m_l^2|1+\xi|^2}{2}\right)\frac{q\cdot G}{q\cdot p_l}\right.$$
$$\left.+\sum_\epsilon\left(\frac{p_l\cdot\epsilon}{p_l\cdot q}\right)^2\left[\left(G^2+\frac{m_l^2|1+\xi|^2}{4}\right)(m_K^2-m_l^2-G^2)+2m_l^2G^2\,\mathrm{Re}(1+\xi)\right]\right\}, \tag{14}$$

where

$$m_\pi^2\leqslant G^2\equiv(H-q)^2\leqslant H^2, \tag{15}$$

and $q=(q_0,\mathbf{q})$ is the photon four-momentum.

As no provision is made experimentally to detect inner bremsstrahlung photons we will integrate Eq. (14) over all kinematically compatible photon momenta. Care must be exercised in allowing the photon a "fictitious" mass λ when performing the integrations and the polarization sum. After a somewhat lengthy calculation we obtain the result

$$\left(\frac{d\Gamma}{dE}\right)_{\text{IB}}=\frac{|f_+|^2}{(2\pi)^3}\frac{p}{2m_K}\frac{\alpha}{\pi}\left\{\left[p_l\cdot H\left(H^2+\frac{m_l^2|1+\xi|^2}{4}\right)+m_l^2H^2\,\mathrm{Re}(1+\xi)\right]I_0\right.$$
$$+\frac{I_1}{2}\left[\frac{(H^2-m_\pi^2)^2}{2}-m_\pi^2(H^2-m_\pi^2)+m_\pi^4\ln\frac{H^2}{m_\pi^2}-\left(2p_l\cdot H+2m_l^2\,\mathrm{Re}(1+\xi)-\frac{m_l^2|1+\xi|^2}{4}\right)\right.$$
$$\left.\times\left(\frac{H^4-m_\pi^4}{H^2}-2m_\pi^2\ln\frac{H^2}{m_\pi^2}\right)\right]+\frac{I_2}{4m_Kp}\left[\frac{(H^2-m_\pi^2)^3}{6}-\frac{m_\pi^2(H^2-m_\pi^2)^2}{2}-m_\pi^4\left(H^2-m_\pi^2-H^2\ln\frac{H^2}{m_\pi^2}\right)\right.$$
$$\left.\left.+\frac{m_l^2|1+\xi|^2}{4}\left(\frac{(H^2-m_\pi^2)^2}{2}+3m_\pi(H^2-m_\pi^2)-m_\pi^2(2H^2+m_\pi^2)\ln\frac{H^2}{m_\pi^2}\right)\right]\right\}, \tag{16}$$

where,

$$I_0=\left(1-\frac{m_\pi^2}{H^2}\right)^2\left[I_1\left(\ln\frac{H^2(H^2-m_\pi^2)}{m_\pi^2m_Km_l}-1-\ln\frac{\lambda}{m_l}\right)-I_3-I_4+I_5\right]+I_1\left(1-\frac{m_\pi^2}{H^2}-\ln\frac{H^2}{m_\pi^2}\right), \tag{17}$$

and

$$I_1=2[(E/p)\ln[(E+p)/m_l]-1], \tag{18}$$

$$I_2=\ln[(E+p)/m_l]+\tfrac{1}{2}\ln[(m_K-E+p)/(m_K-E-p)], \tag{19}$$

$$I_3=(E/p)[\tfrac{1}{2}\,\mathrm{Li}_2[(E-p)/2E]-\tfrac{1}{2}\,\mathrm{Li}_2[(E+p)/2E]+\ln[(E+p)/m_l]\ln(2E/m_l)], \tag{20}$$

$$I_4=(E/p)[\mathrm{Li}_2[(E-p)/m_K]-\mathrm{Li}_2[(E+p)/m_K]]+1-\ln(H^2/m_K^2)$$
$$+(m_l^2/m_Kp)\ln[(E+p)/m_l]-[(m_K^2-m_l^2)/2m_Kp]\ln[(m_K-E+p)/(m_K-E-p)], \tag{21}$$

$$I_5=1+(E/p)[\ln[(E+p)/m_l]-\mathrm{Li}_2(p/E)+\mathrm{Li}_2(-p/E)]. \tag{22}$$

Combining Eqs. (11), (12), and (16) gives the leptonenergy spectrum

$$d\Gamma/dE=(d\Gamma/dE)_0+(d\Gamma/dE)_{\text{virtual}}+(d\Gamma/dE)_{\text{IB}}\equiv(d\Gamma/dE)_0+(d\Gamma/dE)_{\text{RC}}, \tag{23}$$

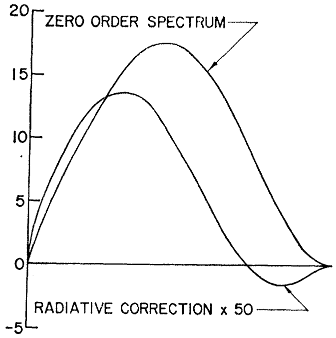


FIG. 2. $\pi_{e3}^{\pm}$ (pion beta-decay): zero-order electron spectrum and radiative corrections for $\Lambda=m_p$. In Figs. 2–6 the abscissa is the lepton energy, which runs from $E_{\min}=m_l$ to $E_{\max}=(m_K^2+m_l^2-m_\pi^2)/(2m_K)$; the ordinate is proportional to $d\Gamma/dE$.

where the last two terms represent the radiative correction of order α. It is to be noted that the infrared divergent terms in Eqs. (12) and (16) exactly cancel, as they should. However, the result still contains a dependence on the ultraviolet cutoff Λ.

Equation (23) is directly applicable to experiments which measure the shape of the lepton energy or momentum spectrum, and we have analyzed the various terms in it numerically, by computer. The results are presented below. It is also of some theoretical interest to calculate the radiative correction to the decay rate Γ (or equivalently the lifetime $\tau\equiv 1/\Gamma$) which we obtain by integrating Eq. (23) over E. In principle, this could be done analytically (as is easily done with the zero-order spectrum[12]) however, since the total correction is small, and in any case, not unambiguous due to the presence of the cutoff, a numerical integration is more than adequate. In the Appendix we give the result of an analytic integration of Eq. (23) in the limit of vanishing lepton mass. (This limit applies to K_{e3}^+ decays quite accurately, but much less accurately to π_{e3}^+ decays. In π_{e3}^+ decay approximation $E=p$ is poor since $E_{\max}$ is only 4.5 MeV.)

We should mention that radiative corrections to pion beta decay have been published twice previous to the present effort.[13] However, both of these previous papers are marred by misprints, and are further limited by several approximations. Dr. Chang has kindly supplied us with the relevant parts of his thesis, and his unpublished result is in agreement with the $m_l\to 0$ case given in the Appendix. This provided a valuable check on our work.

We now turn to the numerical evaluation of the terms in Eq. (23) and the integral of these over the lepton energy. These results (largely self-explanatory) are presented in Figs. (2) through (6). It is convenient to discuss the three cases, π_{e3}^+, K_{e3}^+, and $K_{\mu3}^+$ separately.

The radiative corrections to the electron spectrum in pion beta decay are shown in Fig. 2. As this decay is quite rare, only the correction to the lifetime is of experimental interest at present. The fractional change in the lifetime is

$$\Delta\tau/\tau=(\tau-\tau_0)/\tau=1-\Gamma/\Gamma_0=-\Gamma_{\rm RC}/\Gamma_0. \qquad (24)$$

Some numerical values of the fractional change in lifetime for different values of the cutoff Λ are given below:

$\Delta\tau/\tau$	Λ
-1.1%	$m_\rho=769$ MeV
-1.2%	$m_p=938$ MeV
-1.4%	$2m_p$

The dependence of the radiative correction on the cutoff Λ is quite transparent from Eqs. (11) and (12), where Λ occurs only in A and B. In the approximation $m_e\to 0$ the effect of varying Λ from some initial value, say one proton mass m_p, is to add a multiple of the zero-order spectrum, namely

$$((3\alpha/2\pi)\ln(\Lambda/m_p))(d\Gamma/dE)_0 \simeq (0.0035\ln(\Lambda/m_p))(d\Gamma/dE)_0. \qquad (25)$$

For $\Lambda=2m_p$ this is just 0.24% of the zero-order spectrum. The effect on the fractional change in lifetime is just the negative of the above factor, i.e.,

$$\Delta\tau/\tau=(\Delta\tau/\tau)_{\Lambda=m_p}-(3\alpha/2\pi)\ln(\Lambda/m_p), \qquad (26)$$

again, about 0.24% for $\Lambda=2m_p$. It is interesting to note that the same "universal" term is present in the radiative correction[14] for nucleon leptonic decay. From Eqs. (A8) and (A1) we see that this term is approximately

$$-(3\alpha/2\pi)\ln(m_p/\eta m_{\pi^+})\to-(3\alpha/2\pi)\times\ln(m_p/E_{\max})+O(\alpha/\pi), \qquad (27)$$

where $E_{\max}$ is the end-point energy. The contribution from the "universal" term is about -1%. The results for this case agree with the earlier results of Chang,[13] and were useful in debugging our computer program.

The radiative correction to the electron spectrum from K_{e3}^+ decays at rest is shown in Fig. 3 and in the

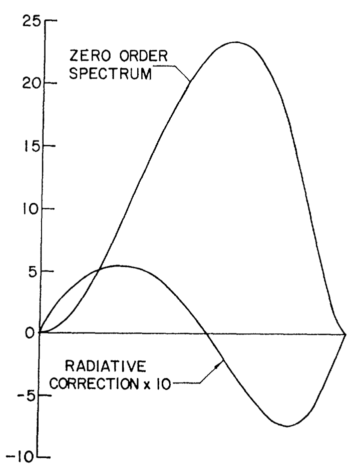


FIG. 3. $K_{e3}^{\pm}$: zero-order electron spectrum and radiative corrections for $\Lambda=m_p$.

[12] A. Fujii and M. Kawaguchi, Phys. Rev. **113**, 1156 (1959).

[13] N.-P. Chang, Phys. Rev. **131**, 1272 (1963). G. Da Prato and G. Putzolu, Nuovo Cimento **21**, 541 (1961).

[14] S. M. Berman and A. Sirlin, Ann. Phys. (N.Y.) **20**, 30 (1962).

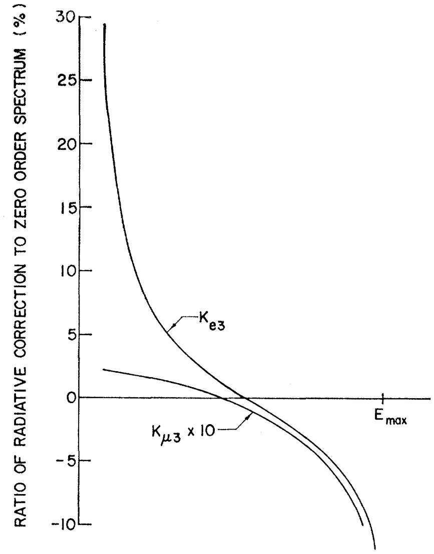


FIG. 4. Ratio of radiative corrections to zero-order lepton spectrum for $\Lambda = m_p$. The $K_{\mu 3}^{\pm}$ spectrum is for $\xi = 0$.

upper curve in Fig. 4. The correction is considerable, averaging about 5% in absolute magnitude for the energy range 50 to 229 MeV. As the above discussion for the π_{e3}^{+} case shows, the spectrum is insensitive to variations in the cutoff over a reasonable range (except where the spectrum crosses through zero). Doubling the cutoff (from m_p to $2m_p$) adds only 0.24% of the zero-order spectrum. The shape of the spectrum due to the radiative corrections is interesting in so far as there is almost equal amounts of positive and negative corrections. In fact, the integral of the radiative correction very nearly vanishes, the fractional change in lifetime for $\Lambda = m_p$ is only $\Delta\tau/\tau = -0.03\%$. This is much smaller than the -0.24% ambiguity resulting from the ultraviolet cutoff. From Eq. (27) we see that in this case the "universal" term is in fact small since $E_{\max}$ is large (229 MeV). Therefore, although the radiative corrections to the electron spectrum are substantial and quite insensitive to reasonable variations in the cutoff, we get a completely ambiguous result for the correction to the

FIG. 5. $K_{\mu 3}^{\pm}$: zero-order muon spectrum and radiative corrections for $\Lambda = m_p$, $2m_p$ and $\xi = 0$.

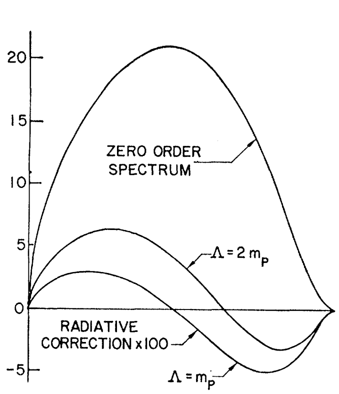
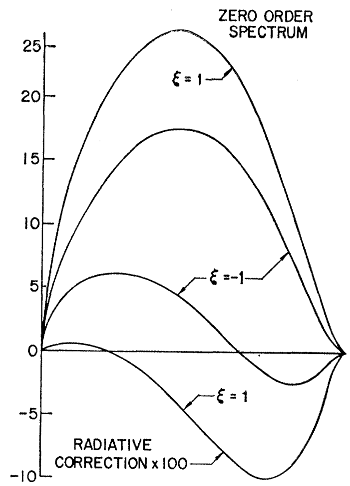


FIG. 6. $K_{\mu 3}^{\pm}$: zero-order muon spectrum and radiative corrections for $\Lambda = m_p$, and $\xi = \pm 1$.



decay rate. The most we can say about the latter is that it is probably small.

The corrections to the $K_{\mu 3}^{+}$ spectrum are shown in Figs. (4)–(6) for various real values of ξ. The correction to the spectrum averages only a fraction of a percent through the entire energy range. It is to be expected that the radiative corrections are much smaller in the $K_{\mu 3}^{+}$ case than in the K_{e3}^{+} case owing to the difference in mass between the muon and the electron whereas the energy available is about the same. In the $K_{\mu 3}^{+}$ case, a variation of Λ from m_p to $2m_p$ will add an amount to the correction comparable to the correction at $\Lambda = m_p$ as shown in Fig. 5. Some values of $\Delta\tau/\tau$ for different values of ξ are given below for $\Lambda = m_p$,

ξ	$\Delta\tau/\tau$ (%)	ξ	$\Delta\tau/\tau$ (%)
1.0	0.25	−0.1	0.025
0.5	0.15	−0.5	−0.053
0.1	0.065	−1.0	−0.19
0.0	0.045		

The sensitivity of these values to the cutoff is great. For $K_{\mu 3}^{+}$ decay we have, analogous to Eq. (26),

$$\Delta\tau/\tau = (\Delta\tau/\tau)_{\Lambda = m_p} - (3\alpha/2\pi) g(\xi) \ln(\Lambda/m_p) , \quad (28)$$

where $g(-1) = 1$ and $g(0) = 0.8$. Therefore, our numerical results for the radiative corrections to $K_{\mu 3}^{+}$ decay are meaningful only as regards to their general order of magnitude.

In conclusion, we have calculated the radiative corrections to $K_{l3}^{\pm}$ decays assuming a phenomenological weak Hamiltonian and using perturbation theory. Our result, applied to pion beta decay, gives a fractional change in lifetime of -1.2% and this is not sensitive to reasonable variations in the cutoff. The radiative correction to the lepton spectrum from $K_{e3}^{\pm}$ decays is substantial, averaging about 5% in absolute magnitude over most of the measurable energy range and is insensitive to the cutoff. The correction to the $K_{e3}^{\pm}$ lifetime, and in general to $K_{\mu 3}^{\pm}$ decays, is probably small (a fraction of a percent) although the numerical estimates are sensitive to the cutoff.

We wish to thank Paul Eschstruth and Dr. E. B. Hughes for encouraging this calculation and for many discussions. A conversation with Dr. N.-P. Chang was also very helpful. We are also grateful to Professor Henry Primakoff for his comments on the manuscript.

APPENDIX

In this Appendix we consider the radiative corrections in the limit $m_l \to 0$ analytically. With the definitions

$$\eta = 1-(m_\pi^2/m_K^2)\,, \quad \text{and} \quad x=E/E_{\max}\,, \tag{A1}$$

Eqs. (11), (12), and (16) become

$$\left(\frac{d\Gamma}{dx}\right)_0 = \frac{|f_+|^2}{4\pi^3}\left(\frac{m_K}{2}\right)^5 \frac{\eta^5 x^2(1-x)^2}{(1-\eta x)}\,, \tag{A2}$$

$$\begin{aligned}\left(\frac{d\Gamma}{dx}\right)_{\mathrm{RC}} = \frac{|f_+|^2}{4\pi^3}\left(\frac{m_K}{2}\right)^5 \frac{\alpha}{\pi}\Bigg\{ &\frac{\eta^5 x^2(1-x)^2}{(1-\eta x)}\left[\tfrac{3}{2}\ln\frac{\Lambda}{m_l}-1+2\,\mathrm{Li}_2(\eta x)-\frac{\pi^2}{3}+I_1\left(\ln\frac{(1-x)(1-\eta x)}{x(1-\eta)}-1\right)+\ln\eta x\,\ln(1-\eta x)\right]\\ &-\eta^4 x(1-x)^2\ln(1-\eta x)+\eta^2 x I_1\left[\frac{(1-\eta)^2}{1-\eta x}-1+\eta+\frac{\eta^2}{2}(1-x)^2+\{(1-\eta)^2+\eta x(1-2\eta)+\eta^2x^2\}\ln\left(\frac{1-\eta x}{1-\eta}\right)\right]\\ &+\eta I_2\left[\frac{\eta^3}{6}(1-x)(x^2-5x-2)+\frac{\eta^2}{2}(1-x)(3+x)-\eta(1-x)+(1-\eta)^2(1-\eta x)\ln\left(\frac{1-\eta x}{1-\eta}\right)\right]\Bigg\}\,,\end{aligned} \tag{A3}$$

where

$$I_1 = 2[\ln(m_K\eta x/m_l)-1]\,, \tag{A4}$$

and

$$I_2 = \ln(m_K\eta x/m_l)-\tfrac{1}{2}\ln(1-\eta x)\,. \tag{A5}$$

The integral of Eq. (A2) gives the zero-order decay rate

$$\Gamma_0 = (|f_+|^2/4\pi^3)(m_K/2)^5 I\,, \quad I \equiv -(1-\eta)^2\ln(1-\eta)-\eta+\tfrac{3}{2}\eta^2-\tfrac{1}{3}\eta^3-\tfrac{1}{12}\eta^4\,. \tag{A6}$$

Then integral of Eq. (A3) gives the lowest order radiative correction to Γ_0, and can be written in the form

$$\Gamma_{\mathrm{RC}} = (\alpha/\pi)\Gamma_0\cdot I_{\mathrm{RC}}\,, \tag{A7}$$

where

$$\begin{aligned}I_{\mathrm{RC}} = \tfrac{3}{2}\ln\frac{\Lambda}{m_K\eta}-\tfrac{2}{3}\pi^2-\tfrac{1}{2}+\frac{1}{I}\Bigg\{ &-\frac{343}{72}\eta+\frac{173}{24}\eta^2-\frac{125}{72}\eta^3-\frac{21}{64}\eta^4\\ &+(1-\eta)^2\left[3\,\mathrm{Li}_3(1-\eta)-3\,\mathrm{Li}_3(1)-\left(\frac{\pi^2}{3}+\mathrm{Li}_2(1-\eta)\right)\ln(1-\eta)\right]+\left(-\frac{85}{72}+\frac{23}{18}\eta+\frac{3}{4}\eta^2-\frac{25}{36}\eta^3-\frac{11}{72}\eta^4\right)\ln(1-\eta)\\ &+\left(\frac{43}{12}-\frac{52}{6}\eta+6\eta^2-\frac{2}{3}\eta^3-\frac{1}{6}\eta^4\right)\mathrm{Li}_2(\eta)+\left[\left(\frac{25}{12}-\frac{14}{3}\eta+3\eta^2-\frac{1}{3}\eta^3-\frac{1}{12}\eta^4\right)\ln(1-\eta)\right.\\ &\left.+\frac{25}{12}\eta-\frac{25}{8}\eta^2+\frac{31}{36}\eta^3+\frac{13}{144}\eta^4\right]\ln\eta\Bigg\}\,.\end{aligned} \tag{A8}$$

In Eq. (A8) $\mathrm{Li}_3(x)$ denotes the trilogarithm function defined by[11]

$$\mathrm{Li}_3(x) \equiv \int_0^x \frac{\mathrm{Li}_2(z)}{z}dz = \sum_{n=1}^{\infty}\frac{x^n}{n^3}\,. \tag{A9}$$

The fractional change in lifetime due to the lowest order radiative corrections is just

$$\Delta\tau/\tau = -(\alpha/\pi)I_{\mathrm{RC}}\,. \tag{A10}$$