Coupled Equations for Rearrangement Collisions*

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An exact set of coupled inhomogeneous equations for rearrangement collisions, analogous to the coupled equations used in ordinary scattering, is derived. The boundary conditions are obtained from the integral equation for scattering. The solutions describing rearrangement are the particular integrals and contain none of the complementary function. Both truncation procedures and distorted-wave approximations are discussed.

INTRODUCTION

THE coupled equations method is well known as a technique for treating elastic and inelastic scattering.[1] It is quite straightforward using this technique to isolate the channels of interest and write down a set of coupled equations for the chosen channels, while ignoring the remaining channels. As pointed out, for example by Mittleman,[2] a similar procedure in the case of rearrangement collisions, while formally allowed, has not seemed practicable because of the complexity of the projection operators involved in selecting the initial and final channels. The reason for this complexity is that the projection operators in the initial and final states are not orthogonal. Mittleman has given another method for selecting the initial and final channels,[2] but his method suffers from the same problem as previously encountered, viz, the complexity of the required projection operator. In fact, to use this method, as Mittleman has noted, requires solving two integral equations as a preliminary to solving the pair of coupled channel equations describing the rearrangement.

We present in this paper an alternative means for obtaining coupled equations for rearrangement collisions that appears to be much less complex than other methods. Although we also use projection-operator techniques, the operators themselves are the same as those encountered in nonrearrangement problems. The equations are seen to be no harder to solve for the rearrangement case than for the case of scattering, since the same projection operators are used. The usual uncertainty of only imprecisely known wave functions and/or interactions is no more of a problem for rearrangements than for ordinary scattering, and hence the method will permit one to calculate results as accurately as in this latter case.

The method we use is derived from, and is an extension of, a technique used to derive coupled equations for the scattering of a particle by a system of identical particles.[3] In that work, it was found that the exchange contribution to the scattering was entirely contained in *inhomogeneous* terms in the coupled equations. Since exchange effects are equivalent to rearrangements (because of the relabeling), one might expect that a pure rearrangement process could also be discussed in terms of an inhomogeneous equation. We show that this is indeed so, using two alternative, but related, means in the following work.

We do not isolate in our method one initial and one final channel coupled together.[4] Instead we retain the incident wave of the initial channel as the "source" of an inhomogeneity in the coupled equations for all final state channels. In effect, we expand the *scattered* wave in eigenstates of the final unperturbed Hamiltonian and then solve. That is, we describe a situation in which a true rearrangement necessarily occurs and is measured, and we use the states of the final unperturbed Hamiltonian as our expansion set, since one (or more) of these states will be detected.

For most of our discussion, we ignore the possibility that the projectile and target and also the reaction products may contain identical particles. Exchange effects are briefly discussed at the end of this work. Only two-body collisions are considered.

INTEGRAL-EQUATION FORMULATION AND BOUNDARY CONDITIONS

We assume that the Hamiltonian for the system may be broken up in a variety of ways into perturbed and unperturbed portions:

$$H=H_a+V_a=H_b+V_b=\cdots, \tag{1}$$

where a and b denote specific groupings of particles and

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[1] A comprehensive discussion is given by N. F. Mott and H. B. W. Massey, *Theory of Atomic Collisions* (Oxford University Press, London, 1949). For a more recent treatment with applications to the optical model, see H. Feshbach, Ann. Phys. (N. Y.) **5**, 357 (1958) and M. H. Mittleman and R. Pu, Phys. Rev. **126**, 370 (1962).

[2] M. H. Mittleman, Ann. Phys. (N. Y.) **28**, 430 (1964). The literature on rearrangement collisions is quite large. Selected references to this literature may be found in M. L. Goldberger and K. M. Watson, *Collision Theory* (John Wiley & Sons, New York, 1964); T. Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1965); or A. Dalgarno, in *Proceedings of the Third International Conference of Atomic Electronic Collisions*, edited by M. R. C. McDowell (North-Holland Publishing Company, Amsterdam, 1964), p. 609.

[3] F. S. Levin, Phys. Rev. **140**, B1099 (1965). We denote this work by I.

[4] This is the treatment followed by Mittleman (Ref. 3) and also in many atomic-collision processes (such as charge transfer) when the projectile and target have comparable masses.

V_a, V_b are the interactions between the pairs in these groupings. As a specific example, we may imagine that a corresponds to $d+C^{12}$ and b to $\alpha+B^{10}$. We also assume that

$$H_a=h_a+T_a \quad \text{and} \quad H_b=h_b+T_b, \tag{2}$$

where h is a Hamiltonian for the internal motion and T is the kinetic-energy operator for the relative motion of the pair described by h. Corresponding to the Hamiltonian h are complete sets of states $\{\varphi\}$ and energies ϵ:

$$h_a\varphi_a=\epsilon_\alpha\varphi_\alpha \quad \text{and} \quad h_b\varphi_\beta=\epsilon_\beta\varphi_\beta. \tag{3}$$

We have $(\varphi_\alpha,\varphi_{\alpha'})=\delta_{\alpha\alpha'}$, $(\varphi_\beta,\varphi_{\beta'})=\delta_{\beta\beta'}$, and $(\varphi_\alpha,\varphi_\beta)\neq 0$. Finally, we define the projection operators P_β,Q_β:

$$P_\beta=\sum{}'|\varphi_\beta><\varphi_\beta|, \qquad Q_\beta=1-P_\beta, \tag{4}$$

where the prime on the summation in (4) means that we choose a restricted set of states (or channels) among the $\{\varphi_\beta\}$. Generally, P_β will contain only *bound* states of h_b. A similar pair of operators P_α,Q_α can also be defined, though we shall not need them.

We suppose that the scattering event is initiated by the collision of a pair in state φ_{α_0}. The initial state is then $\Phi_0=\varphi_{\alpha_0}\times\exp(i\mathbf{k}\cdot\mathbf{r}_a)$, where k is the incident wave number defined by $\hbar^2k^2/2\mu_a=E-\epsilon_{\alpha_0}$ with E the total energy and μ_a the reduced mass for the grouping in a, and $\mathbf{r}_a$ is the relative coordinate vector of the pair forming a. The total scattering wave function Ψ is generated from[5]

$$\Psi=i\epsilon(E^+-H)^{-1}\Phi_0, \tag{5}$$

where $E^+=E+i\epsilon$ and $\lim\epsilon\to 0$ is to be taken after relevant integrations have been performed.

If we use the expansion

$$(E^+-H)^{-1}=(E^+-H_a)^{-1}[1+V_a(E^+-H)^{-1}], \tag{6}$$

valid for operators with inverses, then substitution of (6) into (5) leads to the usual Lippmann-Schwinger equation[6] for Ψ, which is appropriate for a coupled channels analysis of the nonrearrangement problem. Since we wish to consider the case of a rearrangement, we expand $(E^+-H)^{-1}$ in terms of $(E^+-H_b)^{-1}$, the propagator appropriate to the rearranged system. That is, as noted above, we assume that the scattering system is detected in states of h_b. Thus, changing the subscript a to b in Eq. (6) and substituting into (5) leads to

$$\Psi=\Phi_0+(E^+-H_b)^{-1}(V_a-V_b)\Phi_0+(E^+-H_b)^{-1}V_b\Psi. \tag{7}$$

Equation (7) was first derived by Lippmann to describe rearrangement collisions.[7] We now show how to convert it into a set of coupled, inhomogeneous, differential equations in the final state group of channels, which we have denoted by β. We project with P_β and Q_β on both sides of Eq. (7), yielding

$$P_\beta\Psi=P_\beta\eta+(E^+-H_b)^{-1}P_\beta V_b(P_\beta+Q_\beta)\Psi \tag{8a}$$

and

$$Q_\beta\Psi=Q_\beta\eta+(E^+-H_b)^{-1}Q_\beta V_b(P_\beta+Q_\beta)\Psi, \tag{8b}$$

where we have followed the notation of I, and have defined

$$\eta=\Phi_0+(E^+-H_b)^{-1}(V_a-V_b)\Phi_0. \tag{9}$$

We now assume specifically that P_β contains only two-body bound states. For this case, as indicated in I, $P_\beta\eta=0$. The vanishing of $P_\beta\eta$ (for bound states occurring in P_β) was first shown by Lippmann,[7,8] when he derived the amplitude $T_{\beta\alpha}$ for the rearrangement collision process:

$$T_{\beta\alpha}=<\Phi_\beta|V_b|\Psi>, \tag{10}$$

with $\Phi_\beta=\varphi_\beta\exp(i\mathbf{k}_\beta\cdot\mathbf{r}_b)$, in a notation analogous to that used for Φ_0.

Equation (8a) gives us the asymptotic boundary condition on Ψ in the bound β channels:

$$P_\beta\Psi\sim\text{outgoing waves only} \tag{11}$$

since $P_\beta\eta=0$. Similarly, $Q_\beta\Psi$ obeys an identical boundary condition if any open β channels are included in Q_β; otherwise, $Q_\beta\Psi$ decays exponentially. We have assumed, of course, that at least one channel in P_β is open. If three-body channels were open and were included in P_β, then (11) would be modified to include a plane-wave component in the three-body channels. It is evident that the amplitude (10) can be derived from (8a) and (11).

The exact set of coupled equations for the $P_\beta\Psi$ follows on solving (8b) for $Q_\beta\Psi$. The procedure is straightforward and yields

$$Q_\beta\Psi=[1+(E^+-H_b-Q_\beta V_b)^{-1}Q_\beta V_b]Q_\beta\eta+(E^+-H_b-Q_\beta V_b)^{-1}Q_\beta V_bP_\beta\Psi. \tag{12}$$

Substituting Eq. (12) into Eq. (8a) then gives

$$P_\beta\Psi=(E^+-H_b)^{-1}P_\beta U(b)P_\beta\Psi+(E^+-H_b)^{-1}P_\beta U(b)Q_\beta\eta, \tag{13}$$

where

$$U(b)=V_b+V_b(E^+-H_b-Q_\beta V_b)^{-1}Q_\beta V_b \tag{14}$$

is the optical-potential operator for distinguishable particle scattering.[9] The differential form of Eq. (13) is

$$P_\beta[E-H_b-U(b)]P_\beta\Psi=P_\beta U(b)Q_\beta\eta. \tag{15}$$

In I, we showed that the term $Q_\beta\eta$, for the case of a

[5] This equation is given by M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).
[6] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).
[7] B. A. Lippmann, Phys. Rev. **102**, 264 (1956).
[8] The term $Q_\beta\eta$ is not zero, as discussed in I.
[9] See, e.g., H. Feshbach (Ref. 1) or H. Feshbach, Ann. Rev. Nucl. Sci. **8**, 49 (1958).

single particle in the projectile, could be transformed. The method used in I is applicable in the general case, if it is assumed, as we have done, that P_β contains only bound two-body states. The result of applying this method[10] is

$$P_\beta U(b) Q_\beta \eta = P_\beta [E - H_b - U(b)] P_\beta \Phi_0 + P_\beta \bar{U}(b) \Phi_0, \quad (16)$$

where

$$\bar{U}(b) = V_a + V_b (E^+ - H_b - Q_\beta V_b)^{-1} Q_\beta V_a. \quad (17)$$

Comparison of Eqs. (14) and (17) shows that $U(b)$ and $\bar{U}(b)$ differ only in the interactions appearing on the extreme left and right. This is natural, since V_a is the proper interaction for the state Φ_0. Substituting (16) into (15) finally yields the set of coupled equations of interest:

$$P_\beta [E - H_b - U(b)] P_\beta (\Psi - \Phi_0) = P_\beta \bar{U}(b) \Phi_0, \quad (18)$$

which is recognized as an *inhomogeneous* differential equation, for the *scattered* wave $\Psi - \Phi_0$ in the channels of P_β.

A formal solution to (18) is easily obtained. Let $P_\beta G P_\beta$ be the outgoing wave Green's function defined by

$$P_\beta G P_\beta = P_\beta [E^+ - H_b - U(b)]^{-1} P_\beta.$$

Then the solution describing the *rearrangement* process is given by

$$P_\beta \Psi = P_\beta \Phi_0 + P_\beta G P_\beta \bar{U}(b) \Phi_0, \quad (19)$$

which is recognized as the *particular* solution to Eq. (18). The complementary function is not included since we do *not* have an incident source of waves in the channels P_β. We also note that $P_\beta \Phi_0$ gives no flux asymptotically, since P_β has only bound states of h_b and we have implicity assumed that in the transition $\alpha_0 \to \beta$ there has been either a transfer of at least one particle or a relabeling of particles. That is, $P_b \Phi_0$ decays exponentially in *all* of the channels included in P_β.

By iterating Eq. (13) after Eq. (16) has been used to eliminate the $Q_\beta \eta$ term, a solution identical to Eq. (19) is obtained. In this latter form the solution (19) may be more easily understood, since the boundary conditions are automatically specified in the integral-equation form of (13). It is evident from (13) and (16) that $P_\beta \Psi$ has no incident waves in the two-body bound-state β channels. Since the iteration of (13) produces (19), it is thus clear that we must, because of the boundary condition, choose (19) as the proper solution to Eq. (18).

Equations (18) and (19) are seen to verify our comments in the Introduction that the rearrangement collision problem may be treated in terms of an inhomogeneous equation. In such a framework, of course, we do not need to consider the fact that $(\varphi_\beta, \varphi_\alpha) \neq 0$. That is, the right-hand side of (18) is assumed to be known and we need only use the property $(\varphi_\beta, \varphi_{\beta'}) = \delta_{\beta\beta'}$ to obtain equations for $P_\beta(\Psi - \Phi_0)$. One quadrature then provides $T_{\beta\alpha}$. This is also true of approximations, as we shall see.

The amplitude $T_{\beta\alpha}$ for the transition $\alpha_0 \to \beta$ is obtained directly from (19):

$$T_{\beta\alpha} = \langle u_{k_\beta}^{(-)} \varphi_\beta | \bar{U}(b) | \Phi_0 \rangle, \quad (20)$$

where $u_{k_\beta}^{(-)}$ is the ingoing-wave solution of

$$P_\beta [E - H_b - U(b)] P_\beta (\Psi - \Phi_0) = 0$$

that has a plane wave in channel β with energy $E_{k_\beta} = \hbar^2 k_\beta^2 / 2\mu_b = E - \epsilon_\beta$. Equation (20) is an alternative to Eq. (10); these two forms for $T_{\beta\alpha}$ are easily shown to be identical. If we assume that $\bar{U}(b) \approx V_a$, then we have

$$T_{\beta\alpha} \approx \langle u_{k_\beta}^{(-)} \varphi_\beta | V_a | \Phi_0 \rangle. \quad (21)$$

This is similar to a distorted-wave amplitude in the theory of direct nuclear reactions, which we shall consider in the next section.

In practice, one will not be able to determine either $u_{k_\beta}^{(-)}$ or $\bar{U}(b)$ exactly and approximations will be used instead. One approximation, common to nonrearrangement collision problems, is to include only certain channels in the calculation; i.e., carry out a truncated coupled channels calculation for functions $P_\beta \chi$ that will approximate the $P_\beta \Psi$. That is, we suppose that the channels in Q_β are the ones to be ignored. Then the set of coupled equations to be used when only the channels in P_β are considered is obtained from Eq. (18) by dropping the terms with Q_β, giving

$$P_\beta [E - H_b - V_b] P_\beta \chi = P_\beta V_a \Phi_0. \quad (22)$$

We again assume that only two-body bound states occur in P_β so that the $P_\beta \Phi_0$ terms may be ignored, as before. This approximation will be valid when both the $Q_\beta V_b P_\beta$ and $Q_\beta V_a \Phi_0$ matrix elements are small.

If the inhomogeneity in (22) were set equal to zero, we would simply have the usual truncated set of coupled equations for scattering processes. It is the presence of the inhomogeneity that specifies the rearrangement. Of course, we must also impose the same boundary condition on $P_\beta \chi$ as we did on $P_\beta \Psi$:

$$P_\beta \chi \sim \text{outgoing waves only.}$$

Thus, in analogy to Eq. (19), we may write

$$P_\beta \chi = P_\beta \mathcal{G} P_\beta V_a \Phi_0, \quad (23)$$

where

$$P_\beta \mathcal{G} P_\beta = P_\beta (E^+ - H)^{-1} P_\beta \quad (24)$$

is the full Green's function for the problem in the restricted space of the channels included in P_β. The solution (23) corresponds to keeping only the particular solution to Eq. (22), analogous to Eq. (19). The same result is obtained from iterating Eq. (13) if first Eq. (16) is used and then all terms in Q_β are dropped.

The rearrangement amplitude $T_{\beta\alpha}{}^P$ arising from Eq.

[10] The steps involved in this transformation are as follows: First, write $U(b) Q_\beta = U(b) - U(b) P_\beta$. Then use an equation similar to (6) to transform $U(b) Q_\beta \eta$ to $[V_b - V_a + \bar{U}(b)] \Phi_0$. Using the properties of P_β to write $P_\beta (V_b - V_a) \Phi_0 = P_\beta (E - H_b) \Phi_0$, and collecting all terms, gives (16).

(23) is found to be

$$T_{\beta\alpha}{}^{P}=\langle v_{k_\beta}{}^{(-)}\varphi_\beta|\,V_a|\Phi_0\rangle\,, \tag{25}$$

with $v_{k_\beta}{}^{(-)}$ the solution of $P_\beta[E-H]P_\beta\chi=0$ analogous to $u_{k_\beta}{}^{(-)}$. In practice, using available electronic computers, the amplitude $T_{\beta\alpha}{}^{P}$, or its partial-wave components, should be no harder to obtain than the phase shifts or S-matrix elements of the simpler scattering problem given by $P_\beta[E-H]P_\beta\chi=0$. These latter scattering parameters will, of course, be present in $T_{\beta\alpha}{}^{P}$ through the solution of $v_{k_\beta}{}^{(-)}$. Precautions will have to be taken to ensure that none of the complementary function is present in the numerical solution, but this can be done using the known boundary conditions and normalization on the $v_{k_\beta}{}^{(-)}$ to subtract out any such components.

SCHRÖDINGER-EQUATION FORMULATION

In deriving the results of the preceding section, we required P_β to contain only bound states. Let us now assume that Eq. (18) holds for arbitrary states of H_b included in P_β. If we include all states of H_b in P_β, so that $P_\beta\to 1$ and $Q_\beta\to 0$, then $U(b)\to V_b$, $\bar{U}(b)\to V_a$, and Eq. (18) becomes

$$(E-H)(\Psi-\Phi_0)=V_a\Phi_0. \tag{26}$$

But if we note that $(E-H_a)\Phi_0=0$, then (26) is immediately seen to be an alternative form for the Schrödinger equation $(E-H)\Psi=0$. This suggests that (18) may have a greater range of validity than implied by our derivation. We now show that (18) [and (22)] may also be derived from the Schrödinger equation, using (26).

The method we follow is obvious. We use $(E-H)\Psi=0$, and add to each side of this equation the term $V_a\Phi_0$, where Φ_0 is the incident wave, so that $\Psi-\Phi_0$ is the scattered wave. We thus regain Eq. (26). Inserting a complete set of states $\sum_\beta|\varphi_\beta\rangle\langle\varphi_\beta|=P_\beta+Q_\beta$ into the left-hand side of (26) and operating on both sides of the resulting equation with P_β gives

$$P_\beta[E-H_b-V_b](P_\beta+Q_\beta)(\Psi-\Phi_0)=P_\beta V_a\Phi_0. \tag{27}$$

Similarly, we may obtain

$$Q_\beta[E-H_b-V_b](P_\beta+Q_\beta)(\Psi-\Phi_0)=Q_\beta V_a\Phi_0. \tag{28}$$

Equations (27) and (28) are a set of coupled channel equations for rearrangements which are to be solved according to the boundary conditions discussed in the preceding section. The analog of the usual truncation procedure of scattering theory is to drop the Q_β terms in Eq. (27), which gives Eq. (22):

$$P_\beta[E-H_b-V_b]P_\beta\chi=P_\beta V_a\Phi_0. \tag{22}$$

The set of equations given by (22) was discussed in the preceding section and is the basic truncated set of coupled equations for the rearrangement collision problem. However, we also easily rederive Eq. (18) by solving (28) for $Q_\beta(\Psi-\Phi_0)$ using the outgoing-wave boundary condition and substituting the result into Eq. (27). We then obtain Eq. (18) with $U(b)$ and $\bar{U}(b)$ as defined in the preceding section. We note now that in the preceding derivations, we have nowhere used the assumption that P_β contains only two-body bound states [this assumption will only enter the specification of the boundary conditions on $P_\beta(\Psi-\Phi_0)$]. Hence, Eq. (18) appears to be valid for arbitrary final-state channels.

It is perhaps important to stress that although the approximation of ignoring the channels in Q_β, which gives Eq. (22), can lead to tractable problems, there is no guarantee that inclusion of more and more states in P_β will give a result that rapidly converges to Ψ. We are unable to assess either the rate of convergence, or the accuracy of the approximation $\Psi\approx P_\beta\Psi$. It is to be hoped that inclusion of highly excited states is unnecessary when $E-\epsilon_{\alpha_0}$ is small, since such states are far off the energy shell, but this will have to be determined in particular cases, if it is indeed feasible at all.

One possible means for improving the accuracy of the approximation $\Psi\approx P_\beta\Psi$ is to attempt to include some of the interactions, or effects of the interactions, in the initial wave function Φ_0. This may be done using distorted waves. Let $\mathcal{U}(r_a)$ be a single-particle potential depending on r_a and let $\lambda^{(\pm)}$ be scattering solutions in this potential:

$$[T_a+\mathcal{U}(r_a)]\lambda^{(\pm)}=(E-\epsilon_{\alpha_0})\lambda^{(\pm)}.$$

Now define $\Lambda_0{}^{(\pm)}\equiv\varphi_{\alpha_0}\lambda^{(\pm)}$. Then, in analogy to Eq. (26), we may write

$$(E-H)(\Psi-\Lambda_0{}^{(\pm)})=(V_a-\mathcal{U})\Lambda_0{}^{(\pm)}. \tag{29}$$

This equation may be handled exactly as was Eq. (26). We now find for the truncated set of coupled equations

$$P_\beta[E-H-V_b]P_\beta\zeta=P_\beta(V_a-\mathcal{U})\Lambda_0{}^{(+)}, \tag{30}$$

with $P_\beta\zeta$ denoting the approximate wave function and P_β assumed to contain no three-body channels. Equation (30) is to be solved in the same manner as Eq. (22).

The full solution to (29) for $P_\beta(\Psi-\Lambda_0{}^{(+)})$ is obtained from

$$P_\beta[E-H_b-U(b)]P_\beta(\Psi-\Lambda_0{}^{(+)})=P_\beta W(b)\Lambda_0{}^{(+)}, \tag{31}$$

where

$$W(b)=[1+(E^{+}-H_b-Q_\beta V_b)^{-1}](V_a-\mathcal{U}). \tag{32}$$

Using the boundary conditions specified in the preceding section, we find the exact rearrangement amplitude $T_{\beta\alpha}$ to be

$$T_{\beta\alpha}=\langle u_{k_\beta}{}^{(-)}\varphi_\beta|\,W(b)\,|\Lambda_0{}^{(+)}\rangle\,, \tag{33}$$

where the definition of $u_{k_\beta}{}^{(-)}$ is given following Eq. (20). Comparing Eqs. (20) and (33), we deduce the relation $W(b)\Lambda_0{}^{(+)}=\bar{U}(b)\Phi_0$, which is the analog of the well-known T-operator equation, which in our notation is $T_a\Phi_0=V_a\Psi$.

The distorted-wave approximation to $T_{\beta\alpha}$ of Eq. (33) is given by keeping only the term $(V_a-\mathcal{U})$ in $W(b)$. Denoting the resulting amplitude by $T_{\beta\alpha}{}^{DW}$, we have

$$T_{\beta\alpha}{}^{DW}=\langle u_{k_\beta}{}^{(-)}\varphi_\beta|\, V_a-\mathcal{U}\,|\Lambda_0{}^{+}\rangle\,, \tag{34}$$

which is the familiar form used in deriving, for example, the (d,p) amplitude in the distorted-wave approximation.[11] Equation (34) should be a good approximation whenever the matrix elements $Q_\beta(V_a-\mathcal{U})\Lambda_0{}^{(+)}$ are small compared to $P_\beta(V_a-\mathcal{U})\Lambda_0{}^{(+)}$. Judicious choice of $\mathcal{U}$ may help to effect such a situation. It should be noted that $u_{k_\beta}{}^{(-)}$ is a wave function in a complex potential, the imaginary part arising from the possibility of real transitions to states $\varphi_{\beta'}$, $\beta'\neq\beta$.

We have not commented on the possible difficulties that may be encountered by expanding Eq. (10), (20), (25), (33), or (34) as a power series[12] of amplitudes in the interactions V_b and V_a. It is our contention that such a discussion is unnecessary here, since the only practical calculations we have in mind would be based on Eq. (25) or (34), as modified by the replacement $u_{k_\beta}{}^{(-)}\rightarrow v_{k_\beta}{}^{(-)}$. These amplitudes could be obtained either by solving the homogeneous portions of Eq. (22) or (30), and then carrying out a single quadrature, or else by solving Eq. (22) or (30) directly, using in both cases a computer to determine solutions and to extract T-matrix elements. Since divergences of the sort found to occur in expansions of rearrangement amplitudes do not seem to occur in ordinary coupled channel calculations of scattering, they should similarly not occur in the type of coupled channel calculations we are advocating.

So far, we have ignored the effects of the Pauli principle. We now indicate how our results are generalized to include exchange effects. The change in the equations so far derived is trivial, as we shall see, but the actual task of evaluating the additional matrix elements arising from exchange is seen to be no less of a problem than at present.

We follow the methods used above where we subtracted the incident wave from Ψ, only now we assume that all particles in the projectile and target are identical. The incident wave must, therefore, be antisymmetrized:

$$\Phi_0\rightarrow\Phi_0{}^{A}\,, \tag{35}$$

where $\Phi_0{}^{A}$ is the antisymmetric part of Φ_0. Explicit formulas for $\Phi_0{}^{A}$ in terms of the relabeling of states φ_{α_0} are easily determined. In place of (26), we now have, using (35),

$$(E-H)(\Psi-\Phi_0{}^{A})=\mathcal{A}\{V_a\Phi_0\}\,, \tag{36}$$

where V_a is the interaction for a specific labeling in Φ_0 and $\mathcal{A}$ is an antisymmetrizing operator. All the terms in $\Phi_0{}^{A}$ are source terms (in one or another of the different sets of channels defined by the relabelings); we thus are subtracting from Ψ the total incident wave. The scattered wave $\Psi-\Phi_0{}^{A}$, projected onto the β channels, can now be treated exactly as above to give either the exact set of coupled equations and scattering amplitudes, or any approximation thereto of the sort already discussed. In this case also, only the particular integral should be retained as the solution to the inhomogeneous rearrangement equations. A similar analysis can be made starting with the integral equation $\Phi_0{}^{A}$ as the source term. With very few modifications, this approach can be used to discuss elastic and inelastic scattering of identical particles. This latter topic will be treated elsewhere.[13]

[11] The distorted-wave model of direct nuclear reactions has been reviewed recently in many places: W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961); N. Austern, in *Selected Topics in Nuclear Theory*, edited by F. Janouch (International Atomic Energy Agency, Vienna, 1963); N. Glendenning, Ann. Rev. Nucl. Sci. **13**, 191 (1963); G. R. Satchler, Nucl. Phys. **55**, 1 (1964).

[12] R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961). Other problems that arise in connection with $\lim \epsilon\rightarrow 0$ of Eqs. (5) or (7) have been noted by L. L. Foldy and W. Tobocman, *ibid.* **105**, 1099 (1957) and S. T. Epstein, *ibid.* **106**, 598 (1957). The discussion of these latter authors, in our terms, deals with the ambiguities that can arise when solutions of the homogeneous portions of (16) [or (22)] are not excluded from (19) [or (23)]. Extensions and amplifications of these works are given by E. Gerjuoy, *ibid.* **109**, 1806 (1958) and Ann. Phys. (N. Y.) **5**, 58 (1958). We differ, of course, with Gerjuoy's consistency requirement that η of Eq. (9) vanish for *all* β channels (see the first of the articles cited above). As we have argued in I, only $P_{\beta\eta}=0$ when P_β contains only bound two-body channels.

[13] F. S. Levin (unpublished).