Theory of the Triton Wave Function*

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The exact bound state of the three-nucleon system with separable interactions between pairs is studied. The method for extracting the bound-state wave function from the equations for the scattering amplitude is outlined. The wave function is used to study the proton- and neutron-body form factors of the triton. S-wave spin-dependent forces between nucleon pairs are used and a parameter represents the effects of tensor forces. Although this theory gives good answers for the binding energy of the triton and for low-energy neutron-deuteron scattering, rms triton radii are slightly small and the form factors too large at large momentum transfers. This difficulty is due to the neglect of a short-range repulsion. The experimental splitting of the proton and neutron form factors due to spin dependence of the forces is present. The percent of mixed-symmetry S state is calculated.

I. INTRODUCTION

In this paper we calculate the form factor of the three-nucleon bound state. We use a wave function which is the exact solution of the three-body problem with separable interactions. This work is a continuation of our investigations in the three-nucleon system with these interactions.[1–3] We use a simple two-body interaction which fits low-energy S-wave nucleon-nucleon scattering and is spin-dependent. We have already studied the triton binding energy, the neutron-deuteron scattering lengths and neutron-deuteron scattering up to 14 MeV—all with surprisingly good results, particularly when one parameter is introduced in the nucleon-nucleon spin-triplet channel to account for the relative reduction of the force in this channel due to the tensor force and short-range repulsion.[2,3] We now extract from these calculations the three-body wave function. The properly symmetrized three-body wave function is a complex function of many variables which we only know numerically. There is presumably little to be learned by trying to list its values. Rather it should be used to calculate some property of the bound state and we use it here to calculate the proton and neutron body form factors of the triton. Although our triton has nearly the exact binding energy, and in spite of the previous successes with the scattering data, the form factors do not agree very well with experiment.[4] In particular our triton is too small; that is the form factor is too large. On the other hand, our form factors do display qualitatively the splitting of the proton and neutron form factor which is characteristic of the experiment. We have this splitting because of the spin dependence of our force. The reason our triton is small is the absence of saturating effects in the force. The triton is collapsing because there is no short-range repulsion to keep it spread out in our theory. This effect is already known from the deuteron where the Hulthén wave function fitted to the binding energy gives too large a form factor.[5] A short-range repulsion pushes the deuteron out and cures this. Our force in fact gives a Hulthén deuteron and hence is subject to just this difficulty.

Another feature of the three-particle wave function which has received considerable attention recently is the percentage of mixed-symmetry S' state in the triton.[6] The form-factor data seem to require a larger value for this than is commensurate with the rate for thermal-neutron capture in deuterium[7] and the Gamow-Teller matrix element for tritium decay.[8] Although in a simple theory, both these are strongly sensitive to the S' probability, in fact meson effects cloud the issue to the point where it is difficult to say anything. We get $(7\pm1)\%$ in our calculation for this probability. The error arises from a rather crude momentum mesh in the computer and the fact that the probability is proportional to the second difference of rather large numbers. This is a larger value for the probability than is normally obtained, but the meson effects make such estimates very difficult.[6]

Since we have begun our study of the three-body problem by analyzing the scattering equations, the extraction of the three-body wave function is not done by a conventional solution of the Schrödinger equation. In Sec. II we explain qualitatively how the wave function may be extracted from the scattering amplitude and how the various terms in the form factor arise. A proof that this extraction is equivalent to solving the Schrödinger equation is given in Appendix I. The spin and isotopic spin algebra is discussed in Appendix II.

* Supported in part by the National Science Foundation.
† Permanent address.
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[5] See, for example, J. A. McIntyre, Phys. Rev. 103, 1464 (1956).
[6] A discussion of this problem and complete set of references will be found in B. F. Gibson and L. I. Schiff, Phys. Rev. 138, B26 (1965).
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[8] R. J. Blin-Stoyle, Phys. Rev. Letters 13, 55 (1964).

In Sec. III the results for the form factors are presented. In Sec. IV the symmetry of the wave function is discussed and the method of extracting the S' probability presented. Some conclusions and some further problems are discussed in Sec. V.

II. BOUND-STATE WAVE FUNCTION AND FORM FACTORS

A straightforward approach to finding the bound-state wave function of the three-nucleon system would be to solve the Schrödinger equation. In fact it was in this context that Mitra first observed that the problem was manageable with separable two-body interactions.[9] However we have approached the three-body problem through the integral equation for the scattering amplitude in momentum space,[10] and we wish to extract the bound-state wave function from that work. To get the binding energy of the three-body bound state we used the fact that the scattering amplitude has a pole when the total energy is equal to the binding energy. To find the wave function, we must study the residue at that pole as a function of momentum. This can be done since we are studying the scattering amplitude off the energy shell so the momentum variables are not related to the total energy.

To understand the relation of the residue and the wave function, let us begin with the Lippmann-Schwinger[11] equation for the scattering of a single particle by a potential v. The scattering amplitude t may be written

$$t=v+v[1/(E-H)]v \tag{1}$$

or

$$t=v+v[1/(E-H_0)]t, \tag{2}$$

where $H=H_0+v$ and H_0 is the kinetic energy. E is the total energy variable and need have no connection with momenta. Equation (1) contains the exact Green's function. If there is a bound state $|B\rangle$ of the system satisfying $H|B\rangle=-B|B\rangle$, then from (1) we see that t will have a pole at $E=-B$ with residue $v|B\rangle\langle B|v$. If we take matrix elements of (2), let E approach $-B$, and equate residues, we obtain

$$\langle k|v|B\rangle\langle B|v|k'\rangle=\int\frac{\langle k|v|p\rangle\langle p|v|B\rangle\langle B|v|k'\rangle}{-B-p^2}d^3p \tag{3a}$$

or

$$\langle k|v|B\rangle=-\int\frac{\langle k|v|p\rangle\langle p|v|B\rangle}{B+p^2}d^3p, \tag{3b}$$

where we have taken units such that $H_0|p\rangle=p^2|p\rangle$. Now using the fact that $v=H-H_0$ and that the state $|B\rangle$ is bounded so that we may integrate by parts, we get

$$\langle k|v|B\rangle=-(k^2+B)\langle k|B\rangle; \tag{4}$$

putting this into (3) we obtain

$$k^2\langle k|B\rangle+\int\langle k|v|p\rangle\langle p|B\rangle d^3p=-B\langle k|B\rangle, \tag{5}$$

which is just the Schrödinger equation for the bound-state wave function $\langle k|B\rangle$, in momentum space.

In more complex problems, such as the three-body problem, many of these results remain valid. It is still possible from the equation analogous to (1) (this is the Low equation[12]) to show that the amplitude has a pole at the bound state and that the residue at the pole is separable. That is, it factors into a function of the initial momenta times a function of the final momenta. These functions are vertex functions for the bound state and may be written in a form analogous to (4) to that they may be interpreted in terms of Fourier transforms of the bound-state wave function. In the three-body problem we study an equation analogous to (2) which when expressed in terms of the residues at the bound state pole gives the homogeneous vertex function equation analogous to (3b). These connections between the bound-state wave function and the residues in the three-particle case are made explicitly in Appendix I. In the remainder of this section we shall rest on the qualitative discussion presented above and turn to the three-nucleon problem.

We begin by reviewing briefly our equations for the scattering amplitude for nucleon-deuteron scattering[3] in the doublet channel (the bound state has spin $\frac{1}{2}$). They are a set of coupled equations representing the amplitude for nucleon-deuteron elastic scattering and nucleon-deuteron goes to a nucleon plus a two nucleon system in the singlet state (which we call φ). The equations in the center-of-mass system are ($\hbar=2M_n=1$)

$$\langle\mathbf{k},d|t(E)|\mathbf{k}',d\rangle=\tfrac{1}{2}\langle\mathbf{k},d|B(E)|\mathbf{k}',d\rangle+\frac{1}{2(2\pi)^3}\int d^3p\langle\mathbf{k},d|B(E)|\mathbf{p},d\rangle P_d(p^2,E)\langle\mathbf{p},d|t(E)|\mathbf{k}',d\rangle$$
$$+\frac{3}{2(2\pi)^3}\int d^3p\langle\mathbf{k},d|B(E)|\mathbf{p},\phi\rangle P_\phi(p^2,E)\langle\mathbf{p},\phi|t(E)|\mathbf{k}',d\rangle. \tag{6a}$$

$$\langle\mathbf{k},\phi|t(E)|\mathbf{k}',d\rangle=\tfrac{3}{2}\langle\mathbf{k},\phi|B(E)|\mathbf{k}',d\rangle+\frac{1}{2(2\pi)^3}\int d^3p\langle\mathbf{k},\phi|B(E)|\mathbf{p},\phi\rangle P_\phi(p^2,E)\langle\mathbf{p},\phi|t(E)|\mathbf{k}',d\rangle$$
$$+\frac{3}{2(2\pi)^3}\int d^3p\langle\mathbf{k},\phi|B(E)|\mathbf{p},d\rangle P_d(p^2,E)\langle\mathbf{p},d|t(E)|\mathbf{k}',d\rangle. \tag{6b}$$

[9] A. N. Mitra, Nucl. Phys. **32**, 529 (1962).
[10] R. D. Amado, Phys. Rev. **132**, 485 (1963).
[11] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[12] F. W. Low, Phys. Rev. **97**, 1392 (1955).

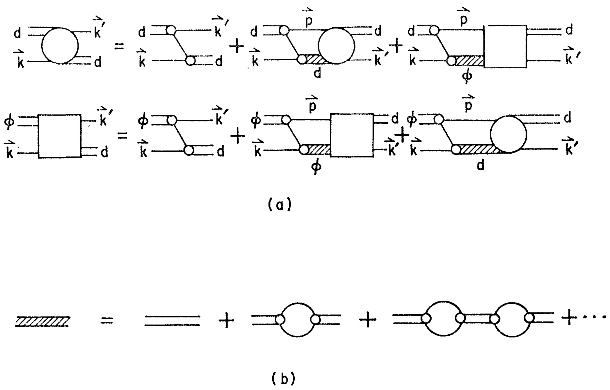


FIG. 1. (a) Coupled integral equations for neutron-deuteron scattering. (b) Perturbative representation of the propagator. The single line represents a nucleon, the double line a correlated pair φ or d. The small circle is the nucleon-nucleon vertex, the large circle the amplitude for $nd \to nd$, and the box the amplitude for $nd \to n\varphi$.

The momentum labels the nucleon momentum and ϕ or d label which pair the nucleon is incident on. B is the Born approximation representing the exchange of a nucleon between the pairs and P is the full propagator in the intermediate state for an interacting pair and a free particle. These equations, like most linear scattering equations, involve going off the energy shell, and E is the total energy variable. It need have no connection with any momentum appearing in the equations. The Born term is given by

$$\langle \mathbf{k},\alpha | B(E) | \mathbf{k}',\beta\rangle = \frac{\gamma_\alpha \gamma_\beta f_\alpha((\mathbf{k}'+\mathbf{k}/2)^2) f_\beta((\mathbf{k}+\mathbf{k}'/2)^2)}{E-k^2-k'^2-(\mathbf{k}+\mathbf{k}')^2}. \quad (7)$$

Here α and β stand for d or ϕ. γ_α is the coupling constant of the two nucleons to the α pair state. $f_\alpha(k^2)$ is the vertex function of the pair state. The propagators are

$$P_d(p^2,E) = \left\{ (\sigma+\epsilon) \times \left[Z - \frac{\gamma_d^2}{(2\pi)^3} \int \frac{d^3n\, f_d^2(n^2)}{(\epsilon+2n^2)^2(\sigma-2n^2)} \right] \right\}^{-1}, \quad (8a)$$

and

$$P_\phi(p^2,E) = -\left[1 + \frac{\gamma_\phi^2}{(2\pi)^3} \int \frac{d^3n\, f_\phi^2(n^2)}{\sigma-2n^2} \right]^{-1}, \quad (8b)$$

where

$$\sigma = E - \tfrac{3}{2}p^2,$$

ϵ is the deuteron binding energy, and Z is the wave-function renormalization constant of the deuteron. Further discussion of these equations is presented in Ref. 3. They are exact three-body equations for nucleon-deuteron scattering with separable interactions. The role of Z is to reduce the triplet central nucleon-nucleon force and give the correct triton energy and n-d scattering lengths. The equations are represented graphically in Fig. 1. The amplitudes have a pole at $E=-B$, the three particle binding energy. The residue at this pole is factorable according to

$$\langle \mathbf{k},d | \Gamma | T\rangle \langle T | \Gamma | \mathbf{k}',d\rangle \quad \text{for Eq. (6a)}$$

and

$$\langle \mathbf{k},\phi | \Gamma | T\rangle \langle T | \Gamma | \mathbf{k}',d\rangle, \quad \text{for Eq. (6b)},$$

where Γ is the vertex operator and $|T\rangle$ is the triton. Each of these vertex functions satisfies an equation exactly analogous to (3b), that is

$$\langle \mathbf{k},d | \Gamma | T\rangle = \frac{1}{2(2\pi)^3} \int d^3p \langle \mathbf{k},d | B(-B) | \mathbf{p},d\rangle P_d(p^2,-B) \langle \mathbf{p},d | \Gamma | T\rangle + \frac{3}{2(2\pi)^3} \int d^3p \langle \mathbf{k},d | B(-B) | \mathbf{p},\phi\rangle P_\phi(p^2,-B) \langle \mathbf{p},\phi | \Gamma | T\rangle. \quad (9a)$$

$$\langle \mathbf{k},\phi | \Gamma | T\rangle = \frac{1}{2(2\pi)^3} \int d^3p \langle \mathbf{k},\phi | B(-B) | \mathbf{p},\phi\rangle P_\phi(p^2,-B) \langle \mathbf{p},\phi | \Gamma | T\rangle + \frac{3}{2(2\pi)^3} \int d^3p \langle \mathbf{k},\phi | B(-B) | \mathbf{p},d\rangle P_d(p^2,-B) \langle \mathbf{p},d | \Gamma | T\rangle \quad (9b)$$

which equations are represented graphically in Fig. 2. These vertices are not then full wave functions, but rather only the vertices for finding a nucleon and a two-body singlet or triplet state. To get the full three particle vertex we must let each of the two particle correlated states propagate and disassociate. This is accomplished by appending a propagator and a two-particle vertex to each so that the three-particle vertex is given by

$$\langle T | \Gamma | \mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\rangle = \langle T | \Gamma | \mathbf{k}_1,d\rangle P_d(k_1^2,-B) \gamma_d f_d([(\mathbf{k}_3-\mathbf{k}_2)/2]^2) + \langle T | \Gamma | \mathbf{k}_1,\phi\rangle P_\phi(k_1^2,-B) \gamma_\phi f_\phi([(\mathbf{k}_2-\mathbf{k}_3)/2]^2). \quad (10)$$

This equation is represented graphically in Fig. 3. The three-particle wave function, in momentum space is

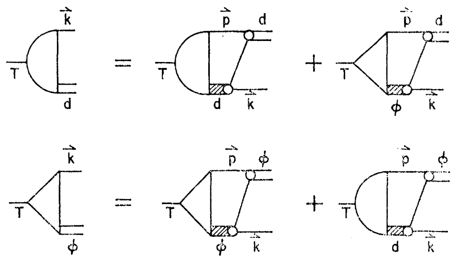


FIG. 2. The coupled equations for the vertex functions $T \to nd$ represented by the semicircle and $T \to n\varphi$ by the triangle.

then given by

$$\langle T|\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\rangle = \langle T|\Gamma|\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3\rangle/(k_1^2+k_2^2+k_3^2+B), \quad (11)$$

analogously to Eq. (4). This wave function does not possess the proper symmetry. We have also suppressed spin and isospin indices. If we reintroduce them (call them α), we can construct the properly antisymmetric state of the triton as follows:

$$|T\rangle = N\int \langle T|\mathbf{k}_1\alpha_1,\mathbf{k}_2\alpha_2,\mathbf{k}_3\alpha_3\rangle d^3k_1 d^3k_2 d^3k_3 \times\psi_{\alpha_1}{}^\dagger(\mathbf{k}_1)\psi_{\alpha_2}{}^\dagger(\mathbf{k}_2)\psi_{\alpha_3}{}^\dagger(\mathbf{k}_3)|0\rangle. \quad (12)$$

Where the $\psi_\alpha{}^\dagger(\mathbf{k})$ is the nucleon field creation operator for momentum $\mathbf{k}$ and spin and isospin α and obeys the usual anticommutation relation. The integral includes a sum over α. N is some normalization. Strictly we should also include the component of the three-body wave function corresponding to a nucleon and bare deuteron, since we keep $Z\neq 0$. We take the attitude, however, that since the actual triton has no such component, we should only take the form factor of the three-particle component. We therefore normalize this component to 1. In any case the nucleon and bare-deuteron component presumably has a very small probability.

We wish now to use (12) to calculate the body form factors for protons or neutrons in the triton. These are given by

$$F_{p/n}(q^2) = \frac{1}{(2\pi)^3}\int \sum_m \langle T_{-\mathbf{q}/2}|\psi_{m,\pm}{}^\dagger(\mathbf{p}-\tfrac{1}{2}\mathbf{q}) \times\psi_{m,\pm}(\mathbf{p}+\tfrac{1}{2}\mathbf{q})|T_{\mathbf{q}/2}\rangle d^3p, \quad (13)$$

where we have separated the spin and isospin indices on ψ; and used $+$ for proton, $-$ for neutron, and m for the ordinary spin, $|T_\mathbf{p}\rangle$ means a state of the triton with center-of-mass momentum $\mathbf{p}$. To calculate the form factors one now substitutes the expression for $|T\rangle$ from (12) into (13) (with the appropriate center-of-mass momentum), manipulates away the field operators to get a series of delta functions, and integrates over all the internal moments. There are many different factor pairings of the operators, and hence many different terms in the expression for the form factors. These various terms are most easily listed graphically. They are so presented in Fig. 4. From the factor pairings one finds that the form factors contain one of each of the A terms, two of the B, four of the C, and two of the C'. Each of the terms can be factored into a part containing the spin and isospin information, and a part involving scalar functions of momenta. The spin and isospin factors can be calculated, for example, by using standard Clebsch-Gordan algebra. Our method for doing this is outlined in Appendix II. This then leaves the diagrams of Fig. 4, stripped of spin and isospin factors, to be calculated. Putting in the spin and isospin factors and the weights from the factor pairings we get for the proton body form factor in the triton

$$F_p(q^2) = \tfrac{2}{3}A_\phi+\tfrac{1}{3}B_\phi+B_d-2C_{\phi d}+\tfrac{2}{3}C_{\phi\phi} -\tfrac{1}{6}C_{\phi\phi}{}'+\tfrac{1}{2}C_{dd}{}'-C_{\phi d}{}', \quad (14)$$

and for the neutron

$$F_n(q^2) = A_d+\tfrac{1}{3}A_\phi+B_d+(5/3)B_\phi+C_{dd}-C_{\phi d} -3C_{d\phi}+\tfrac{1}{3}C_{\phi\phi}+\tfrac{2}{3}C_{\phi\phi}{}'-2C_{\phi d}{}', \quad (15)$$

where now the letters A, B, C, C' stand for the functions obtained for the diagrams without spin and iso-

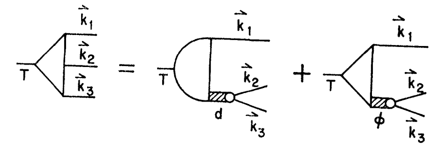


FIG. 3. The relation of the vertex $T \to 3n$ to the two-particle vertices.

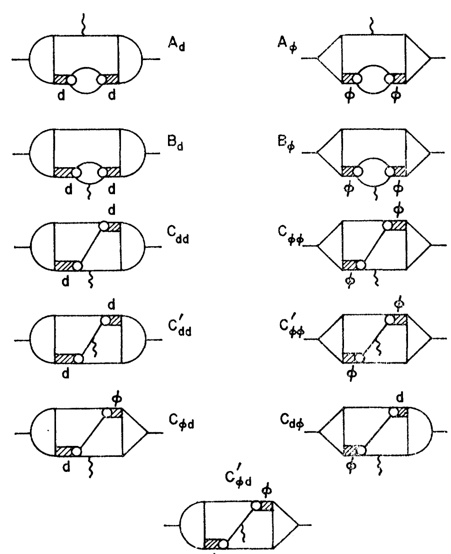


FIG. 4. The eleven terms entering the triton form factor. The wavy line represents the virtual photon.

spin. If we use the fact that at $q^2=0$

$$A_\phi=B_\phi,\quad A_d=B_d,\quad C_{dd}=C_{dd}',\quad C_{\phi\phi}=C_{\phi\phi}' \qquad C_{\phi d}=C_{d\phi}=C_{\phi d}' \tag{16}$$

we get $F_n(0)=2F_p(0)$, which reassures us that there are two neutrons and one proton in the triton.

The remaining problem in calculating the form factors is to obtain A, B, C, and C' as functions of q^2. To do this we must obtain the wave function of Eq. (11) which in turn requires the residue of the scattering amplitude as a function of momentum. Rather than obtain this by extrapolation, which can be treacherous on a digital computer, we calculate it directly on the computer. To do this we first convert the integrals of (6) to sums using Gaussian quadratures.[13] This replaces the integral equations by a set of simultaneous algebraic equations. The position of the bound state is given by the value of the energy which makes the determinant of the coefficients in the equation zero. For the nonsingular case the equations are inverted and the value of the determinant is calculated using the IBM share routine MATINV. We have modified this routine so that when the magnitude of the determinant is less than a prescribed amount, the residue matrix is calculated, and hence we get the residues or the vertex for finding an n and d in the triton or an n and φ. These are each functions of the relative n-d or n-φ momentum in the T rest frame. Call them $R_d(n^2)$ and $R_\varphi(n^2)$. In terms of these residues, the terms in the form factors are

$$A_\alpha=\frac{\gamma_\alpha^2}{(2\pi)^6}\int\frac{d^3pd^3k\,R_\alpha(p^2)R_\alpha((\mathbf{p}+\frac{2}{3}\mathbf{q})^2)P_\alpha(p^2,-B)P_\alpha((\mathbf{p}+\frac{2}{3}\mathbf{q})^2,-B)f_\alpha{}^2(k^2)}{(B+\frac{3}{2}p^2+2k^2)[B+\frac{3}{2}(\mathbf{p}+\frac{2}{3}\mathbf{q})^2+2k^2]}, \tag{17a}$$

$$B_\alpha=\frac{\gamma_\alpha^2}{(2\pi)^6}\int\frac{d^3pd^3k\,R_\alpha(p^2)R_\alpha((\mathbf{p}+\frac{1}{3}\mathbf{q})^2)P_\alpha(p^2,-B)P_\alpha((p+\frac{1}{3}\mathbf{q})^2,-B)f_\alpha(k^2)f_\alpha((k+\frac{1}{2}\mathbf{q})^2)}{(B+\frac{3}{2}p^2+2k^2)[B+\frac{3}{2}(\mathbf{p}+\frac{1}{3}\mathbf{q})^2+2(k+\frac{1}{2}\mathbf{q})^2]}, \tag{17b}$$

$$C_{\alpha\beta}'=\frac{\gamma_\alpha\gamma_\beta}{(2\pi)^6}\int\frac{d^3pd^3k\,R_\alpha(p^2)R_\beta(k^2)P_\alpha(p^2,-B)P_\beta(p^2,-B)f_\alpha((\mathbf{k}+\frac{1}{2}\mathbf{p}+\frac{1}{3}\mathbf{q})^2)f_\beta((\mathbf{p}+\frac{1}{2}\mathbf{k}-\frac{1}{3}\mathbf{q})^2)}{[B+\frac{3}{2}p^2+2(\mathbf{k}+\frac{1}{2}\mathbf{p}+\frac{1}{3}\mathbf{q})^2][B+\frac{3}{2}k^2+2(\mathbf{p}+\frac{1}{2}\mathbf{k}-\frac{1}{3}\mathbf{q})^2]}, \tag{17c}$$

$$C_{\alpha\beta}=\frac{\gamma_\alpha\gamma_\beta}{(2\pi)^6}\int\frac{d^3pd^3kR_\alpha(p^2)R_\beta(k^2)P_\alpha(p^2,-B)P_\beta(k^2,-B)f_\alpha((\frac{1}{2}\mathbf{p}-\mathbf{k}+\frac{2}{3}\mathbf{q})^2)f_\beta((\mathbf{p}-\frac{1}{2}\mathbf{k}+\frac{1}{3}\mathbf{q})^2)}{[B+\frac{3}{2}k^2+2(\mathbf{p}-\frac{1}{2}\mathbf{k}+\frac{1}{3}\mathbf{q})^2][B+\frac{3}{2}p^2+2(\frac{1}{2}\mathbf{p}-\mathbf{k}+\frac{2}{3}\mathbf{q})^2]}, \tag{17d}$$

where α and β stand for φ or d. The rest of the quantities are defined after Eq. (6). Equations (17a) to (17d) may be obtained diagrammatically or by direct substitution. In either case use is made of the translation invariance of the vertex functions and in getting C an C' some variable changes have been made. These are useful since in doing the integrals of (17c) and (17d) as sums we will need only the residues at the mesh points. In (17a) and (17b) this does not happen and interpolation is used to find the residue between mesh points.

For simple two-body vertex functions f, the d^3k integral of A can be done analytically. All other integrals are done on the computer using the same Gaussian quadrature mesh as is the inversion for the momenta and using a different Gaussian quadrature mesh for the angular integrals. No sophisticated multiple-integral methods were used, but rather the integrals were done as sequential one-dimensional integrals. This is tedious and with only 21 points for the momentum mesh and 7 for the angular integrals, the entire calculation including obtaining the residues took 26 min on the IBM 7094 at the UCLA Mathematical Sciences Computing Center.

III. FORM-FACTOR RESULTS

To calculate the form-factor terms of Sec. II, we must specify the quantities that enter. As in our previous calculation,[3] we take a Hulthén form for the two-body vertex both for the d and the φ:

$$f_\alpha(q^2)=1/(q^2+\beta_\alpha{}^2). \tag{18}$$

This form gives the deuteron a Hulthén wave function.[14] In each two-body spin channel there are now two parameters, β and γ which are essentially the range and strength of the force. In the singlet channel we determine these by fitting to the scattering length and effective range. These give

$$\beta_\phi=\frac{3}{2r_s}\left[1+\left(1-\frac{16r_s}{9a_s}\right)^{1/2}\right], \tag{19}$$

and

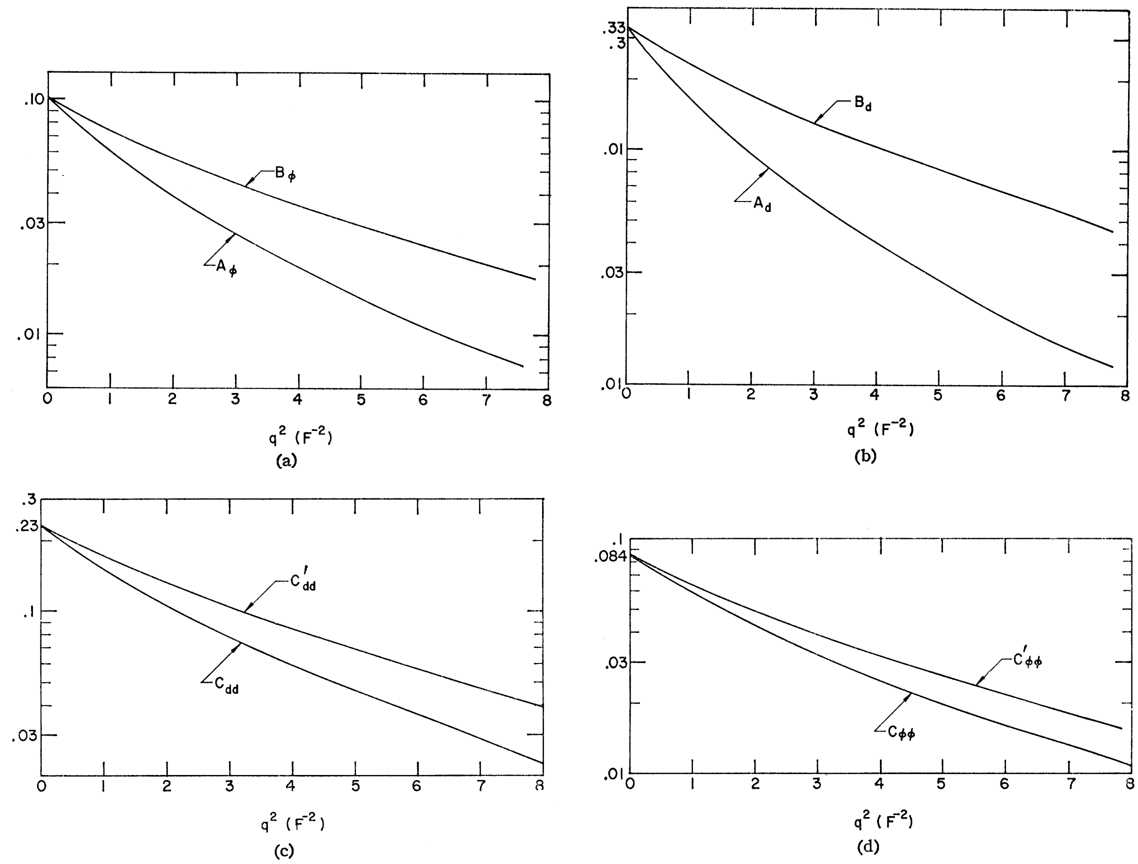
$$\gamma_\phi{}^2=16\pi\beta_\phi{}^4a_s/(a_s\beta_\phi-2).$$

We use $a_s=-23.78$ fermis and $r_s=2.67$ fermis.[15] In the triplet channel we fit the deuteron binding energy of $\epsilon=-2.226$ MeV, and the scattering length $a_d=5.411$

[13] See for example V. I. Krylov, *Approximate Calculation of Integrals*, translated by A. H. Stroud (The Macmillan Company, New York, 1962).

[14] L. Hulthén and M. Sugawara, in *Encylopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

[15] M. J. Moravcsik, *The Two-Nucleon Interaction* (Oxford University Press, New York, 1963).



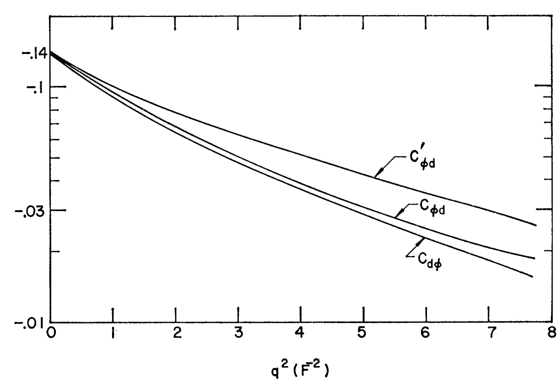


FIG. 5. The terms entering the triton form factor as a function of q^2 in inverse fermis squared. (a) A_φ, B_φ. (b) A_d, B_d. (c) C_{dd}, C_{dd}'. (d) $C_{\varphi\varphi}$, $C_{\varphi\varphi}'$ and (e) $C_{d\varphi}$, $C_{\varphi d}$, $C_{\varphi d}'$. The terms are correctly normalized to use directly in Eqs. (14) and (15).

fermis,[15] according to

$$a_t^{-1}=\frac{16\pi\alpha_d{}^2\beta_d{}^4Z}{\gamma_d{}^2}+\frac{\gamma_d\beta_d(\alpha_d+2\beta_d)}{[2(\beta_d+\alpha_d)^2]},\tag{20}$$

$$\gamma_d=32\pi\alpha_d\beta_d(\alpha_d+\beta_d)^3(1-Z),$$

where $\alpha_d{}^2=\frac{1}{2}\epsilon$. We keep Z different from zero to represent the relative weakening of the triplet force in the triton due to the tensor force, hard cores, etc. We take Z from our previous work, $Z=0.0488$.[2] This gives us the correct doublet nucleon-deuteron scattering length. With these parameters we find a triton binding energy of -8.576 MeV. (The experimental value is 8.49 MeV.) For this energy the determinant of the Eq. (6) is -1.07×10^{-7}. With the parameters so determined we calculate the form factor terms of Fig. 4. The results are shown in Fig. 5 and the combinations appropriate to the proton and neutron body form factors of the triton are shown in Fig. 6. They are compared there with the experimental results from Stanford analyzed according to Collard *et al.*[4] The analysis assumes the charge form factors of He^3 and H^3 can be simply related to the charge form factors of the proton and neutron

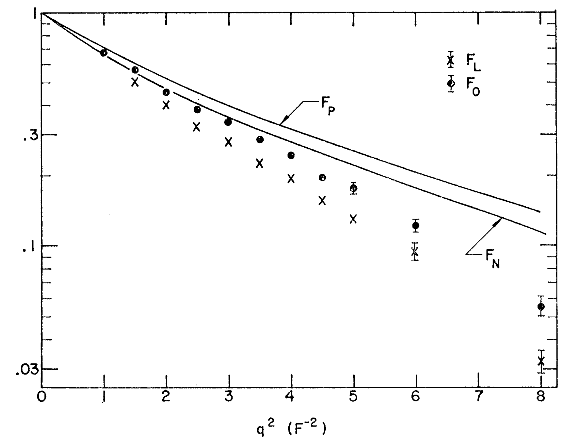


FIG. 6. The theoretical proton and neutron body form factors of the triton as a function of q^2 in inverse fermis squared, both normalized to 1. The experimental points F_0 and F_L are from Collard *et al.*[4] and have been analyzed according to their prescription. We should have $F_p=F_0$ and $F_N=F_L$.

and the body form factors of the odd nucleon, F_0, and the like pair, F_L, in the three-body system by

$$F_{\mathrm{ch}}(\mathrm{He}^3)=F_{\mathrm{ch}}(p)F_L+\tfrac{1}{2}F_{\mathrm{ch}}(n)F_0\,, \tag{21a}$$

$$F_{\mathrm{ch}}(\mathrm{H}^3)=F_{\mathrm{ch}}(p)F_0+2F_{\mathrm{ch}}(n)F_L\,. \tag{21b}$$

Forms for the neutron and proton form factors are taken from Hughes *et al.*[16] It is these odd- and like-pair body form factors, all normalized to one, that are plotted in Fig. 6. We should have $F_0=F_p$ and $F_L=F_n$. We do not. Although our form factors reproduce the splitting $F_p>F_n$, they are too large. This means our triton is too concentrated at its center.

We can also use our form factors to extract a mean-square radius according to

$$F(q^2)\cong 1-q^2r^2/6 \tag{22}$$

for small q^2. We get a proton radius of 1.45 F and a neutron radius of 1.62 F. Experimentally the charge radius of H^3 is 1.70 ± 0.05 F and of He^3 1.87 ± 0.05 F.[4] We analyze these using Eq. (21) expanding for small q^2 and using a proton charge radius of 0.80 ± 0.01 F.[17] For the neutron we take a radius of zero since the slope of the three body form factors is determined experimentally at values of q^2 above the very small q^2 for which the neutron charge form factor slope is known to be important. We get 1.69 ± 0.05 F for the rms radius of the like nucleon distribution and 1.50 ± 0.05 F for the odd nucleon. Any contribution from a neutron radius would increase these radii. They are slightly larger than our theoretical values and give further evidence that our triton is too compressed. Recently Dalitz and Thacher[18] have also analyzed the data and obtained rms proton radii of 1.47 ± 0.07 F for H^3 and 1.66 ± 0.07 F for He^3, using slightly different parameters. These are in better agreement with our values, but are still on the large side.

These results are at first surprising and disappointing in view of our excellent results for the binding energy, scattering lengths, and n-d cross sections. It is probably neither correct nor honest to blame the discrepancy on the form factors taken from experiment. There is some leeway in the method of extracting F_L and F_0 from experiment. There is, for example, the question of whether the measured form factors can be expressed as the product of nucleon form factors and body form factors. There is also the question of relativistic corrections, particularly since the object we calculate is entirely nonrelativistic. However, our discrepancy is larger than any reasonable latitude in this analysis.

We are thus forced to accept the discrepancy and account for it. If we look at Fig. 5, we see that the relatively large contributions at large q are coming from these terms in which we "measure" the two-body correlations. The A terms, for example, fall very rapidly with q, and the C' and B terms fall least rapidly with the C in between. In the B and C' we are taking the form factor of the correlated pairs, and C is sort of the square root of them. Unfortunately, because of the importance of the exchange term C', we cannot simply factor the two-body form factor out of these terms and take it from experiment. Thus, we conclude that our two-body form factor is too large at large q: that is, our pairs are getting too close together. This is already seen in the Hulthén deuteron which gives too large a form factor—that is, too small a deuteron.[5] This difficulty is repaired by introducing short-range repulsion in the two-body force. This pushes the deuteron out a little and so reduces the form factor to the experimental value. It is easy to see in the two-body system that the effect of the short-range repulsion will be to introduce oscillations in the form factor and also to make it tend to zero more rapidly at large momentum transfers. Consider the definition of the form factor

$$F(q^2)=\int\psi^2(r)\exp(i\mathbf{q}\cdot\mathbf{r}/2)d^3r\,, \tag{23}$$

where $\psi(r)$ is the internal two-body wave function. Then we have

$$\int F(q^2)d^3q\propto\psi^2(0)\,, \tag{24}$$

assuming that we may interchange orders of integration. Thus, if $\psi^2(0)$ is zero, $F(q^2)$ must oscillate. Further, if we expand F for large q^2 in inverse powers of q^2, we find that the coefficient of the leading power of q^{-2n} is $\psi^2(0)$. Therefore, for large q^2 F will go to zero more rapidly if $\psi^2(0)$ is zero. This is of course an asymptotic result and we are not in the asymptotic region. In fact,

[16] E. B. Hughes, T. A. Griffy, M. R. Yearian, and R. Hofstadter, Phys. Rev. **139**, B458 (1965).

[17] Cf. R. R. Wilson and J. S. Levinger, *Ann. Rev. Nucl. Sci.* **14**, 135 (1964).

[18] R. H. Dalitz and T. W. Thacher, Phys. Rev. Letters **15**, 204 (1965).

for the two-particle case, the form factor will change sign at a q of the order of π/r_c, where r_c is some core radius, and we are interested in q well below this value. But the fact that it must vanish in the repulsive case clearly pushes the form factor below the case with no repulsion, since it does not vanish. This pushing out effect and concomitant reduction of the form factor must also be present in the singlet pair, where, in fact, the scattering evidence for repulsion at high momentum is better than in the triplet state.

The effect of the nonvanishing wave-function normalization constant for the deuteron on the form factor is small and in the wrong direction. Although making Z not zero is a way of weakening the attractive force, the effect of Z on the form factor is to increase it rather than decrease it.

Turning back to the triton, we may blame our failure to account for its size on the absence of a short-range repulsion, which acts to push the correlated particle pairs apart.[19] In more traditional nuclear physics language, we may say our force does not saturate. Our phenomenological artifice Z for weakening the effective triplet force has given us the correct binding energy, but has not provided the dynamical repulsion needed to give the correct wave function.[19a]

In a sense this result is both obvious and reassuring. It is obvious because of the importance of saturation in complex nuclei, of which the triton is the first, is well known. It is reassuring because it shows that we may learn something about the details of nuclear forces, off-energy shell properties, three-body forces, etc., from the three-body system. The previous results on binding energies and low-energy scattering did so well with so little input that they left little space for learning about these details. But it is well known that wave functions are a much more sensitive test of a theory than energies or scattering parameters.

The recalculation of the binding energy and form factors with short-range repulsion will require considerable time and effort. Tabakin's fit of nucleon-nucleon scattering with separable interactions indicates that there are a number of effects, tensor forces, attractive d waves for example, that are of the same order of importance as the short-range repulsion.[20] Including all of them would add many new equations to the coupled set (6a) and (6b) and many new form factor terms. In this context it should be recalled that it requires at least two separate interactions in a channel to represent long-range attraction and short-range repulsion. This greatly added complexity probably could be tolerated in computer memory space for the bound-state problem since the kernels are purely real and rather smooth in this case. Neither is true for the scattering problem and thus to add so many forces in that case is probably beyond present computer fast-memory capabilities. Even in the bound-state problem, however, the calculation of the form factor would take a great deal of time. A simpler possibility would be to take a two-body vertex that fits the deuteron form factors better than the Hulthén form and use it both for the singlet and triplet case. This can always be done. A single separable interaction resulting from such a form will never truly be repulsive at short distances, that is the phase shift will never change sign. This is the case even if one puts a "hole" in the deuteron wave function. In that case the phase shift oscillates at large energy, has many zeros, but is always of one sign. Modifying our calculation with the introduction of a more complex two-body vertex is presently under study. A simple wave function that fits the deuteron form factor without introducing oscillation in the phase shift has been given by Durand.[21]

IV. BOUND-STATE WAVE-FUNCTION SYMMETRIES

The traditional way in which the triton wave function is analyzed is in terms of its symmetries.[22] It is well known, for example, that the three-nucleon bound state consists mostly of a totally symmetric S state of isotopic spin $\frac{1}{2}$. This is the state that would arise from a pure Wigner force, that is no spin or isotopic spin dependence. It would lead to no splitting of the odd and like pair form factors and a vanishing cross section for thermal neuteron capture of deuterium.[7] The next most likely state is the S state of mixed symmetry, but still isospin $\frac{1}{2}$—usually called the S' state. This state arises with a singlet-triplet spin dependence of nuclear forces. It splits the form factors and gives rise to thermal neutron capture. The inclusion of tensor forces and spin-orbit forces gives states of more complex symmetry and states of nonzero total orbital angular momentum. These states and also states of isotopic-spin $\frac{3}{2}$ coming from isotopic spin violating forces have been discussed recently.[6] In particular they have been invoked since the amount of S' needed to fit the form factor splitting seems to give too much thermal neutron capture. Unfortunately this simple analysis is confused by the question of meson exchange currents. It is already known from the magnetic moments of H^3 and He^3 that these effects are present.[22] In the thermal capture rate they interfere destructively with the S' contribution and therefore it might be possible to tolerate larger S' probabilities than usually considered without doing violence to the neutron capture rate.[7]

Another approach to the wave function symmetries is

[19] Such effects of the dependence on repulsion have been obtained in a phenomenological model by Y. C. Tang and R. C. Herndon, Phys. Letters **18**, 42 (1965). I would like to thank L. I. Schiff for calling this work to my attention in preprint.

[19a] *Note added in proof.* The need for short-range repulsion is also evident in the work of Dalitz and Thacker (Ref. 18). We thank Professor Dalitz for a letter clarifying this work.

[20] F. Tabakin, Ann. Phys. **30**, 51 (1964).

[21] L. R. Durand, Phys. Rev. **123**, 1393 (1961).

[22] See, for example, R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953).

through the Gamow-Teller matrix element for tritium beta decay.[8] The purely symmetric S state gives $\sqrt{3}$ for this matrix element, and any other reasonable state including S' would subtract from this. The experimental number is slightly larger than this.[8] Again this gives slight evidence for exchange effects. Until these can be calculated we see little hope of using the matrix element to find the S' probability. We hope, of course, that these effects are not very important in form factors, or at least have been properly removed in the analysis of the data. This question of the meson effects on the form factors is a complex one which we have neither the space nor the competence to discuss.

In our theory, since we have only charge-independent central forces but do have spin dependence, our triton will be a pure L-S eigenstate with total orbital angular momentum zero and spin and isospin $\frac{1}{2}$. Both the completely symmetric and mixed symmetry S state will be present. We wish now to calculate the S' probability. Not having our states expressed explicitly as product wave functions in space, spin, and isotopic spin, we shall calculate the probability by calculating the Gamow-Teller matrix element and then comparing with conventional results. We want the reduced matrix element between the triton and He^3 of $\sum \boldsymbol{\sigma}\tau^\dagger$. We take σ_z and the states of the triton and He^3 with Z component $\frac{1}{2}$ and divide by the appropriate Clebsch-Gordan coefficient to get the reduced matrix element. This gives when the $\sum\sigma\tau^\dagger$ is expressed in terms of field operators

$$\mathfrak{M}=-\frac{1}{(2\pi)^3}\int\langle T\; m=\tfrac{1}{2}|\psi_{1/2-}{}^\dagger(\mathbf{k})\psi_{1/2+}(\mathbf{k}) -\psi_{-1/2-}{}^\dagger(\mathbf{k})\psi_{-1/2+}(\mathbf{k})|\,\mathrm{He}^3\; m=\tfrac{1}{2}\rangle d^3k/\langle\tfrac{1}{2}\,\tfrac{1}{2}|10\tfrac{1}{2}\,\tfrac{1}{2}\rangle, \quad (25)$$

where $\langle jm|j_1m_1j_2m_2\rangle$ is the Clebsch-Gordan coefficient for adding $j_1+j_2=j$ and $m_1+m_2=m$. The minus sign is an over-all phase to make us the same as Blin-Stoyle.[8] This matrix element may be expressed in terms of the various factors that enter the form factor calculation. In fact the diagrams of Fig. 4 may be interpreted as beta-decay diagrams with the photon replaced by the lepton pair. Only the spin algebra must be redone. Also since the operator in spin space is σ now rather than 1, a new term appears in the B sequence. The correlative pair can change from φ to d via the decay. So we need a term $B_{\varphi d}$. All terms are, of course, to be taken at zero momentum transfer. Doing the spin algebra by the method indicated in the Appendix, and using the relation

$$\mathfrak{M}/\sqrt{3}=1-P_{S'}/3\,, \quad (26)$$

where $P_{S'}$ is the S' probability we obtain

$$P_{S'}=2[(B_\phi+B_d+2B_{\phi d})-(C_{dd}+C_{\phi\phi}+2C_{\phi d})]. \quad (27)$$

In obtaining this relation use has been made of the equalities of Eq. (16). Calculation similar to those outlined in (17) gives $B_{\varphi d}=-0.1788$. Both $B_{\varphi d}$ and $C_{\varphi d}$ are negative. Hence each of the combination in round brackets are differences. Each of these differences clearly vanish when the singlet-triplet force difference is turned off. $P_{S'}$ is furthermore a difference of these differences. It is therefore possible that there is a considerable singlet-triplet force difference, leading to relatively large values of $B_\varphi+B_d+2B_{\varphi d}$ and the corresponding C expression but that $P_{S'}$ is small. The form factors, measuring as they do a different combination of these factors and measuring them at nonzero momentum transfer, could therefore be quite different for proton and neutron, and $P_{S'}$ be quite small. Our value is $P_{S'}=(7\pm1)\%$. This value is larger than usually allowed but may be compatible with the Gamow-Teller matrix element and the thermal neutron capture rate when due allowance is made for exchange currents. The error in our answer comes from the computing error. We measure this by the extent to which the equalities of (16) are satisfied. Due to our relatively coarse integration mesh they are only valid to 1%. This error is vastly magnified in $P_{S'}$ because of the many differences of large numbers.

V. CONCLUSION

We have seen how the exact three-particle wave function may be extracted from our equation for nucleon-deuteron scattering with separable interactions by studying the residue of this equation at the triton pole as a function of momentum, and we have seen how to calculate the proton and neutron-body form factor of the triton from this wave function. Many terms enter because of antisymmetry. We have calculated all these terms for our simple theory with only low-energy triplet and singlet nucleon-nucleon interactions. In spite of the good value we get for the binding energy, our triton is too small because of the absence of short-range repulsion to keep the particles apart. The same difficulty would occur with our deuteron. On the other hand, the difference, seen experimentally between the proton and neutron body form factors is reproduced qualitatively. To correct the trouble with the form factor by including short-range repulsion in a consistent manner would be quite complicated—but not impossible. To correct it in a more phenomenological way would be easier but perhaps of less fundamental interest.

It is satisfactory, in a way, that our results are not very good. The surprisingly good results we obtained in the binding energy and scattering calculation[2,3] left little leeway for learning about the refinements of the nucleon-nucleon force such as its short-range behavior, off-energy shell effects and three-body forces. As we would have expected, the wave function is a much more sensitive test for these effects.

We have also calculated the probability of the mixed symmetry state. The expression we obtain is the (small) second difference of large numbers. Hence very small changes such as a short-range repulsion could make it very small indeed. We obtain $(7\pm1)\%$, the error arising from the computer mesh and the differences of large numbers. This probability is rather larger than usually estimated[6–8] but the effects of meson currents make estimates of this probability from experiments treacherous. Whether reducing our form factors by including repulsion will also reduce this probability remains to be seen. It will be interesting for us to calculate the rate of capture of thermal neutrons in deuterium to compare with other estimates and get further information on this state.

The problem of meson effects is a knotty one. It is even present in the form factor analysis, where hopefully it has been properly tamed. It may in fact be the limiting feature in an attempt to calculate the wave function with great accuracy and comparing with experiment, setting in to cloud the issue before the fine points of, for example, multiparticle forces can be unraveled.

ACKNOWLEDGMENTS

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APPENDIX I. THE WAVE FUNCTION

Using a field-theoretic formalism in which we introduce an elementary particle for each of the φ and d, we may write the renormalized Hamiltonian for the system in second-quantized form:

$$\begin{aligned}H=&\sum_{\mathbf{k},\alpha} k^2\psi_\alpha{}^\dagger(\mathbf{k})\psi_\alpha(\mathbf{k})+\sum_{\mathbf{k},m}[\epsilon_d(k)+\delta_d(d)]D_m{}^\dagger(\mathbf{k})D_m(k)Z_d+\sum_{\mathbf{k},\tau}[\epsilon_\phi(\mathbf{k})+\delta_\phi(\mathbf{k})]\Phi_\tau{}^\dagger(\mathbf{k})\Phi_\tau(\mathbf{k})Z_\phi\\&+\frac{\gamma_d}{\sqrt{2}}\sum_{\mathbf{p},\mathbf{k},M,\alpha,\alpha'} f_d(k^2)C_{d,M\alpha\alpha'}[D_M{}^\dagger(\mathbf{p})\psi_\alpha(\tfrac{1}{2}\mathbf{p}-\mathbf{k})\psi_{\alpha'}(\tfrac{1}{2}\mathbf{p}+\mathbf{k})+\psi_{\alpha'}{}^\dagger(\tfrac{1}{2}\mathbf{p}+\mathbf{k})\psi_\alpha{}^\dagger(\tfrac{1}{2}\mathbf{p}-\mathbf{k})D_M(\mathbf{p})]\\&+\frac{\gamma_\phi}{\sqrt{2}}\sum_{\mathbf{p},\mathbf{k},\tau,\alpha,\alpha'} f_\phi(k^2)C_{\phi,\tau\alpha\alpha'}[\Phi_\tau{}^\dagger(\mathbf{p})\psi_\alpha(\tfrac{1}{2}\mathbf{p}-\mathbf{k})\psi_{\alpha'}(\tfrac{1}{2}\mathbf{p}+\mathbf{k})+\psi_{\alpha'}{}^\dagger(\tfrac{1}{2}\mathbf{p}+\mathbf{k})\psi_\alpha{}^\dagger(\tfrac{1}{2}\mathbf{p}-\mathbf{k})\Phi_\tau(\mathbf{p})],\end{aligned}\tag{A1}$$

where the $\psi_\alpha(\mathbf{k})$ are the creation and annihilation operators for the nucleons of momentum $\mathbf{k}$ and α is a spin and isotopic spin index, they satisfy

$$\{\psi_\alpha(\mathbf{k}),\psi_{\alpha'}(\mathbf{k}')\}=0,\quad \{\psi_\alpha{}^\dagger(\mathbf{k}),\psi_{\alpha'}(\mathbf{k}')\}=\delta_{\alpha,\alpha'}\delta_{\mathbf{k},\mathbf{k}'}.\tag{A2}$$

The $D_M(\mathbf{k})$ and $\Phi_\tau(\mathbf{k})$ are the corresponding operators for the D and ϕ particles with third component of spin and isospin M and τ, respectively. They satisfy

$$\begin{aligned}[D_M(\mathbf{k}),D_{M'}(\mathbf{k}')]=[\Phi_\tau(\mathbf{k}),\Phi_{\tau'}(\mathbf{k}')]=0,\\ [D_M(\mathbf{k}),D_{M'}{}^\dagger(\mathbf{k}')]=\delta_{M,M'}\delta_{\mathbf{k},\mathbf{k}'}/Z_d,\\ [\Phi_\tau(\mathbf{k}),\Phi_{\tau'}{}^\dagger(\mathbf{k}')]=\delta_{\tau,\tau'}\delta_{\mathbf{k},\mathbf{k}'}/Z_\phi.\end{aligned}\tag{A3}$$

The Z_d and Z_φ are wave-function renormalization constants. Z_d is called Z in the body of the paper. After deriving the equations we set $Z_\varphi=0$.[23] All other combinations of ψ, D, and Φ commute. $\epsilon_d(k)$ is the renormalized energy of a d particle of momentum $\mathbf{k}$, and similarly for ϵ_φ. The $\delta_d(k)$ and $\delta_\varphi(k)$ terms are the energy renormalizations—the counterparts of mass counter terms. The coefficients $C_{d,M\alpha\alpha'}$ and $C_{\varphi,\tau\alpha\alpha'}$, are coupling coefficients. Re-expressed in terms of the components of spin μ and isospin i, they are

$$C_{d,M\alpha\alpha'}=\langle 1M|\tfrac{1}{2}\mu\tfrac{1}{2}\mu'\rangle\langle 00|\tfrac{1}{2}i\tfrac{1}{2}i'\rangle\tag{A4}$$

and

$$C_{\phi,\tau\alpha\alpha'}=\langle 1\tau|\tfrac{1}{2}i\tfrac{1}{2}i'\rangle\langle 00|\tfrac{1}{2}\mu\tfrac{1}{2}\mu'\rangle,$$

where the bracket $\langle jM|j_1M_1j_2M_2\rangle$ is the Clebsch-Gordan coefficient for adding $j=j_1+j_2$ and $M=M_1+M_2$. The coefficients of (A4) have the properties

$$\begin{aligned}\sum_{\alpha,\alpha'} C_{d,M\alpha\alpha'}C_{d,M'\alpha\alpha'}=\delta_{M,M'},\\ \sum_{\alpha,\alpha'} C_{\phi,\tau\alpha\alpha'}C_{\phi,\tau'\alpha\alpha'}=\delta_{\tau,\tau'},\\ \sum_{\alpha,\alpha'} C_{\phi,\tau\alpha\alpha'}C_{d,M\alpha\alpha'}=0,\end{aligned}\tag{A5}$$

and are odd under the interchange of α and α'. The γ's are then renormalized coupling constants and the f's the two-particle vertex functions.

We now look for a state $|T\rangle$ such that in the T rest frame

$$H|T\rangle=-B|T\rangle\tag{A6}$$

and write

$$\begin{aligned}|T\rangle=&Z_d\sum_{\mathbf{k},M,\alpha} g_{d,M\alpha}(k)D_M{}^\dagger(\mathbf{k})\psi_\alpha{}^\dagger(-\mathbf{k})|0\rangle+Z_\phi\sum_{\mathbf{k}\tau\alpha} g_{\phi,\tau\alpha}(k)\Phi_\tau{}^\dagger(\mathbf{k})\psi_\alpha{}^\dagger(-k)|0\rangle\\&+\sum_{\mathbf{k},\alpha} h_{\alpha_1,\alpha_2,\alpha_3}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2,-\mathbf{k}_3}\psi_{\alpha_1}{}^\dagger(\mathbf{k}_1)\psi_{\alpha_2}{}^\dagger(\mathbf{k}_2)\psi_{\alpha_3}{}^\dagger(\mathbf{k}_3)|0\rangle,\end{aligned}\tag{A7}$$

[23] M. T. Vaughn, R. Aaron, and R. D. Amado, Phys. Rev. **124**, 1258 (1961).

where we have suppressed dependence of the g's on the spin and isotopic spin of the triton. We now insert this form for $|T\rangle$ into the Schrödinger equation (A6) and use the commutation relations. Equating the coefficients of $D^\dagger\psi^\dagger$, $\Phi^\dagger\psi^\dagger$, and $\psi^\dagger\psi^\dagger\psi^\dagger$ separately to zero, we get

$$h_{\alpha_1,\alpha_2,\alpha_3}(\tfrac{1}{2}\mathbf{p}+\mathbf{k},\,\tfrac{1}{2}\mathbf{p}-k,\,-\mathbf{p})=(\tfrac{3}{2}p^2+2k^2+B)^{-1}\left[\frac{\gamma_d}{\sqrt{2}}f_d(k^2)C_{d,M\alpha_1\alpha_2}g_{d,M\alpha_3}(p)+\frac{\gamma_\phi}{\sqrt{2}}f_\phi(k^2)C_{\phi,\tau\alpha_1\alpha_2}g_{\phi,\tau\alpha_3}(p)\right] \quad \text{(A8)}$$

and

$$g_{d,M\alpha}(p)=\sum\frac{\sqrt{2}\gamma_d f_d(k^2)C_{d,M\alpha_1\alpha_2}}{Z_d(p^2+\epsilon_d(p)+\delta_d(p)+B)}\big[h_{\alpha_1,\alpha_2,\alpha}(\tfrac{1}{2}\mathbf{p}+\mathbf{k},\,\tfrac{1}{2}\mathbf{p}-\mathbf{k},\,-\mathbf{p})$$
$$+h_{\alpha_2,\alpha,\alpha_1}(\tfrac{1}{2}\mathbf{p}+\mathbf{k},\,-\mathbf{p},\,\tfrac{1}{2}\mathbf{p}-\mathbf{k})+h_{\alpha,\alpha_1,\alpha_2}(-\mathbf{p},\,\tfrac{1}{2}\mathbf{p}-\mathbf{k},\,\tfrac{1}{2}\mathbf{p}+\mathbf{k})\big], \quad \text{(A9)}$$

and a corresponding expression for g_φ. If we substitute (A8) into (A9) and use (A5), we get

$$g_{d,M\alpha}(p)=2\left[Z_d(p^2+\epsilon_d(p)+\delta_d(p)+B)-\gamma_d^2\sum_{\mathbf{k}}\frac{f_d^2(k^2)}{(\frac{3}{2}p^2+2k^2+B)}\right]^{-1}$$
$$\times\left\{\gamma_d^2\sum_{\mathbf{k},\alpha_1,\alpha_2}\frac{f_d(k^2)f_d((\frac{3}{4}\mathbf{p}+\frac{1}{2}\mathbf{k})^2)C_{d,M\alpha_1\alpha_2}C_{d,M'\alpha_2\alpha}g_{d,M\alpha_1}(|\frac{1}{2}\mathbf{p}-\mathbf{k}|)}{(\frac{3}{2}p^2+2k^2+B)}\right.$$
$$\left.+\gamma_d\gamma_\phi\sum\frac{f_d(k^2)f_\phi((\frac{3}{4}\mathbf{p}+\frac{1}{2}\mathbf{k})^2)C_{d,M\alpha_1\alpha_2}C_{\phi,\tau\alpha_2\alpha}g_{\phi,\tau\alpha_1}(|\frac{1}{2}\mathbf{p}-\mathbf{k}|)}{(\frac{3}{2}p^2+2k^2+B)}\right\} \quad \text{(A10)}$$

and a corresponding equation for g_φ. The first factor is the full propagator (8a). The vertex defined in (6) is g without this factor and (A10) when expressed as an equation for this vertex is just the integral equation (9a). Hence the prescription given in Sec. II for the wave function, h, is the same as that given by Eq. (A8) with g from (A10).

APPENDIX II. SPIN ALGEBRA

In this Appendix we present, by example, our method for obtaining the spin and isospin weights accompanying the form factor terms. Let us consider first a very simple term, A_φ. The diagram is rewritten in Fig. (7a), with the Z component of spin and isospin for each line labeled. We take the triton to have $M_Z=\frac{1}{2}$. From each vertex representing disassociation of the triton into φ and a particle we get a Clebsch-Gordan coefficient in spin and in isospin. These are contained implicitly in the g functions of the previous Appendix. We also get a set of Clebsch-Gordan coefficients for each φ disassociation. Thus the spin factors going with the diagram give

$$\sum_{m,m_1,m_2}|\langle\tfrac{1}{2}\,\tfrac{1}{2}|00\tfrac{1}{2}m\rangle\langle00|\tfrac{1}{2}m_1\tfrac{1}{2}m_2\rangle|^2=1 \quad \text{(B1)}$$

and the isospin factors give

$$\sum|\langle\tfrac{1}{2}-\tfrac{1}{2}|1i\tfrac{1}{2}i_3\rangle\langle1i|\tfrac{1}{2}i_1\tfrac{1}{2}i_2\rangle|^2=|\langle\tfrac{1}{2}-\tfrac{1}{2}|1i\tfrac{1}{2}i_3\rangle|^2, \quad \text{(B2)}$$

where we are to take $i_3=\frac{1}{2}$ for the proton term and $-\frac{1}{2}$ for the neutron and this gives the coefficients of A_φ in (14) and (15).

Now consider a more complex term. Let us take $C_{\Sigma,\Sigma'}$ where Σ and Σ' stand for φ or d. This diagram is shown in Fig. (7b) with the third component labels. The spin contribution is

$$\sum\langle\tfrac{1}{2}\,\tfrac{1}{2}|\Sigma\sigma\tfrac{1}{2}m_3\rangle\langle\Sigma\sigma|\tfrac{1}{2}m_1\tfrac{1}{2}m_2\rangle\langle\tfrac{1}{2}\,\tfrac{1}{2}|\Sigma'\sigma'\tfrac{1}{2}m_1\rangle\langle\Sigma'\sigma'|\tfrac{1}{2}m_3\tfrac{1}{2}m_2\rangle$$
$$=-[(2\Sigma+1)(2\Sigma'+1)]^{1/2}W(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2};\Sigma\Sigma'), \quad \text{(B3)}$$

where W is the usual Racah coefficient.[24] For the isospin one gets just the same sort of term except there is no

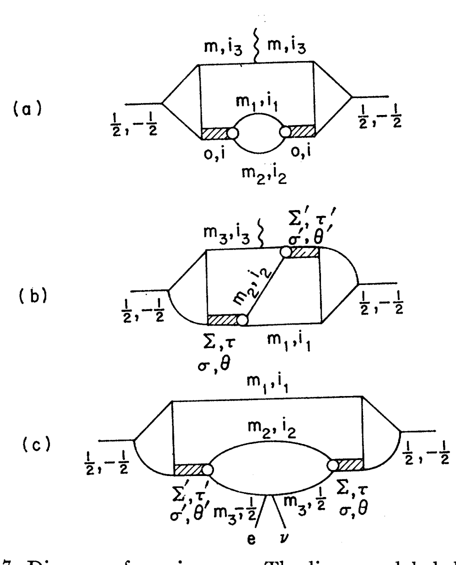

FIG. 7. Diagrams for spin sums. The lines are labeled by the third component of spin and isospin, respectively. For (b) and (c) we do not specify which correlated pair is involved and call the total spin and isospin of the pairs Σ and τ, with third components σ and θ. (a) The diagram A_φ. (b) A "C" diagram. (c) A "B" diagram for the beta-decay case.

[24] See, for example, M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

sum over i_3. Hence the final spin and isospin factor is

$$[(2\Sigma+1)(2\Sigma'+1)(2\tau+1)(2\tau'+1)]^{1/2}W(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2};\Sigma\Sigma') \\ +W(\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2};\tau\tau')|\langle\tfrac{1}{2}-\tfrac{1}{2}|\tau\theta\tfrac{1}{2}i_3\rangle|^2. \quad \text{(B4)}$$

The other factors are obtained in corresponding ways.

For the calculation of the Gamow-Teller matrix element one requires a different set of factors. Consider, for example, the B-matrix element shown in Fig. (7c). The isospin part gives

$$\sum_{i_1,i_2\theta\theta'} \langle\tfrac{1}{2}\,\tfrac{1}{2}|\tau\theta\tfrac{1}{2}i_1\rangle\langle\tau\theta|\tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}i_2\rangle \\ \times\langle\tau'\theta'|\tfrac{1}{2}-\tfrac{1}{2}\,\tfrac{1}{2}i_2\rangle\langle\tfrac{1}{2}-\tfrac{1}{2}|\tau'\theta'\tfrac{1}{2}i_1\rangle, \quad \text{(B5)}$$

which is not conveniently further reduced, but which sum we do explicitly. For the spin factors we get

$$\sum_{m_1m_2m_3\sigma\sigma'} \langle\tfrac{1}{2}\,\tfrac{1}{2}|\Sigma\sigma\tfrac{1}{2}m_1\rangle\langle\Sigma\sigma|\tfrac{1}{2}m_3\tfrac{1}{2}m_2\rangle \\ \times\langle\tfrac{1}{2}\,\tfrac{1}{2}|\Sigma'\sigma'\tfrac{1}{2}m_1\rangle\langle\Sigma'\sigma'|\tfrac{1}{2}m_3\tfrac{1}{2}m_2\rangle m_3, \quad \text{(B6)}$$

where the factor of m_3 comes from the matrix element of σ_z. This sum can be converted to a sum over Clebsch-Gordan coefficients by making use of the identity

$$m_3=(r_3/2)(\langle\tfrac{1}{2}m_3|10\tfrac{1}{2}m_3\rangle) \quad \text{(B7)}$$

we get for the spin sum (B6)

$$[(2\Sigma+1)(2\Sigma'+1)]^{1/2}W(1\Sigma\tfrac{1}{2}\,\tfrac{1}{2};\Sigma'\tfrac{1}{2})W(1\tfrac{1}{2}\Sigma'\tfrac{1}{2};\tfrac{1}{2}\Sigma). \quad \text{(B8)}$$

The other sums for the Gamow-Teller element are done in a similar way.

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Microscopic Theory of the Optical-Model Potential and the Hole-Particle Model in Nuclear Spectroscopy

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Approximate expressions for the optical-model potential $\mathcal{V}_{\rm opt}$ in terms of the nucleon-nucleon interaction are discussed. The identity of all the $A+1$ nucleons of the scattering problem is approximately taken into account in our final formulas. The imaginary part of $\mathcal{V}_{\rm opt}$ is calculated for the case of ^{12}C and the incident-nucleon energy of 20 MeV. The spherical-model random-phase-approximation (RPA) eigenvalues and eigenvectors are assumed for the complete set of intermediate nuclear states involved. The results are rather insensitive to the exchange-force mixture assumed for our zero-range nucleon-nucleon potential. The antisymmetrization of our V-matrix elements is extremely important as it reduces our $\mathrm{Im}\mathcal{V}_{\rm opt}$ by a factor of 2–3. For a reasonable set of our RPA intermediate eigenvalues and eigenvectors we obtain a semiquantitative agreement with the best phenomenological $\mathrm{Im}\mathcal{V}_{\rm opt}$ available for our case. The most important contribution to $\mathrm{Im}\mathcal{V}_{\rm opt}$ corresponds to the first excited $T=0$, 2^+ state in ^{12}C.

I. INTRODUCTION

A GREAT variety of attempts have been undertaken to calculate the optical-model potential from basic two-body forces. Several independent definitions of the optical potential have been employed which are not exactly equivalent. One common fundamental difficulty of a microscopic derivation of the potential is the exact antisymmetry in the $(A+1)$-particle system, i.e., the Pauli exclusion principle. It leads to many pitfalls. In fact, most early attempts to incorporate the Pauli exclusion principle in the derivation turned out to be failures.[1] In the following we shall not review these attempts, nor shall we discuss all the different approaches to the general problem of the microscopic theory of $\mathcal{V}_{\rm opt}$. Only a few treatments related more closely to our calculations will be mentioned.

One of these is the Watson multiple-scattering formalism. The corresponding solution constructed for $\mathcal{V}_{\rm opt}$ is a rather complicated infinite series of terms, and only partly considers the indistinguishability of the projectile ("0") from the target nucleons.[2] This approach employs the concept of the two-nucleon t matrix, which we shall refer to in our treatment. Most applications and

[1] J. S. Bell, in *Lectures on the Many-Body Problem*, edited by E. R. Caianiello (Academic Press Inc., New York, 1962), p. 91; in this reference the following papers are criticized: (a) F. Coester and H. Kümmel, Nucl. Phys. **9**, 225 (1958); (b) H. Rollnik, Z. Naturforsch. **13a**, 59 (1958); (c) L. M. Frantz and R. L. Mills, Nucl. Phys. **15**, 16 (1960); (d) L. M. Frantz, R. L. Mills, R. G. Newton, and A. M. Sessler, Phys. Rev. Letters **1**, 340 (1959).

[2] K. M. Watson, Phys. Rev. **89**, 575 (1953); (a) N. C. Francis and K. M. Watson, *ibid.* **92**, 291 (1953); (b) G. Takeda and K. M. Watson, *ibid.* **97**, 1336 (1955); K. M. Watson, *ibid.* **105**, 1388 (1957).