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Relativistic Corrections to the Impulse Approximation in Elastic Electron-Deuteron Scattering*

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Equations are derived for the deuteron form factor in the impulse approximation which include corrections to order M^{-2}, where M is the nucleon mass. The corrections are of two types, arising from (1) the expansion of the nucleon current to order M^{-2} and (2) the treatment of the deuteron wave function in a relativistic manner so as to retain terms of order M^{-2} which describe the distortion of the wave function of a moving deuteron. While some of the former corrections have been discussed by several people, detailed results for the latter corrections have not been given before. All of the results are expressed as functionals of the nonrelativistic S- and D-state wave functions, so that numerical results may be calculated for any deuteron model. Numerical estimates of the charge correction term indicate that it is large enough to substantially affect the determination of the neutron-charge form factor from this experiment, making the form factor more positive than results obtained from the uncorrected theory. Corrections to the magnetic moment are also discussed.

1. INTRODUCTION AND SUMMARY

IN this paper we derive equations for the deuteron form factor in the impulse approximation, accurate to order M^{-2}, where M is the nucleon mass. The method we use for treating loosely bound systems relativistically has been discussed in an earlier paper,[1] and most of the discussion has been given there. We will however indulge in some further discussion in Sec. 4.

The principal idea behind the method is to rearrange the perturbation series for the deuteron form factor so that all those terms which differ only in their contribution to the deuteron or nucleon structure are lumped together, and the proper vertex function which results is then either taken from experimental data (as is the nucleon form factor) or determined from some model, as are the deuteron wave functions. One then has a reduced perturbation series, in which just those diagrams which are irreducible (i.e., cannot be partitioned into nontrivial pieces by cutting only 2 nucleon lines) are included and all point vertices are to be replaced by proper "bubbles" and all internal propagators are to be regarded as "dressed".

The impulse approximation is then just the first term in this series (see discussion in Sec. 4 and Fig. 1), and a method for calculating it approximately was discussed in A. This involved examining the singularity structure in the internal energy, and eliminating this troublesome variable by integrating over it and retaining only the largest term, which one finds is accurate to order M^{-2}.

In earlier attempts to treat the deuteron form factor relativistically we made use of dispersion relations.[2] This approach is very elegant, and gives one interesting insights into the behavior of loosely bound systems. In particular, it demonstrates the importance of the singularity structure of the Feynman diagrams, and lead eventually to the method being used in this paper. However, one of the properties of the dispersion method is that it tends to mix up the internal structure of the interacting particles (deuteron wave functions and nucleon form factors in this case) with the over-all interaction (the deuteron current). Since the calculation of the general form of the deuteron current can be expected to be easier and more reliable than the calculation of deuteron wave functions, it is convenient to express the final results in terms of the usual S- and D-state wave functions, so that numerical results can be obtained for any deuteron model. The present formalism is better suited to this task than is the dispersion formalism, where the weight functions of the wave functions enter directly (see I and II) and the role of the wave function tends to be obscured. Finally, interpretation of the dispersion theory results is greatly facilitated by comparison of the formalism with potential theory, so that a formalism closer to potential theory would seem to be a better choice.

Although it may be somewhat hazardous to generalize, our results indicate that there are advantages in a return to relativistic wave functions. A similar point of view has been taken recently by other people,[3] and the use of spectral representations in the relative energy, a feature of the approximate method discussed in A, may be helpful in treating other strong-interaction problems.[4]

In Sec. 2 we present a wave function for a deuteron of total momentum $\mathbf{d}$ and relative internal momentum $\mathbf{r}$. This wave function was obtained in A by reducing a relativistic Bethe-Salpeter–type wave function to

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[1] F. Gross, Phys. Rev. 140, B410 (1965). Referred to as A in the text.

[2] F. Gross, Phys. Rev. 134, B405 (1964) referred to as I; Phys. Rev. 136, B140 (1964) referred to as II.

[3] See for example, C. Schwartz and C. Zemach, Phys. Rev. 141, 1456 (1966).

[4] J. G. Taylor (private communication).

Pauli form and retaining terms up to order M^{-2}. It is phenomenological in that it is determined by requiring that it reduce to the nonrelativistic wave function when $\mathbf{d}=0$, and hence it may be used with any deuteron model.

In Sec. 3 this wave function is used with the relativistic nucleon current to derive expressions for the deuteron form factor. In Sec. 4, we estimate the size of the charge corrections to the form factor by calculating the slope of the correction term at $q^2=0$. This slope is of the right sign to make the neutron-charge form factor more positive (in agreement with thermal neutron data), and is large enough to substantially effect the results. We also estimate the correction to the static magnetic moment, and discuss the accuracy of the formalism.

For convenience, we outline our principal results here. The cross section[5] for elastic electron-deuteron scattering is:

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\Omega}\bigg|_{\text{N.S.}}\bigg\{G_C^2+\frac{q^2}{18M_d^2}G_Q^2-\frac{q^2}{6M_d^2}\times\left[1+2\left(1+\frac{q^2}{4M_d^2}\right)\tan^2\theta/2\right]G_M^2\bigg\}, \quad (1.1)$$

where the 3 form factors G_C, G_Q, and G_M are invariant functions of the 4-momentum transfer $q^2=q_0^2-\mathbf{q}^2$, which are defined in terms of the relativistic deuteron current (see II). However, if we work in the Breit frame and reduce the current to nonrelativistic form we have an equally good definition. Introducing $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ as nonrelativistic deuteron polarization vectors,

$$\langle D'|j^0|D\rangle=G_C\boldsymbol{\xi}'^*\cdot\boldsymbol{\xi}+(1/6M_d^2)G_Q[3\boldsymbol{\xi}'^*\cdot\mathbf{q}\boldsymbol{\xi}\cdot\mathbf{q}-\mathbf{q}^2\boldsymbol{\xi}'\cdot\boldsymbol{\xi}] \quad (1.2)$$
$$\langle D'|j^k|D\rangle=(1/2M_d)G_M(\xi^k\boldsymbol{\xi}'^*\cdot\mathbf{q}-\xi'^{*k}\boldsymbol{\xi}\cdot\mathbf{q}),\quad k=1,2.$$

In Sec. 3 we will obtain the following results for the form factors:

$$\eta G_C=F_C\int_0^\infty(u^2+w^2)j_0(\tau)dx+F_CI_C(q^2)+(2F_M-F_C)J_C(q^2),$$
$$\eta G_Q=F_C\frac{6\sqrt{2}M_d^2}{\mathbf{q}^2}\int_0^\infty\left(uw-\frac{w^2}{\sqrt{8}}\right)j_2(\tau)dx+F_CI_Q(q^2)+(2F_M-F_C)J_Q(q^2),$$
$$\tfrac{1}{2}\eta G_M=\left[\left(1+\frac{\mathbf{q}^2}{32M^2}\right)F_C+(F_C-F_M)\frac{\mathbf{q}^2}{16M^2}\right]\frac{3}{4}\int_0^\infty\mathcal{W}^2[j_0(\tau)+j_2(\tau)]dx \quad (1.3)$$
$$+\left(1+\frac{\mathbf{q}^2}{32M^2}\right)F_M\left[\int_0^\infty(\mathcal{U}^2-\tfrac{1}{2}\mathcal{W}^2)j_0(\tau)dx+\frac{1}{\sqrt{2}}\int_0^\infty\left(\mathcal{U}\mathcal{W}+\frac{\mathcal{W}^2}{\sqrt{2}}\right)j_2(\tau)dx\right]$$
$$+F_CI_M{}^1(q^2)+F_MI_M{}^2(q^2)+F_MJ_M{}^1(q^2)+(F_C-F_M)J_M{}^2(q^2),$$

$$\eta=\frac{D_0}{M_d}\cong1+\frac{\mathbf{q}^2}{32M^2}.$$

In Eqs. (1.3), F_C and F_M are the usual isotopic scalar form factors of the nucleon, while j_n is the spherical Bessel function of order n [defined in Eq. (2.9)], $\tau=\frac{1}{2}|\mathbf{q}|x$, $u=u(x)$ and $w=w(x)$ are the S- and D-state wave functions of the deuteron and

$$\mathcal{U}=u(x)+(1/4M^2)u''(x),$$
$$\mathcal{W}=w(x)+(1/4M^2)(w''(x)-(6/x^2)w(x)). \quad (1.4)$$

We use the prime to denote differentiation with respect to x.

The functions I are correction terms of order M^{-2} which arise from the dependence of the deuteron wave functions on the *total* momentum of the deuteron. The J functions are correction terms which originate from corrections to the usual nonrelativistic current. All of these functions are of order M^{-2} with respect to the leading terms. If one neglects all of these terms, then $\mathcal{U}=u$ and $\mathcal{W}=w$ and our results reduce to the standard nonrelativistic results.

A complete tabulation of the rather lengthy functionals I and J in terms of u and w are given in the Appendix. In the Appendix we have also collected together the remaining expressions relevant to this work (the current and the deuteron wave function).

We will now turn to a derivation of the results.

2. THE DEUTERON WAVE FUNCTION

In A, we introduced the following relativistic deuteron wave function:

$$\phi_d(\mathbf{r})=\phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})u_\alpha(\tfrac{1}{2}\mathbf{d}+\mathbf{r})u_\beta{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r})+\phi_{\alpha\beta}{}^{-+}(\mathbf{r},\mathbf{d})v_\alpha(-\tfrac{1}{2}\mathbf{d}-\mathbf{r})u_\beta{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r}), \quad (2.1)$$

where $u_\alpha(\mathbf{p})$ is the free Dirac spinor representing a nucleon of momentum $\mathbf{p}$ and polarization α (normalized

[5] M. Gourdin, Nuovo Cimento 28, 553 (1963).

to unity),[6] $\mathbf{d}$ is the total 3-momentum of the deuteron, and $\mathbf{r}$ is one-half the difference of the 3-momenta of the two nucleons in the deuteron (or simply the "relative" momentum). Summation over repeated indices is implied.

In A we discussed how the functions ϕ^{++} and ϕ^{-+} are related to the four invariant functions of the deuteron-nucleon (d-N) vertex with one nucleon off the mass shell. There we also discussed their physical interpretation; ϕ^{++} is that part of the bound state in which both nucleons have positive energy, and ϕ^{-+} corresponds to the part in which one of the nucleons has negative energy. The corresponding functions ϕ^{+-} and ϕ^{--} would also contribute had we not restricted one of the nucleons to the mass shell. To restrict one nucleon to the mass shell is an approximation which considerably simplifies the calculation, and yet still allows one to get an accurate calculation to order M^{-2} and hence improve the usual nonrelativistic treatment of the deuteron. This has been discussed in great detail in A, and will be discussed further in the following sections. In this section we will take (2.1) as our starting point for a relativistic deuteron wave function and re-express it in terms of the usual S- and D-state deuteron wave functions. Our results will be accurate to order M^{-2}.

The method by which (2.1) can be determined from the nonrelativistic wave function was also discussed in A. We assume that when $\mathbf{d}=0$, the deuteron is completely and accurately described by the nonrelativistic wave function, and that since this clearly makes no allowance for the negative energy states, ϕ^{-+} must be zero when $\mathbf{d}=0$. Hence[7]

$$\phi_{\alpha\beta}{}^{++}(\mathbf{r},0)=\phi_{\alpha\beta}{}^{\mathrm{NR}}(\mathbf{r}) \qquad (2.2\mathrm{a})$$

$$\phi_{\alpha\beta}{}^{-+}(\mathbf{r},0)=0. \qquad (2.2\mathrm{b})$$

This procedure is justified by the fact that the phenomenological nonrelativistic wave functions are calculated from potentials which have been chosen to reproduce actual nucleon-nucleon scattering data (in the center of momentum system), and hence should already simulate the negative energy effects as well as possible with a nonrelativistic theory.

In view of (2.2), the primary significance of (2.1) is that it tells us what the wave function of the deuteron is in a *system in which it is not at rest.* Since it is never possible to find a system in which *both* deuterons are at rest, (except when $\mathbf{q}^2=0$), it is essential to have this information in our calculation of the elastic e-d cross section. Omitting this information introduces systematic errors of order M^{-2}.

Equations (2.2) are four equations for the four d-N vertex invariants. Using (2.2b) to eliminate two of them (see A), we emerge with

$$\phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})=(1+\lambda)\bar{u}_\alpha(\tfrac{1}{2}\mathbf{d}+\mathbf{r})\Delta^\nu C\bar{u}_\beta{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r})\xi_\nu, \quad (2.3)$$

where

$$(2\pi)^{1/2}\Delta^\nu=A(t)\gamma^\nu-B(t)(P^\nu/M)(\boldsymbol{P}+M/2M). \quad (2.4)$$

Here C is the charge conjugation matrix and ξ is the 4-polarization of the deuteron subject to the conditions $\xi_\mu\xi^\mu=1$ and $\xi_\mu d^\mu=0$. The 4-vector P is determined by the prescription that one of the particles is on the mass shell (see A) so that $P=(P_0,\mathbf{P})$, where

$$P_0=[M_d{}^2+\mathbf{d}^2]^{1/2}-[M^2+(\tfrac{1}{2}\mathbf{d}-\mathbf{r})^2]^{1/2},$$
$$\mathbf{P}=\tfrac{1}{2}\mathbf{d}+\mathbf{r}. \quad (2.5)$$

Finally,

$$t=P^2=P_0{}^2-\mathbf{P}^2,$$
$$\lambda=P_0/[M^2+(\tfrac{1}{2}\mathbf{d}+\mathbf{r})^2]^{1/2}. \quad (2.6)$$

The functions A and B are regarded as unknown scalar functions of t for the time being,[8] but will shortly be determined from (2.2a).

The nonrelativistic wave function in momentum space is:

$$\phi_{\alpha\beta}{}^{\mathrm{NR}}(\mathbf{r})=(2\pi)^{1/2}\chi_\alpha{}^*\mathcal{O}_{\mathrm{NR}}i\sigma^2\chi_\beta, \quad (2.7)$$

where[9]

$$\mathcal{O}_{\mathrm{NR}}=u_0(r)\sigma\cdot\xi+(1/\sqrt{2})w_2(r)[3\sigma\cdot\mathbf{rr}\cdot\xi/r^2-\sigma\cdot\xi], \quad (2.8)$$

and for any function $f(x)$

$$f_l(r)=\int_0^\infty f(x)xj_l(rx)dx$$
$$j_l(y)=(-y)^l\left(\frac{1}{y}\frac{d}{dy}\right)^l\frac{\sin y}{y}. \quad (2.9)$$

In these expressions $u(x)$ and $w(x)$ are the x-space $^3S_1$ and $^3D_1$ deuteron wave functions, respectively, while $j_l(y)$ is the spherical Bessel function of order l.

We will now use (2.2a) to determine A and B in terms of u_0 and w_2. First we have a straightforward matter of reducing (2.3) to two component Pauli form. In making this reduction we retain terms to order M^{-2} in A, and M^{-4} in B. This is necessary if we wish to have all terms to order M^{-2} in w_2 in the end, and is directly justified by the fact that B is about 20 times the size of A (see II). The final result of the reduction is:

$$\phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})=(2\pi)^{1/2}\chi_\alpha{}^*\mathcal{O}(\mathbf{r},\mathbf{d})i\sigma^2\chi_\beta, \quad (2.10)$$

[6] We have adopted the conventions used in J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1964).

[7] The identification given in (2.2) above differs by a factor $(M/E_0)^{1/2}$ from that used in A, as discussed in the next section. We believe the identification used in this paper to be the most correct.

[8] These functions are related to the d-N vertex invariants as described in A, so that they could be calculated theoretically if desired.

[9] For the remainder of the paper we write $r=|\mathbf{r}|$, and 4-vectors will always be explicitly indicated with Greek subscripts.

where

$$\mathcal{O}(\mathbf{r},\mathbf{d})=\phi_1\sigma\cdot\xi+\phi_2[3\sigma\cdot\mathbf{r}\mathbf{r}\cdot\xi/r^2-\sigma\cdot\xi]+\phi_3 i\frac{\mathbf{r}\times\mathbf{d}\cdot\xi}{rd}+\phi_4\left[\frac{\sigma\cdot\mathbf{d}\xi\cdot\mathbf{r}+\sigma\cdot\mathbf{r}\xi\cdot\mathbf{d}}{rd}\right], \quad (2.11a)$$

$$\begin{aligned}
\phi_1&=A(t)[1-(\alpha^2/2M^2)]+\phi_2,\\
\phi_2&=(B(t)r^2/3M^2)[1-(r^2/2M^2)-(\alpha^2/2M^2)]-(A(t)r^2/6M^2), \quad (2.11b)\\
\phi_3&=-(rd/4M^2)A(t),\\
\phi_4&=B(t)(rd/4M^2)[(r^2/M^2)-(\mathbf{r}\cdot\mathbf{d}/2M^2)+(\alpha^2/M^2)].
\end{aligned}$$

Here $\alpha^2=M\epsilon$ where ϵ is the deuteron binding energy. Note that when the deuteron is at rest ($\mathbf{d}=0$) neither ϕ_3 nor ϕ_4 contribute to the wave function and ϕ^{++} has the same structure as in the nonrelativistic case.

The next step in our reduction is to note that the argument of A and B depends on $\mathbf{d}$. To take this properly into account we expand t to order M^{-2}:

$$t\cong M^2-2(r^2+\alpha^2)+\tfrac{1}{2}M^{-2}\{[\mathbf{d}\cdot\mathbf{r}-r^2-\alpha^2]^2+2\alpha^4\}. \quad (2.12)$$

The $\mathbf{d}$ dependence in the term of order M^{-2} will make corrections to the wave function of the same order as we have already considered,[10] and hence must be retained. This can be easily done by expanding $A(t)$ and $B(t)$ about the point t_0, the value of t for $\mathbf{d}=0$. With this modification (2.11b) becomes

$$\begin{aligned}
\phi_1(t)&\cong A(t_0)\left[1-\frac{\alpha^2}{2M^2}\right]+A'(t_0)(t-t_0)+\phi_2(t),\\
\phi_2(t)&\cong\frac{B(t_0)r^2}{3M^2}\left[1-\frac{r^2}{2M^2}-\frac{\alpha^2}{2M^2}\right]+\frac{B'(t_0)(t-t_0)r^2}{3M^2}-\frac{A(t_0)r^2}{6M^2}, \quad (2.13)\\
\phi_3(t)&\cong\phi_3(t_0),\\
\phi_4(t)&\cong\phi_4(t_0),
\end{aligned}$$

where $A'(t_0)=dA(t)/dt|t_0$ and ϕ_3 and ϕ_4 are unchanged up to order M^{-2}. Finally, Eq. (2.2a) gives

$$u_0(r)=\phi_1(t_0), \quad (1/\sqrt{2})w_2(r)=\phi_2(t_0). \quad (2.14a)$$

To lowest order these are

$$\begin{aligned}
u_0(r)&=A(t_0)+(B(t_0)r^2/3M^2)\\
(1/\sqrt{2})w_2(r)&=(B(t_0)r^2/3M^2), \quad (2.14b)
\end{aligned}$$

and hence to lowest order the derivatives of A and B are easily evaluated in terms of the wave functions:

$$\begin{aligned}
A'(t_0)&\cong\frac{dA(t_0)}{dt_0}\cong-\frac{1}{2}\frac{d}{dr^2}\left[u_0(r)+\frac{1}{\sqrt{2}}w_2(r)\right]\\
B'(t_0)&\cong-\frac{1}{2\sqrt{2}}3M^2\frac{d}{dr^2}\left[\frac{w_2(r)}{r^2}\right].
\end{aligned} \quad (2.15)$$

Putting Eqs. (2.12)–(2.15) together gives finally

$$\begin{aligned}
\phi_1&=u_0+\tfrac{1}{2}\Delta(u_0'+\sqrt{2}w_2'-(1/\sqrt{2})(w_2/r^2)),\\
\sqrt{2}\phi_2&=w_2+\tfrac{1}{2}\Delta(w_2'-(w_2/r^2)), \quad (2.16)\\
\phi_3&=-(rd/4M^2)(u_0-(w_2/\sqrt{2})),\\
\sqrt{2}\phi_4&=(rd/4M^2)w_2[3-\tfrac{3}{2}(\mathbf{r}\cdot\mathbf{d}/r^2)+(3\alpha^2/r^2)],
\end{aligned}$$

where

$$\begin{aligned}
\Delta&=t_0-t=M^{-2}(r^2+\alpha^2)\mathbf{d}\cdot\mathbf{r}-((\mathbf{d}\cdot\mathbf{r})^2/2M^2),\\
u_0'&=\frac{du_0(r)}{dr^2}=\frac{1}{2r}\frac{du_0(r)}{dr}.
\end{aligned} \quad (2.17)$$

It will be convenient to have the x-space representation of the wave function when we turn to the calculation of the form factor. This Fourier transform is

$$\begin{aligned}
\psi_{\alpha\beta}(\mathbf{x},\mathbf{d})&=\frac{1}{(2\pi)^3}\int e^{i\mathbf{x}\cdot\mathbf{r}}\phi_{\alpha\beta}^{++}(\mathbf{r},\mathbf{d})d^3r\\
&=(8\pi)^{-1/2}\chi_\alpha^*\hat{O}(\mathbf{x},\mathbf{d})i\sigma^2\chi_\beta,
\end{aligned} \quad (2.18)$$

where

$$\begin{aligned}
x\hat{O}(\mathbf{x},\mathbf{d})=&[u+(d^2/M^2)\psi_1(x,d,z)]\sigma\cdot\xi-\frac{1}{\sqrt{2}}\left[w+\frac{d^2}{M^2}\psi_2(x,d,z)\right]\left[\frac{3\sigma\cdot\mathbf{x}\mathbf{x}\cdot\xi}{x^2}-\sigma\cdot\xi\right]\\
&+\psi_3(x,d)\frac{\mathbf{x}\times\mathbf{d}\cdot\xi}{M^2x^2}+\frac{i}{\sqrt{2}}\psi_4(x,d,z)\times\left[\frac{\xi\cdot\mathbf{x}\sigma\cdot\mathbf{d}+\xi\cdot\mathbf{d}\sigma\cdot\mathbf{x}}{M^2x^2}\right]. \quad (2.19)
\end{aligned}$$

The wave functions ψ_i are given as functionals of u and w in the Appendix. The variable z is the cosine of the angle between $\mathbf{d}$ and $\mathbf{x}$.

3. THE DEUTERON FORM FACTOR

Now that we have the correct relativistic wave function, we calculate the deuteron form factor in the impulse approximation to order M^{-2}. We begin with the expression derived in A:

$$\begin{aligned}
G_D{}^\mu(q)&=\eta\langle D'|j^\mu|D\rangle\\
&=\frac{e}{(2\pi)^3}\int d^3p\left(\frac{M}{E_0}\right)\times\mathrm{tr}\{\bar{\phi}_{\frac{1}{2}\mathbf{q}}(\mathbf{p}+\tfrac{1}{4}\mathbf{q})F^\mu(q)\phi_{-\frac{1}{2}\mathbf{q}}(\mathbf{p}-\tfrac{1}{4}\mathbf{q})\}, \quad (3.1)
\end{aligned}$$

[10] We wish to thank Professor D. R. Yennie for interesting conversations in which this point was brought out.

where $F^\mu(q)$ is the nucleon isotopic scalar form factor

$$F^\mu(q)=F_1(q^2)\gamma^\mu-i(F_2(q^2)\sigma^{\mu\nu}/2M)q_\nu, \qquad (3.2)$$

normalized so that

$$F_1(0)=1; \quad F_2(0)=-0.12. \qquad (3.3)$$

The factor $E_0=[M^2+\mathbf{p}^2]^{1/2}$ appears as a natural consequence of the reduction of an invariant 4-dimensional integration to 3 dimensions. The relativistic wave functions have been discussed in the preceeding section except that

$$\bar{\phi}_d(\mathbf{r})=-C\gamma^5\phi_d{}^T(\mathbf{r})(C\gamma^5)^{-1}. \qquad (3.4)$$

In this paper we ignore the off-mass-shell dependence of the nucleon form factor. The principal reason for this is that it is probably small (see the next section) and is difficult to calculate.

Furthermore, as discussed in the last section and A, we ignore the antiparticle contributions ϕ^{-+} to the wave function. From Eq. (3.4) and (2.1), it follows that

$$\begin{aligned}\bar{\phi}_d(\mathbf{r})&=-\phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})C\gamma^5u_\beta(\tfrac{1}{2}\mathbf{d}-\mathbf{r})(C\gamma^5u_\alpha)^T(\tfrac{1}{2}\mathbf{d}+\mathbf{r})\\&=-\phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})\bar{u}_{\beta'}{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r})\bar{u}_{\alpha'}(\tfrac{1}{2}\mathbf{d}+\mathbf{r})\\&=\bar{\phi}_{\beta'\alpha'}{}^{++}(\mathbf{r},\mathbf{d})\bar{u}_{\beta'}{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r})\bar{u}_{\alpha'}(\tfrac{1}{2}\mathbf{d}+\mathbf{r}), \qquad (3.5)\end{aligned}$$

provided we obtain the polarization α' from α by

$$\chi_{\alpha'}\equiv-i\sigma^2\chi_\alpha{}^* \qquad (3.6)$$

and define

$$\bar{\phi}_{\alpha\beta}{}^{++}\equiv-\phi_{\beta'\alpha'}{}^{++}. \qquad (3.7)$$

Hence

$$\bar{\phi}_{\alpha\beta}{}^{++}=-(2\pi)^{1/2}\chi_\beta i\sigma^2\mathcal{O}\chi_\alpha{}^*=(2\pi)^{1/2}\chi_\alpha{}^*\mathcal{O}^Ti\sigma^2\chi_\beta, \qquad (3.8)$$

where $\mathcal{O}$ was introduced in (2.10).

Using (3.5), Eq. (3.1) for the form factor becomes

$$G_D{}^\mu(q)=\frac{e}{(2\pi)^3}\int d^3p\left(\frac{M}{E_0}\right)[\bar{\phi}_{\alpha\beta}{}^{++}(\mathbf{p}+\tfrac{1}{4}\mathbf{q},\tfrac{1}{2}\mathbf{q})f_{\beta\gamma}{}^\mu(q)\times\phi_{\gamma\alpha}{}^{++}(\mathbf{p}-\tfrac{1}{4}\mathbf{q},-\tfrac{1}{2}\mathbf{q})], \qquad (3.9)$$

where the Pauli current is

$$f_{\beta\gamma}{}^\mu(q)=\bar{u}_\beta(\mathbf{p}+\tfrac{1}{2}\mathbf{q})F^\mu(q)u_\gamma(\mathbf{p}-\tfrac{1}{2}\mathbf{q}), \qquad (3.10)$$

and we have used the fact that

$$u_\alpha{}^T(-\mathbf{p})\bar{u}_\beta{}^T(-\mathbf{p})=\delta_{\alpha\beta}. \qquad (3.11)$$

The only advantage of (3.9) over (3.1) is that the terms are in 2 component form so that they may be easily compared with the usual nonrelativistic results. The expression (3.10) for the relativistic current has been obtained many times before,[11] and our results are the same as that usually obtained once we have included only the positive energy parts of the wave function.[12] The new result in (3.9) is the appearance of corrections in the wave function as a consequence of our treatment of the bound-state wave function on an equal footing with the current.

The Pauli reduction on the current gives us

$$f_{\alpha\beta}{}^0(q)=\chi_\alpha{}^*\left[F_C\left(1+\frac{p^2}{2M^2}\right)-\frac{(2F_M-F_C)}{4M^2}i(\mathbf{p}\times\mathbf{q})\cdot\sigma\right]\chi_\beta,$$

$$\begin{aligned}f_{\alpha\beta}{}^k(q)=\chi_\alpha{}^*\bigg[&\frac{F_M}{2M}i(\sigma\times\mathbf{q})^k+\frac{F_C}{M}p^k\\&-\frac{F_M}{2M}(i(\sigma\times\mathbf{p})^k+\tfrac{1}{2}q^k)\frac{\mathbf{p}\cdot\mathbf{q}}{2M^2}+(F_M-F_C)\frac{p^k}{2M}\\&\times\left(\frac{-i(\sigma\times\mathbf{p})\cdot\mathbf{q}}{2M^2}+\frac{q^2}{4M^2}\right)\bigg]\chi_\beta, \qquad (3.12)\end{aligned}$$

where F_C and F_M are the charge and magnetic-moment form factors:

$$\begin{aligned}F_C&=F_1-(q^2/4M^2)F_2,\\F_M&=F_1+F_2.\end{aligned} \qquad (3.13)$$

Note that the presence of the term $(1+p^2/2M^2)$ multiplying F_C would mean that if our wave function were normalized to unity beforehand, charge would not be conserved (because $p^2/2M$ makes a small but nonzero contribution to the value of charge form factor at $q^2=0$.) This difficulty can be avoided, of course, by renormalizing the wave functions as has been proposed by Adler,[13] but if our wave function approach is really consistent such renormalization should be unnecessary. In fact, this is true if the factor of M/E_0 in (3.1) is incorporated into the current, for it contains a term which precisely cancels the $p^2/2M$. In A, we choose to incorporate the M/E_0 symmetrically into the wave function; it appears now that this procedure was incorrect and that a preferably procedure is to include it in the current.

A further point to be considered is current conservation, which requires that $q\cdot j=-\mathbf{q}\cdot\mathbf{j}=0$. We will show explicitly that our expressions do satisfy this condition shortly. The result is that, if we choose $\mathbf{q}$ to define the z axis, j^3 is zero. Hence the current becomes (where $k=1, 2$)[14]

$$j^0\equiv\left(\frac{M}{E_0}\right)f_{\alpha\beta}{}^0(q)=\chi_\alpha{}^*\left[F_C-\frac{2F_M-F_C}{4M^2}i(\mathbf{p}\times\mathbf{q})\cdot\sigma\right]\chi_\beta,$$

$$\begin{aligned}j^k=\left(\frac{M}{E_0}\right)f_{\alpha\beta}{}^k(q)=\chi_\alpha{}^*\bigg[&\frac{F_M}{2M}i(\sigma\times\mathbf{q})\left(1-\frac{p^2}{2M^2}\right)\\&-\frac{F_M}{2M}i(\sigma\times\mathbf{p})^k\frac{\mathbf{p}\cdot\mathbf{q}}{2M^2}+F_C\frac{p^k}{M}\left(1-\frac{p^2}{2M^2}\right)\\&+(F_C-F_M)\frac{p^k}{M}\left(\frac{i(\sigma\times\mathbf{p})\cdot\mathbf{q}}{4M^2}-\frac{q^2}{8M^2}\right)\bigg]\chi_\beta. \qquad (3.14)\end{aligned}$$

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[11] See for example M. Gourdin, Nuovo Cimento **35**, 1105 (1965); R. J. Adler, Ph.D Thesis, Stanford University (1965); K. W. McVoy and L. Van Hove, Phys. Rev. **125**, 1034 (1962).

[12] Had we not neglected ϕ^{-+}, we would have had terms involving ϕ^{-+} and the matrix element of the current between v and u spinors etc.

[13] R. J. Adler, Ph.D thesis, Stanford University, 1965 (unpublished).

[14] One might object to retaining terms of order M^{-3} in the spatial current. However, since the leading term is only of order M^{-1} the additional terms still provide corrections of order M^{-2}.

Finally, we are ready to combine the results of Sec. 2 with the current above to calculate the form factor. We have

$$G_D{}^\mu(q)=\frac{-e}{(2\pi)^2}\int d^3p\ \mathrm{tr}\{\mathcal{O}_{\xi'}{}^T(\mathbf{p}+\tfrac{1}{4}\mathbf{q},\tfrac{1}{2}\mathbf{q})\sigma^2 j^\mu(\mathbf{p},\mathbf{q})\times\mathcal{O}_\xi(\mathbf{p}-\tfrac{1}{4}\mathbf{q},-\tfrac{1}{2}\mathbf{q})\sigma^2\},\quad(3.15)$$

where we have added subscripts to $\mathcal{O}$'s to remind us of the fact that the incoming polarization ξ is different from the outgoing polarization ξ'.

We can use the form (3.15) to easily demonstrate current conservation. First, we observe that after the p integration has been performed, the only vectors left out of which we could construct invariants of the form factor are ξ, ξ', and $\mathbf{q}$. Hence the only spatial invariants which can be constructed (up to arbitrary functions of q^2) and which are bilinear in the ξ's are

$$\xi^k\xi'^*\cdot\mathbf{q}\mp\xi'^{*k}\xi\cdot\mathbf{q},\quad \xi'^*\cdot\xi q^k,\quad \xi'^*\cdot\mathbf{q}\xi\cdot\mathbf{q}q^k.\quad(3.16)$$

Hence, to prove current conservation, it is sufficient to show that the spatial part of (3.15) is antisymmetric in ξ and ξ'^*. But, observing that

$$\mathcal{O}(\mathbf{r},\mathbf{d})=\mathcal{O}(-\mathbf{r},-\mathbf{d}),\quad(3.17)$$

and taking the transpose under the trace in (3.15), we obtain

$$G_D{}^\mu(q)=\frac{-e}{(2\pi)^2}\int d^3p\ \mathrm{tr}\{\sigma^2\mathcal{O}_\xi{}^T(\mathbf{p}+\tfrac{1}{4}\mathbf{q},\tfrac{1}{2}\mathbf{q})j^{\mu T}(-\mathbf{p},\mathbf{q})\sigma^2\times\mathcal{O}_{\xi'}(\mathbf{p}-\tfrac{1}{4}\mathbf{q},-\tfrac{1}{2}\mathbf{q})\},\quad(3.18)$$

and hence $G_D{}^k$ is antisymmetric in ξ, ξ'^* provided

$$j^k(\mathbf{p},\mathbf{q})=-\sigma^2 j^{kT}(-\mathbf{p},\mathbf{q})\sigma^2.\quad(3.19)$$

Examination of Eq. (3.12) verifies that this is true. The antisymmetry in ξ and ξ'^* is also a convenience in evaluating the matrix elements.

Equation (3.15) can be put in x space in order to more closely resemble the nonrelativistic results. We have

$$\begin{aligned}G_D{}^\mu(q)&=\frac{-e}{(2\pi)^5}\int d^3r d^3r' d^3x\ e^{i(\mathbf{r}-\mathbf{r}'-\frac{1}{2}\mathbf{q})\cdot\mathbf{x}}\times\mathrm{tr}\{\mathcal{O}^T(\mathbf{r},\tfrac{1}{2}\mathbf{q})\sigma^2 j^\mu\mathcal{O}(\mathbf{r}',-\tfrac{1}{2}\mathbf{q})\sigma^2\}\\&=\frac{-e}{8\pi}\int d^3x\ e^{-i\frac{1}{2}\mathbf{q}\cdot\mathbf{x}}\times\mathrm{tr}\{\hat{O}^T(\mathbf{x},\tfrac{1}{2}\mathbf{q})\sigma^2\hat{j}^\mu\hat{O}(-\mathbf{x},-\tfrac{1}{2}\mathbf{q})\sigma^2\}\\&=\frac{-e}{8\pi}\int d^3x\ e^{-i\frac{1}{2}\mathbf{q}\cdot\mathbf{x}}\times\mathrm{tr}\{\hat{O}^T(\mathbf{x},\tfrac{1}{2}\mathbf{q})\sigma^2\hat{j}^\mu\hat{O}(\mathbf{x},\tfrac{1}{2}\mathbf{q})\sigma^2\},\end{aligned}\quad(3.20)$$

where $\hat{j}^\mu$ is a current operator obtained from j^μ by substituting $i\boldsymbol{\nabla}+\frac{1}{4}\mathbf{q}$ for $\mathbf{p}$ (see the Appendix). The traces can be performed and the results expressed in terms of the wave functions presented in Sec. 2.

When all of the algebra is completed, one emerges with the structure of the results presented in Sec. 1. The explicit form of all of the functions is given in the Appendix.

4. DISCUSSION

While the precise effect of these correction terms will have to await a detailed calculation using realistic wave functions, it is still of interest to obtain a rough estimate of their effect. To this end we examine (1) the slope of the charge term I_C at $q^2=0$ and (2) the value of the correction to the magnetic moment.

Both of the charge correction terms are zero at $q^2=0$. The interest in their slope lies in the mystery surrounding the slope of the neutron-charge form factor. This slope has been quite accurately determined by thermal-neutron scattering data[15] to be about 0.021 F^2, while the electron-deuteron scattering data, *when interpreted with the uncorrected nonrelativistic theory*, suggests that this slope is zero.[16]

Introducing

$$D_C=\int_0^\infty(u^2+w^2)j_0(\tau)dx,\quad(4.1)$$

the nonrelativistic theory *without correction* is

$$G_C=(F_C{}^P+F_C{}^N)D_C,\quad(4.2)$$

and hence by the above assumption

$$G_C'=F_C{}^{P\prime}+D_C'\quad(4.3)$$

since $F_C{}^P(0)=D_C(0)=1$. When we now include the correction term I_C (J_C is negligible) Eq. (4.3) becomes

$$G_C'=F_C{}^{P\prime}+F_C{}^{N\prime}+D_C'+I_C{}'_1\quad(4.4)$$

Since none of the numbers have changed, this means that

$$F_C{}^{N\prime}=-I_C'.\quad(4.5)$$

Hence, if $-I_C'$ is comparable to the known neutron slope of $+0.021$ F^2, then it will help explain the mystery by bringing the electron-deuteron data into closer agreement with the thermal-neutron scattering experiments.

However, examination of I_C (in the Appendix) indi-

[15] See R. Hofstadter, *Nuclear and Nucleon Structure* (W. A. Benjamin, Inc., New York, 1963) for reprints of a number of papers.

[16] Of course this experiment gives practically no information about the slope at $q^2=0$, because the accuracy is not great enough. We take the point of view here that the neutron charge form factor is rather slowly varying, and that its behavior in the $q^2=1-3$F^{-2} region is influenced by its slope at $q^2=0$, so that this experiment gives *indirect* evidence about the slope.

cates that

$$S=\frac{dI_C}{dq^2}\bigg|_{q^2=0}\cong\frac{1}{2M^2}\int_0^\infty dx\bigg\{u\psi_1{}^0+u\psi_1{}^1(\tfrac{1}{3})+\frac{\sqrt{2}}{9}u\psi_4{}^0-\frac{\sqrt{2}}{9}u\psi_4{}^1\bigg\}, \quad (4.6)$$

where we have neglected all functionals bilinear in w (i.e., have retained only u^2 and uw terms).

Substituting from (A2) we have

$$S=\frac{1}{2M^2}\int_0^\infty dx\bigg\{u^2(-\tfrac{1}{12})+uxu'\Big(\frac{5}{24}\Big)+\frac{uw}{\sqrt{2}}\Big(-\frac{13}{12}\Big)+\frac{uxw'}{\sqrt{2}}\Big(-\frac{11}{12}\Big)+u\mathcal{K}u(\tfrac{1}{12})+\frac{u\mathcal{K}w}{\sqrt{2}}(-\tfrac{1}{6})+\frac{uw_I}{\sqrt{2}}\Big(\frac{11}{12}-\frac{\alpha^2x^2}{2}\Big)\bigg\}. \quad (4.7)$$

Observe that

$$\int_0^\infty uxu'dx=-\frac{1}{2}\int_0^\infty u^2dx.$$

To get a rough idea of the size of S, assume that

$$u=3.65w=0.98(e^{-\alpha x}-e^{-\beta x}),$$

where $\beta=7\alpha$ and $\alpha\approx\frac{1}{3}$ in units of pion masses. This choice gives

$$\int_0^\infty(u^2+w^2)dx=1,$$

$$\int_0^\infty(w^2)dx=0.07.$$

Then we have

$$S=(-0.30)/(2M)^2, \quad (4.8)$$

where we have neglected the terms involving w_I which can be expected to give a small positive contribution. Now $M^2\cong25$ F^{-2}, and hence S is seen to account for approximately one-third of the neutron slope!

The above is only a rough estimate, and is model-dependent. And, of course, what one is really interested in is the behavior of the correction terms at finite q^2. We take (4.8) as an indication that these correction terms should be taken into account in extracting the neutron charge form factor from the data.

The size of (4.8) seems quite reasonable. Offhand, one might expect corrections of the order of $(v/c)^2$, which in the Breit frame would be

$$q^2/4M_d{}^2=0.125q^2/2M^2. \quad (4.9)$$

What (4.8) tells us is that these corrections are larger than this estimate (by about a factor of 2, expected because there are 2 deuterons involved), and of the right sign to improve the consistency between experiments.

The corrections to the magnetic moment of the deuteron can also be estimated from our expressions. If we neglect the contribution from $J_M{}^2$, which appears to be small, and look only at those terms which involve u^2, we find from the Appendix that

$$\Delta\mu_d=\frac{e}{2M}\Big(\frac{1}{2M^2}\Big)F_M\int_0^\infty dx\bigg\{u''u(\tfrac{3}{2})+\tfrac{1}{3}u''x^2\Big(\frac{u}{x}\Big)'\bigg\}$$

$$\cong\frac{e}{2M}(-0.011). \quad (4.10)$$

Unless this term is canceled by the combined affect of smaller terms, the correction is negative and tends to increase the discrepancy between a 7% D state and the experimental moment. Of course, the magnetic moment is very sensitive to a great many other processes not included in the impulse approximation. The exchange current contributions are one such processes, and what (4.10) may indicate is that exchange current effects are larger than the present estimates would suggest.[17]

We will now turn to a discussion of the accuracy of our results. First, we point out that we have neglected the dependence of the nucleon current on the masses of the incoming and outgoing nucleons. The approximation resulting from the integration over p_0 leads naturally to the placement of the spectator nucleon on the mass shell, but the interacting nucleon is not on the mass shell, and strictly speaking one must therefore include the off-mass-shell dependence of the nucleon form factors to be completely consistent. We do not include this because we do not know what it is, and because we expect it to be small enough so that our calculation of the correction terms in Sec. 3 are still meaningful. There are several facts which support this view. The first is that the Ward identity guarantees that F_1 at $q^2=0$ is *independent* of the nucleon masses—it is a constant. Hence, one would expect the mass shell dependence of F_1 to be considerably suppressed at small q^2.[18] Although the same need not be true for F_2, the fact that F_2 is small compared to F_1 means that it is less important. One is further helped by the fact that only the isospin $\frac{1}{2}$ N-π scattering amplitude will contribute to the off-mass-shell form factor, and this is essentially zero until quite large energies,[19] suggesting that the off-mass-shell dependence of the form factors is at worst a

[17] R. J. Adler and S. D. Drell, Phys. Rev. Letters 13, 349 (1964).

[18] Imagine expanding F_1 in a power series in q^2, $y=t-M^2$, and $y'=t'-M^2$, where t and t' are the masses of the off-shell nucleons. Then the first few terms would be

$$F_1(q,t,t')=F_1(0,M^2,M^2)+C_1\frac{q^2}{M^2}+C_2\frac{q^2y}{M^4}+C_3\frac{q^2y'}{M^4}+\cdots,$$

and hence our errors in neglecting the y and y' dependence are of order M^{-4}.

[19] Recall that the lowest $I=\frac{1}{2}$ resonance is at 1480 MeV.

slowly varying function of the masses. On the basis of these considerations we guess that these effects are of the order of 20% of those already calculated.[20] However, it would be extremely instructive to have a better estimate of these effects, and at the moment we have no guarantee that these effects are not comparable to those already calculated.

Our second remark concerns the perturbation series of which the impulse approximation is the first term. In Fig. 1 the next few terms of this series are displayed.[21] The diagram in Fig. 1(b) contributes to the exchange current contribution discussed by Adler and Drell.[17] as well as other contributions from the proper $\gamma N \rightarrow \pi N$ scattering amplitude.[22] *A priori*, one would expect these contributions to be of the order of magnitude of the corrections presented here.

The contribution from Fig. 1(d) [also 1(e) and (f) for that matter] are especially interesting. If our proposed perturbation expansion is to make any sense, this contribution should be much smaller than that of Fig. 2, which should be equal[23] to the impulse approximation since it is already included in it and is in fact indistinguishable from it. And indeed, by a judicious examination of the singularities and treatment of the internal energies involved one can show that Fig. 1(d) is of the order of $(\mathbf{p}^2/M^2)^2$ times Fig. 2. Normally this would not be a helpful result, but in our case the rapid convergence of the wave function guarantees that the dominate part of the integral arises when $\mathbf{p}^2/M^2 \ll 1$ and hence 1(d) is down to order M^{-4} of 2. Thus, this perturbation series appears to make some sense, and it might even converge. A similar expansion might be applicable to other problems.

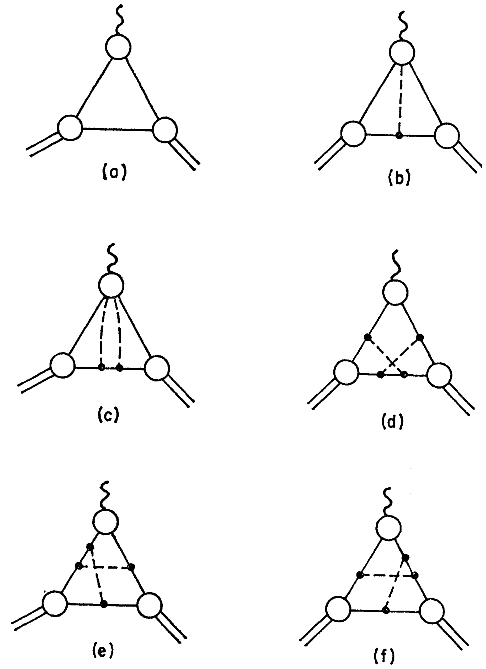

FIG. 1. The first few terms of a rearranged perturbation series for the deuteron form factor. Double solid lines are deuterons, solid lines nucleons, and dashed lines pions. The bubbles are proper vertex functions.

As a final remark we wish to provide additional insight into our treatment of an off-mass-shell fermion as a superposition of mass shell spinors $u(\mathbf{p})$ and $v(-\mathbf{p})$. Since the deuteron wave function discussed in Sec. 2 contains only one off-mass-shell nucleon, it is sufficient to study the reduced wave function θ obtained from $\phi_d(\mathbf{r})$ by multiplying it from the left by $\bar{u}_\beta{}^T(\frac{1}{2}\mathbf{d}-\mathbf{r})$. Hence

$$\theta_d(\mathbf{r}) \equiv \phi_d(\mathbf{r})\bar{u}_\beta{}^T(\tfrac{1}{2}\mathbf{d}-\mathbf{r}) = \phi_{\alpha\beta}{}^{++}(\mathbf{r},\mathbf{d})u_\alpha(\tfrac{1}{2}\mathbf{d}+\mathbf{r}) + \phi_{\alpha\beta}{}^{-+}(\mathbf{r},\mathbf{d})v_\alpha(-\tfrac{1}{2}\mathbf{d}-\mathbf{r}), \quad (4.11)$$

and the reduced wave function θ can be shown to satisfy a Dirac type equation obtained by an approximate reduction of the Bethe-Salpeter equation

$$\mathbf{P}(-M)\theta_d(\mathbf{r}) = V\theta_d(\mathbf{r}), \quad (4.12)$$

where V is some potential operator and

$$P \equiv \tfrac{1}{2}d + r$$

as in Eq. (2.5).

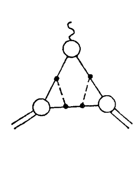

FIG. 2. An example of a term already contained in the impulse approximation, and of the same size as the impulse approximation.

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[20] One obtains this by conjecturing

$$F_2(q^2,t,t') \approx F_2(q^2)\left(\frac{M^2-M^{*2}}{t-M^{*2}}\right)\left(\frac{M^2-M^{*2}}{t'-M^{*2}}\right) \approx F_2(q^2)\left[1-\frac{y+y'}{M^2-M^{*2}}\right].$$

Hence, the correction is

$$\Delta \approx F_2\frac{y}{M^2} \approx -2F_2\frac{p^2}{M^2} \simeq -0.2\frac{p^2}{M^2}.$$

[21] This series appears to bear some resemblence to the first few terms shown in Fig. 3 of I. However the formalism here is quite different from that discussed in I. In particular, in I we focused our attention on only the discontinuities of the diagrams above. Furthermore, the contribution from Fig. 3(d) of I which posed perhaps the greatest threat to the validity of the approximation discussed there, can be shown to be canceled in the anomalous region, and hence need not be considered at all. This is reflected in the fact that there is no such diagram in Fig. 1 above [except those parts already included in 1(a)]. When we integrate over the internal energies in each diagram shown in Fig. 1, and make the approximations discussed in A, it is equivalent to considering the leading singularities as discussed in I, and it is in this way that the two formalisms would appear to lead to the same result.

[22] By "proper" in this context we mean the scattering amplitude excluding the direct and crossed nucleon poles.

[23] More precisely, since the one-pion-exchange contribution represents only one of the contributions to the potential, we may only assert that it, along with the other uncrossed contributions to the potential, must equal the impulse approximation.

Now it is natural to question the choice of the form (4.1) for the wave function. It is not the only choice which can be made,[24] although it leads naturally to the intuitive view of the off mass shell nucleon as a superposition of a mass-shell nucleon of momentum $\mathbf{p}$ and a mass shell antinucleon of momentum $-\mathbf{p}$ as suggested by the hole theory (see A). The crucial question, however, concerns the interpretation of ϕ^{++} and ϕ^{-+} as wave functions, implying that the square of their modulus is a probability density. In what sense is this justified?

The way to correctly obtain positive and negative energy wave functions would presumably be to diagonalize (4.12) with a Foldy-Wouthuysen transformation. Now the interesting but rather trivial fact is that if $V\equiv 0$, then such a diagonalization of (4.12) gives the ϕ^{++} and ϕ^{-+} introduced in Sec. 2. and hence in this case the choice of (4.1) appears to be the appropriate one for comparison with nonrelativistic probability wave functions.

Of course the potential is not zero, and hence it would appear that the correct ϕ^{++} to identify with the nonrelativistic wave function is not the ϕ^{++} introduced in Sec. 2, but differs slightly from it by terms involving the potential V. To pursue this analysis in detail would be difficult, and it is not clear that the additional accuracy thereby achieved would be meaningful, or worth the effort. This is true particularly in view of our neglect of terms like that shown in Figs. 1(d)–1(f), which also contain the potential explicitly. The best way to improve on this approach would be to attempt an explicit calculation of ϕ^{-+}.

ACKNOWLEDGMENT

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APPENDIX

In this Appendix we have collected together the detailed results of the paper.[25] The deuteron wave function is defined in Eq. (2.18) and (2.19). The correction terms can be expressed in terms of u and w (the S- and D-state wave functions) as follows:

$$\begin{aligned}\psi_{1,2}&=\psi_{1,2}{}^0+i\psi_{1,2}{}^1(P_1(z)/xd)+\psi_{1,2}{}^2P_2(z)\,,\\ \psi_4&=\psi_4{}^0+i\psi_4{}^1xdP_1(z)\,,\end{aligned}\tag{A1}$$

where $z=\mathbf{d}\cdot\mathbf{x}/dx$ and P_n is the Legendre polynominal of order n.

$$\begin{aligned}
\psi_1{}^0&=\tfrac{1}{12}u+\frac{1}{24}xu'-\frac{1}{2\sqrt{2}}w-\frac{x}{12\sqrt{2}}w'+\frac{1}{\sqrt{2}}w_I\,,\\
\psi_1{}^1&=-\tfrac{1}{2}u+\tfrac{1}{2}xu'+\tfrac{1}{4}\mathcal{K}u\\
&\qquad-\frac{3}{\sqrt{2}}(w+xw')-\frac{1}{2\sqrt{2}}\mathcal{K}w-\frac{2\alpha^2x^2}{\sqrt{2}}w_I\,,\\
\psi_1{}^2&=\tfrac{1}{12}u-\tfrac{1}{12}xu'-\frac{1}{2\sqrt{2}}w-\frac{x}{6\sqrt{2}}w'\,,\\
\psi_2{}^0&=-\tfrac{1}{6}w-\frac{1}{24}xw'\,,\\
\psi_2{}^1&=-w+xw'+\mathcal{K}w+\tfrac{1}{2}\alpha^2x^2w_I\,,\\
\psi_2{}^2&=\frac{1}{70}w+\frac{1}{420}xw'-\tfrac{1}{4}w_I\,,\\
\psi_3&=-\tfrac{1}{4}u+\tfrac{1}{4}xu'+\frac{1}{2\sqrt{2}}w+\frac{1}{4\sqrt{2}}xw'\,,\\
\psi_4{}^0&=\tfrac{3}{2}w+\tfrac{3}{4}xw'+\tfrac{3}{4}\alpha^2x^2w_I\,,\\
\psi_4{}^1&=-\tfrac{3}{8}(w-w_I)\,.
\end{aligned}\tag{A2}$$

In these equations

$$u'=\frac{d}{dx}u(x)\,,$$

$$\mathcal{K}=x^2\left(\frac{d^2}{dx^2}-\alpha^2\right),\quad\text{where}\quad\alpha^2=M\epsilon\,,$$

$$\epsilon=\text{deuteron binding energy}\,,$$

$$w_I=x\int_x^\infty\frac{w(x')}{x'^2}dx'\,.$$

These results were obtained with use of the identity

$$4\pi\int_0^\infty r^{l+2}j_m(rx)dr\int_0^\infty dx'f(x')x'j_n(rx')\equiv\frac{2\pi^2}{x^{m+1}}(-)^{\frac{1}{2}(m+l-n)}\left[\int_x^\infty\right]^{m+n-l}\frac{1}{x}\left(\frac{d}{dx}\frac{1}{x}\right)^n x^{n+l+m+2}\left(\frac{1}{x}\frac{d}{dx}\right)^m\frac{f(x)}{x^{l+1-m}}\,,\tag{A3}$$

which holds for $m+l-n=$ even integer.

[24] See I. J. McGee and L. Durand, III, Bull Am. Phys. Soc. **10**, 62 (1965), who use off-mass-shell spinors.
[25] In this Appendix all vectors are 3-vectors so that $q^2=\mathbf{q}^2$.

A convenient form for the current $\jmath$ as an operator in x space is:

$$\jmath_0=F_C+(2F_M-F_C)\frac{1}{4M^2}(\overrightarrow{\nabla}\times\mathbf{q})\cdot\boldsymbol{\sigma}\,,$$

$$\jmath^k=\frac{F_M}{2M}i(\boldsymbol{\sigma}\times\mathbf{q})^k\left[1+\frac{\overleftarrow{\nabla}^2}{4M^2}+\frac{\overrightarrow{\nabla}^2}{4M^2}+\frac{q^2}{32M^2}\right]+i\frac{\overrightarrow{\nabla}^k}{M}\left[F_C\left(1+\frac{\overleftarrow{\nabla}^2}{4M^2}+\frac{\overrightarrow{\nabla}^2}{4M^2}+\frac{q^2}{32M^2}\right)+(F_C-F_M)\frac{q^2}{16M^2}\right]$$
$$+\frac{F_M}{2M}\left\{(\boldsymbol{\sigma}\times\overrightarrow{\nabla})^k\left[-\frac{\overleftarrow{\nabla}^2}{M^2}\right]+i(\boldsymbol{\sigma}\times\mathbf{q})^k\frac{\overrightarrow{\nabla}^2}{4M^2}\right\}+(F_C-F_M)\frac{\overrightarrow{\nabla}^k}{M}\frac{\boldsymbol{\sigma}\cdot\mathbf{q}\boldsymbol{\sigma}\cdot\overrightarrow{\nabla}}{4M^2}\,. \quad \text{(A4)}$$

The arrow above the operator indicates to which direction it is to operate. In using (A4) for $\jmath^k$, *only* the terms *antisymmetric* in ξ and ξ'^* are to be retained. Then, if one makes the substitution

$$\overrightarrow{\nabla}=-i\mathbf{r}'=-i(\mathbf{p}-\tfrac{1}{4}\mathbf{q})\,,\quad \overleftarrow{\nabla}=i\mathbf{r}=i(\mathbf{p}+\tfrac{1}{4}\mathbf{q})\,,$$

which is required by Eq. (3.20), (A4) is seen to reduce to (3.14).

Using (A4) and the deuteron wave functions leads to the results given in Sec. 1. The correction functions are

$$I_C=\frac{q^2}{2M^2}\sum\left(u\psi_1+w\psi_2+\frac{2i}{3\sqrt{2}}u\psi_4\frac{P_1(z)}{\tau}-\frac{2i}{3}w\psi_4\frac{P_1(z)}{\tau}\right),$$

$$I_Q=6\sqrt{2}\sum\left\{-\left(u\psi_2+w\psi_1-\frac{w\psi_2}{\sqrt{2}}\right)P_2(z)+\frac{2i}{3}u\psi_4\frac{P_1(z)}{\tau}-i\frac{w\psi_4}{\sqrt{2}}\left(\frac{3}{5}\frac{P_3(z)}{\tau}+\frac{11}{15}\frac{P_1(z)}{\tau}\right)\right\},$$

$$I_M{}^1=\frac{q^2}{2M^2}\sum\left\{\frac{9i}{4}w\psi_2\frac{P_1(z)}{\tau}+\frac{i}{8}w\psi_4{}^1\frac{P_1(z)}{\tau}[1-P_2(z)]+\frac{1}{8}\left(4w\psi_4-xw'\psi_4+xw\frac{\partial\psi_4}{\partial x}\right)\frac{P_2(z)}{\tau^2}\right\}$$
$$+\frac{1}{M^2}\sum\frac{1}{4x^2}\left(5w\psi_4+xw'\psi_4-xw\frac{\partial\psi_4}{\partial x}\right),$$

$$I_M{}^2=\frac{q^2}{2M^2}\sum\left\{u\psi_1-\frac{w\psi_2}{2}-\frac{1}{2\sqrt{2}}(w\psi_1+\sqrt{2}w\psi_2+u\psi_4)P_2(z)-\frac{3i}{10\sqrt{2}}w\psi_3\left(\frac{P_1(z)-P_3(z)}{\tau}\right)+\frac{i}{\sqrt{2}}\left(u+\frac{w}{\sqrt{2}}\right)\psi_4\frac{P_1}{\tau}\right\},$$

$$J_C=(q^2/4M^2)\sum(3i/2)w^2(P_1(z)/\tau)\,,$$

$$J_Q=-6\sqrt{2}\sum\left\{\frac{3}{20}x(uw'-wu')\frac{P_1(z)-P_3(z)}{\tau}+\frac{wu}{20}\left(9\frac{P_1(z)}{\tau}+6\frac{P_3(z)}{\tau}\right)-\frac{w^2}{\sqrt{2}}\left(\frac{6}{10}\frac{P_3(z)}{\tau}+\frac{3}{20}\frac{P_1(z)}{\tau}\right)\right\},$$

$$J_M{}^1=\frac{1}{4M^2}\int_0^\infty dx\left\{(u\hat{u}-\tfrac{1}{2}w\hat{w})j_0(\tau)+\frac{1}{\sqrt{2}}\left(u\hat{w}+\frac{w\hat{w}}{\sqrt{2}}\right)j_2(\tau)\right\}$$
$$+\frac{1}{2M^2}\sum\left\{\hat{u}\left[x^2\left(\frac{u}{x}\right)'-\frac{w}{\sqrt{2}}-\frac{xw'}{2\sqrt{2}}\right]+\frac{\hat{w}}{\sqrt{2}}\left[-\frac{x^2}{2}\left(\frac{u}{x}\right)'-\sqrt{2}xw'+\frac{w}{2\sqrt{2}}\right]\right\}i\frac{P_1(z)}{\tau},$$

$$J_M{}^2=\frac{1}{2M^2}\sum\left\{\left(u+\frac{1}{\sqrt{2}}w\right)\left[\left(\tfrac{1}{3}u''-\frac{u'}{x}+\frac{u}{x^2}-\frac{w''}{6\sqrt{2}}+\frac{w}{\sqrt{2}x^2}\right)P_2(z)+\left(-\frac{u''}{3}+\frac{w''}{6\sqrt{2}}+\frac{w'}{2\sqrt{2}x}\right)\right]\right.$$
$$\left.+\frac{w}{\sqrt{2}}\left[\left(-\frac{u''}{2}+\frac{2u'}{x}-\frac{2u}{x^2}\right)P_2(z)+\left(\frac{u''}{2}-\frac{u'}{2x}+\frac{u}{2x^2}+\frac{w''}{2\sqrt{2}}-\frac{w'}{2\sqrt{2}x}-\frac{4w}{\sqrt{2}x^2}\right)\right]\right\},$$

where

$$\sum\equiv\int\frac{d^3x}{4\pi x^2}e^{-i\tau z}$$

In these expressions $\tau=\frac{1}{2}qx$, $\hat{u}=u''$, $\hat{w}=w''-6w/x^2$. These expressions can be reduced for numerical computation to one-dimensional integrals by expanding the integrands in Legendre polynomials and using the identity

$$\sum P_n(z)f(x)=(-i)^n\int_0^\infty f(x)j_n(\tau)dx\,.$$