Exact Solution of the One-Particle Model of Exchange Scattering in Solids*

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A solution of the Chew-Low equations describing exchange scattering of conduction electrons is presented, which is exact within the one-particle intermediate-state approximation. This solution shares with earlier, approximate ones the feature that for antiferromagnetic exchange the spin-flip scattering amplitude can develop complex poles on the physical sheet. To examine this point further, the theory is generalized to an interaction of finite range, analyticity in momentum transfer being assumed and spin-orbit coupling being neglected. This still does not guarantee removal of the complex poles. However, the forward-scattering amplitude appears to be free from poles in all these cases. Unitarity is satisfied in the appropriate zero-temperature limit.

1. INTRODUCTION

THE logarithmic singularity in the scattering of a conduction electron by an impurity spin, obtained by Kondo[1] in third-order perturbation theory, has recently been examined by the nonperturbative approach of scattering theory,[2,3] and also by a special adaptation of standard many-body perturbation theory.[4] Both approaches strongly suggested that the singularity in the cross section found in finite-order perturbation theory is to be replaced by a scattering resonance. This was indicated by a rather crude approximate solution of the nonlinear scattering equations of Refs. 2 and 3, a solution which was in remarkable agreement with the result of a selective summation of certain perturbation diagrams.[4] This agreement was the more disturbing as the approximate result indicated that the forward scattering amplitude (i.e., the self-energy in the language of Ref. 4) could have a pair of complex-conjugate poles on the physical sheet at sufficiently low temperatures, if the sign of the exchange is antiferromagnetic.

2. EXACT SOLUTION

In Ref. 3, the Chew-Low equations derived in Ref. 2 for impurity spin $\frac{1}{2}$ and zero temperature were generalized to arbitrary spin S and temperature $T(=1/k\beta)$. In the one-particle intermediate-state approximation, these equations connecting the one-electron scattering amplitude t_0 in the $S-\frac{1}{2}$ manifold with the amplitude t_1 in the $S+\frac{1}{2}$ manifold, are (z just above the real axis)

$$t_0(z)=V-\tfrac{1}{2}(S+1)J+\int_{-\epsilon_f}^{+\infty}\frac{|t_0|^2\rho\,dx}{z-x}-\frac{2(S+1)}{(2S+1)^2}\int_{-\epsilon_f}^{+\infty}\frac{|t_0-t_1|^2f(\beta x)\rho\,dx}{z-x},$$
$$t_1(z)=V+\tfrac{1}{2}SJ+\int_{-\epsilon_f}^{+\infty}\frac{|t_1|^2\rho\,dx}{z-x}+\frac{2S}{(2S+1)^2}\int_{-\epsilon_f}^{+\infty}\frac{|t_0-t_1|^2f(\beta x)\rho\,dx}{z-x}. \quad (1)$$

Here V is the spin-independent potential of the impurity, J is the exchange constant, ρ is the density of single-particle states, per particle, $f(\beta x)$ is the Fermi distribution function. (The normalization of the t's differs slightly from that of Ref. 2.) We introduce forward-scattering and spin-flip amplitudes, t and τ, respectively, by the relations

$$t_0=t-2(S+1)\tau\,,\quad t_1=t+2S\tau\,. \quad (2)$$

According to Eqs. (1), the new amplitudes satisfy

$$t(z)=V+\int_{-\epsilon_f}^{+\infty}\frac{|t|^2+4S(S+1)|\tau|^2}{z-x}\rho\,dx\,, \quad (3a)$$

$$\tau(z)=\tfrac{1}{4}J+\int_{-\epsilon_f}^{+\infty}\frac{\tau t^*+\tau^*t-2|\tau|^2\tanh\frac{1}{2}\beta x}{z-x}\rho\,dx\,. \quad (3b)$$

Equation (3b) is in the standard form of a Hilbert problem[5] and is easily solved for τ in terms of a given t. On the other hand, it is difficult to solve (3a) for t in terms of a given, arbitrary τ. However, we will show that (3a) can be solved in terms of a τ whose magnitude is restricted to lie below a certain limit. Assuming τ to have this property, we are then able to solve (3b), and verify that the magnitude of the solution does indeed lie below that limit.

For z just above the real axis and $\mathrm{Re}z>-\epsilon_f$, we see, from (3a), that

$$\mathrm{Im}t=-\pi\rho|t|^2-4S(S+1)\pi\rho|\tau|^2\,. \quad (4)$$

An equation of this form was studied by Ball and Frazer[6] and also by Froissart[7] in connection with the effect of inelastic thresholds on pion-nucleon resonances. Following Ref. 6 we attempt to satisfy (4) with an

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[1] J. Kondo, Progr. Theoret. Phys. (Kyoto) 32, 37 (1964).
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[4] A. A. Abrikosov, paper presented at Novosibirsk Conference on Many-Body Problems, 1965 (unpublished). See also A. A. Abrikosov, Zh. Eksperim. i Teor. Fiz. 48, 990 (1965) [English transl.: Soviet Phys.—JETP 21, 660 (1965)].
[5] N. I. Muskhelishvili, in *Singular Integral Equations*, translated by J. R. M. Radock (P. Noordhoff Ltd., Groningen, The Netherlands, 1953), Chap. 5, para. 37.
[6] J. S. Ball and W. R. Frazer, Phys. Rev. Letters 7, 204 (1961).
[7] M. Froissart, Nuovo Cimento 22, 191 (1961).

ansatz describing inelastic scattering through a complex phase shift. Writing

$$\pi\rho t=-(1/2i)(e^{2i\delta}-1), \tag{5}$$

where $\delta=\delta'+i\delta''$, we see that (5) satisfies (4) if we chose

$$e^{-4\delta''}=1-a|\tau|^2, \tag{6}$$

with $a=16S(S+1)\pi^2\rho^2$. Equation (6) implies that (5) is a possible form only if $a|\tau|^2\leq 1$, but the solution of (3b) will presently be seen to satisfy this condition.

Some care is needed in keeping track of the analytic properties of ρ in the present artificial model which assumes δ-function interaction and essentially constant state density. (In Sec. 4, a more natural model will be dicussed.) For a parabolic-band model we would have $\rho(x)=(x+\epsilon_f)^{1/2}$ (recall that the Fermi level is taken as the zero of energy). In Ref. 6, the definition of the square root is taken to be $\rho(x+i\eta)=\text{sgn}\eta(x+\epsilon_f)^{1/2}$ for $x>-\epsilon_f$. In our present model, in which we wish to consider $x=-\epsilon_f$ to be synonymous with $-\infty$, and to take ρ as constant, we must still write

$$\rho(x+i\eta)=(\text{sgn}\eta)\rho.$$

If we now define the complex phase shift by

$$\delta(z)=\frac{\text{sgn Im}z}{\pi}\int_{-\infty}^{+\infty}\frac{\delta''(x')\,dx'}{(x'-z)}-\delta_v(z), \tag{7}$$

where $\delta_v(z)$ is the phase shift (real on the real axis) corresponding to the ordinary potential-scattering problem

$$t_v=V+\int_{-\infty}^{+\infty}\frac{|t_v|^2}{z-x}\rho\,dx, \tag{8}$$

we see that

$$t(z)=-[1/2i\pi\rho(z)](e^{2i\delta(z)}-1)$$

is a function satisfying the reflection principle, and is acceptable as a scattering amplitude. We disregard the fact that

$$\delta'(x+i\eta)=\frac{1}{\pi}P\int\frac{\delta''(x')dx'}{x'-x-i\eta}-\delta_v(x+i\eta) \tag{9}$$

diverges, a feature which will be seen to disappear when more reasonable, finite-range exchange coupling is considered. We will, in fact, proceed as though $\delta''(x')$ were integrable L^p with $p>1$. Then it follows from Eq. (9) that[8]

$$\delta''(x)=-\frac{P}{\pi}\int\frac{\delta'(x')+\delta_v(x')}{x'-x}dx', \tag{10}$$

where $\delta'(x')$ and $\delta_v(x')$ are evaulated just above the real axis.

Turning to Eq. (3b) we set $\tau=1/w$, and denote by w^+ and w^- the values of w above and below the cut. Then [in the absence of Castillejo-Dalitz-Dyson poles] one obtains from (3b)

$$e^{2i\delta}w^+-e^{-2i\delta^*}w^-=-4\pi i\rho\tanh\tfrac{1}{2}\beta x. \tag{11}$$

Writing $w=hu$, where

$$h(z)=\exp(-2/\pi)\int[\delta'(x)/(x-z)]\,dx$$

is the simplest possible solution of the homogeneous equation corresponding to (11), we see that

$$\begin{aligned}u^+-u^-&=-4\pi i\rho\tanh\tfrac{1}{2}\beta x\\&\quad\times\exp\left\{2\delta''(x)+(2/\pi)P\int dx'\,\delta'(x')/(x'-x)\right\}\\&=-4\pi i\rho\tanh\tfrac{1}{2}\beta x\\&\quad\times\exp\left\{-(2/\pi)P\int dx'\,\delta_v(x')/(x'-x)\right\},\end{aligned} \tag{12}$$

where use has been made of Eq. (10). Since h tends to unity as $z\to\infty$, it follows that

$$u=C-2\int\frac{dx\,\rho\tanh\frac{1}{2}\beta x}{x-z}\times\exp\left\{-(2/\pi)P\int dx'\,\delta_v(x')/(x'-x)\right\}, \tag{13}$$

where C is a constant to be determined later. This result may be simplified by noting that $|t_v|$ and δ_v satisfy the dispersion relation

$$\frac{|t_v|}{V}=\exp\left\{-\frac{1}{\pi}P\int_{-\infty}^{+\infty}dx'\,\delta_v(x')/(x'-x)\right\}.$$

(*Note added in proof.* If the ordinary potential-scattering problem has bound states, this relation must be supplemented by some pole-term factors. Since the poles are located below $-\epsilon_f$, the extra factors do not materially change our results.) Gathering these results, we find, just above the real axis[9]

$$\tau=e^{2i\delta}F, \tag{14}$$

where

$$F(z)=|t_v(z)/V|^2\bigg/\left\{C-2\int_{-\infty}^{+\infty}dx\left|\frac{t_v(x+i\eta)}{V}\right|^2\times\rho\tanh\tfrac{1}{2}\beta x/(x-z)\right\}.$$

Hence

$$|\tau|^2=e^{-4\delta''}|F|^2=(1-a|\tau|^2)|F|^2,$$

[8] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals* (Oxford University Press, Inc., New York, 1937), Chap. 5.

[9] The same result holds in the whole upper half-plane, but with $|t_v(z)/V|^2$ replaced by $t_v(z)t^{II}{}_v(z)/V^2$, where $t_v{}^{II}$ is the analytic continuation of $t_v(x-i\eta)$ into the nonphysical sheet.

so that

$$|\tau|^2=|F|^2/(1+a|F|^2).$$

The constant C is determined by the requirement $\tau(\infty)=\frac{1}{4}J$. Since $t_v/V\to 1$ as $z\to\infty$, $F\to 1/C$ and so

$$C=(16/J^2-a)^{1/2}=(4/J)(1-S(S+1)\pi^2\rho^2J^2)^{1/2}.$$

The required reality of the square root would seem to restrict the solution to sufficiently small values of J. However, this restriction is again a consequence of having chosen a δ function for $J(r)$ (see Sec. 4), and has no significance. The complete solution of the problem on the real axis is thus

$$|\tau|^2=|F|^2/(1+a|F|^2),\quad \tau=e^{2i\delta'}|\tau|,$$
$$\delta'(x)=-\delta_v(x)+(1/4\pi)P\int dx'[\ln(1+a|F|^2)]/(x'-x), \tag{15}$$

and it obviously satisfies the premise $a|\tau|^2\leq 1$.

The problem of complex poles in general still remains. For, at z_1, we have

$$|\tau(z_1)|=|e^{2i\delta(z_1)}||F(z_1)|.$$

Only the imaginary part of $\delta(z_1)$ contributes to the first factor on the right. This imaginary part is

$$-\mathrm{Im}\delta_v(z_1)+(1/\pi)\int\{y_1\delta''(x')/[(x'-x_1)^2+y_1^2]\}dx',$$

where $x_1+iy_1=z_1$ and $y_1>0$. The first of these two terms is finite, and the integral obviously converges. Hence, if $|F(z)|$ has a pair of complex poles, so has τ. For sufficiently low temperatures, $F(z)$ *may* have poles, although this question depends on the precise variation of t_v/V with x. The result is especially simple if V is taken to be zero so that $t_v/V\to 1$, independent of x. A pair of complex poles then appears at a reciprocal transition temperature given by

$$\tfrac{1}{2}C=\int[(\tanh\tfrac{1}{2}\beta_c x)/x]\,dx.$$

and progresses from (0,0) at $\beta=\beta_c$ to $(0,\pm iy_1)$ for $\beta<\beta_c$, where

$$\tfrac{1}{2}C=\int\rho[(x\tanh\tfrac{1}{2}\beta x)/(x^2+y_1^2)]\,dx.$$

It also appears that the forward scattering amplitude t does not develop poles. For, at $z_1=x_1+iy_1$, we have

$$\pi\rho t(z_1)=-\frac{1}{2i}\left(\exp\left[\frac{2i}{\pi}\int_{-\infty}^{+\infty}dx'\frac{(x'-x_1)\delta''(x')}{(x'-x_1)^2+y_1^2}-\frac{2y_1}{\pi}\int\frac{\delta''(x')}{(x'-x_1)^2+y_1^2}dx'-2i\delta_v(z_1)\right]-1\right), \tag{15a}$$

and the exponents evidently remain finite [or would remain finite in a more physical model in which $\delta''(x)\to 0$ as $x\to\infty$].

As a final check on the correctness of the solution (15) we demonstrate that as $\beta\to\infty$, it is consistent with the unitarity of the S matrix. [As explained in Ref. 3, Eqs. (1) describe a unitary process only in the zero-temperature limit. This becomes evident by taking the imaginary part of both sides of Eq. (1), and allowing β to go to infinity. At finite temperatures it is convenient to sacrifice unitarity in favor of the close relation of Eqs. (1) to the Green's-function formulation.] In the low-temperature limit $\tanh\frac{1}{2}\beta x$ tends to the signum function. In the physical region $(x>0)$ we may then write

$$2\pi\rho F=q/(p-iq)=e^{i\Psi}\sin\Psi, \tag{16}$$

where $q=2\pi\rho|t_v/V|^2$, where p is a certain principal value integral, and where $\tan\Psi=q/p$. The S-matrix element in the lower manifold is

$$S_0=1-2\pi i\rho t_0=e^{2i\delta'}e^{-2\delta''}(1+4i(S+1)\pi\rho F).$$

Thus unitarity requires that

$$e^{4\delta''}=|1+4i(S+1)\pi\rho F|^2=|1+2i(S+1)e^{i\Psi}\sin\Psi|^2. \tag{17}$$

Computing the absolute value using (16), and recalling that

$$e^{4\delta''}=(1-a|\tau|^2)^{-1}=1+16S(S+1)\pi^2\rho^2|F|^2=1+4S(S+1)\sin^2\Psi,$$

we see that (17) is, in fact, satisfied. In the same way we verify that

$$|1-4iS\pi\rho F|^2=e^{4\delta''},$$

and so $S_1=1-2\pi i\rho t_1$ is likewise unimodular.

We note also that the solutions (15) and (15a) behave properly in various limits. For example, if we formally set $S=0$ and $\beta=0$, Eqs. (3a) and (3b) immediately give $\tau=\frac{1}{4}J\partial t_v/\partial V$. On the other hand, Eqs. (15) give $\tau=\frac{1}{4}Je^{-i\delta_v}|t_v/V|^2$, which is, in fact, equal to $\frac{1}{4}J\partial t_v/\partial V$. Also, if $V=0$, $(t_v/V=1)$, and $|\tau|^2$ is very small, so that δ'' is very small, Eq. (15a) reduces to

$$\pi\rho t=-\frac{1}{\pi}\int\frac{\delta''(x')}{x'-z}dx'=4S(S+1)\pi\rho^2\int\frac{|\tau|^2}{x'-z}dx',$$

as it should.

3. DISPERSION EQUATIONS IN THE CASE OF FORCES OF FINITE RANGE

The Low equation has so far provided us with a suitable starting point only in the extreme case of a δ-function potential, since only in that case a solution exists that is independent of momentum transfer. For finite-range forces a direct solution of the Low equation is probably out of the question. Further progress becomes possible, however, if one succeeds in proving that the scattering amplitudes are analytic functions of both energy and momentum transfer regarded as independent complex variables (Mandelstam representation). A partial-wave projection then leads to equations for each partial wave that resemble closely the ones previously discussed, except that the Born term is replaced by a function with a branch cut in the energy plane ending to the left of $-\epsilon_f$. (Recall that all energies are measured from the Fermi level at zero.)

A proof of the existence of the Mandelstam representation has so far been given only for potential scattering.[10] However, the possibility of establishing analyticity in a limited domain in the momentum-transfer plane at fixed real energy has been conjectured by Blankenbecler, Goldberger, Khuri, and Treiman.[11] Their inference was based on an analogy between the Low equation and a formal solution to the integral equation for potential scattering. We will argue that such limited analyticity should indeed hold in the present problem and will simply assume that analytic continuation to the whole energy-momentum domain, except for cuts, is possible. We first confine the discussion to the absolute zero of temperature. In the notation of Ref. 2, we write ($\epsilon_k=\epsilon_{k'}$)

$$T_{\omega'k'\omega k}(\epsilon_k+i\eta)=B_{\omega'k',\omega k}+A_1(\omega'k',\,\omega k,\,\epsilon_k+i\eta)+A_2(\omega'k,\,\omega k',\,\epsilon_k+i\eta)\,,$$

where

$$A_1(z)\equiv A_1(\omega'k',\omega k,z)=-\int_0^\infty\frac{\langle\omega'|j^\dagger(k')|n\rangle\langle n|j(k)|\omega\rangle}{E_n-z}\rho_e(E_n)\,dE_n\,,$$

$$A_2(z)\equiv A_2(\omega'k,\omega k',z)=+\int_0^\infty\frac{\langle\omega'|j(k)|n\rangle\langle n|j^\dagger(k')|\omega\rangle}{E_n+z}\rho_h(E_n)\,dE_n\,,$$

where ρ_e and ρ_h, respectively, are the densities of states with one electron or hole plus any number of pairs. (We shall assume that no bound states occur.) Alternatively, in configuration space

$$A_1(z)=-\int d^3r\,d^3r'\times\int_0^\infty\frac{\langle\omega'|j^\dagger(r')|n\rangle\langle n|j(r)|\omega\rangle e^{i(k'r'-kr)}}{E_n-z}\rho_e(E)\,. \quad (18)$$

The plane-wave factor may be written

$$e^{i\Delta\cdot(\mathbf{r}+\mathbf{r}')/2}e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')}$$

where $\Delta=\mathbf{k}'-\mathbf{k}$ is the momentum transfer and $\mathbf{q}=\frac{1}{2}(\mathbf{k}'+\mathbf{k})$. On the energy shell $q^2=2m(\epsilon+\epsilon_f)-\Delta^2/4$ (with $\hbar=1$). In the absence of a preferred direction, A and hence T can depend only on q^2 and Δ^2, i.e., on ϵ and Δ^2. The possibility of extending the above spectral representation for A into the range $\frac{1}{4}\Delta^2>2m(\epsilon+\epsilon_f)$ in which the exponential factor $e^{-\mathbf{q}(\mathbf{r}-\mathbf{r}')}$ diverges at large $r-r'$, depends on the range of the matrix elements of j and $j^\dagger$ in r space. Now these matrix elements may be written as $V(r)$ or $J(r)$ times the expectation value of sums over products of creation and annihilation operators which should show the slowly decreasing oscillatory character typical of density fluctuation in a Fermi gas. Hence the matrix elements of j and $j^\dagger$ should fall off at least as rapidly as $V(r)$ or $J(r)$. If these are essentially of Yukawa type, i.e., decrease as $e^{-\mu r}/r$, then for $\epsilon+\epsilon_f\geq 0$, it will be possible to extend the validity of Eq. (18) for $A_1(\epsilon,\Delta^2)$ up to $\frac{1}{4}\Delta^2<\mu^2$. A similar representation may be written for A_2. By a direct application of the methods of Ref. 12, we are then led to the dispersion relation

$$T(\epsilon+i\eta,\,\Delta^2)=B(\Delta^2)+\frac{1}{\pi}\int_{-\epsilon_f}^\infty\frac{d\epsilon'\,\mathrm{Im}T(\epsilon+i\eta,\,\Delta^2)}{\epsilon'-\epsilon-i\eta}\,, \quad (19)$$

where T and B are still matrices in spin space, where η is infinitesimal, and where the intermediate states have been restricted to single-particle states. [This means that $\rho_h(E_n)=0$ for $E_n<\epsilon_f$.] In arriving at (19) we have assumed that there is no spin-orbit coupling, so that momentum and spin can be treated independently. We note that for local V and J, the Born term depends on the difference of the momenta Δ, and if V and J are spherically symmetric, only on Δ^2. Finally, we assume, without proof, that the integral in (19) is in fact analytic in the product of the complex ϵ,Δ^2 planes cut respectively along the real axes from $-\epsilon_f$ to $+\infty$ and from $-4\mu^2$ to $-\infty$. In addition, it is easily seen that for the $e^{-\mu r}/r$ potential, B is analytic in the Δ^2 plane cut from $-\mu^2$ to $-\infty$. Thus we may write (setting $\Delta^2=\zeta$)

$$T(\epsilon,\zeta)=B(\zeta)+\frac{1}{\pi^2}\int_{-\epsilon_f}^\infty d\epsilon'\int_{4\mu^2}^\infty d\zeta'\frac{\rho(\epsilon',\zeta')}{(\epsilon'-\epsilon)(\zeta'+\zeta)}$$

where ρ is a certain spectral function (or rather spectral matrix in spin space).

This may now be resolved into partial-wave amplitudes. The result is familiar [see Ref. 10]; we find, with

$$T(\epsilon,\zeta)=\sum_{l=0}^\infty(2l+1)T^{(l)}(\epsilon)P_l(1-\zeta/2(\epsilon+\epsilon_f))\,,$$

that each partial wave $T^{(l)}(\epsilon)$ is analytic in the cut ϵ plane from $-\infty$ to $-\frac{1}{4}\mu^2$ (this cut arising from two terms: the Born-term cut from $-\frac{1}{4}\mu^2-\epsilon_f$ to $-\infty$, and another arising from the projection of the double integral onto P_l, which gives a cut from $-\mu^2-\epsilon_f$ to $-\infty$).

[10] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).
[11] See Ref. 10, end of Sec. B.
[12] Y. Nagaoka, Phys. Rev. **138**, A1112 (1965).

The final result for the s wave is

$$T^{(0)}(\epsilon)=B'(\epsilon)+\int_{-\epsilon_f}^{\infty}\frac{\operatorname{Im}T^{(0)}(\epsilon')}{\epsilon'-\epsilon}d\epsilon',$$

where $B'(\epsilon)$ consists of the s-wave projection of B, plus a contribution from ρ. (The important point is that it has a left-hand cut from $-\frac{1}{4}\mu^2-\epsilon_f$ to $-\infty$).

From the last equation we may derive the appropriate generalization of our previous treatment to interactions of finite range. We first note that

$$\int_{-\epsilon_f}^{\infty}\frac{\operatorname{Im}T^{(0)}(\epsilon')}{\epsilon'-\epsilon}d\epsilon'$$
$$=\int_0^{\infty}\frac{\operatorname{Im}T^{(0)}(\epsilon')}{\epsilon'-\epsilon}d\epsilon'-\int_0^{\epsilon_f}\frac{\operatorname{Im}\tilde{T}^{(0)}(\epsilon')}{\epsilon'-\epsilon}d\epsilon'.$$

Both $T^{(0)}(\epsilon)$ and $\tilde{T}^{(0)}(\epsilon)$ in the two integrands are in the physical range; hence the unitarity relation may be applied in their imaginary parts. (*Note added in proof.* Strictly speaking, this use of the unitarity condition restricts the subsequent treatment to $T=0$. This does not matter insofar as the question of complex poles is concerned.)

Finally, the crossing relation (which holds separately for each partial wave) may be applied, with the result that exactly the same equations as in the Chew-Low treatment are obtained, with the exception that V and J are now replaced by functions of ϵ with the above-mentioned left-hand branch cut.[13]

4. SINGLE-POLE APPROXIMATION

Further progress is possible only through numerical computation. However, there is one popular approximation which is amenable to analysis, that in which the left-hand branch cut is replaced by a finite series of poles. We shall analyze the case of one pole in detail. If this does not remove the complex singularities in τ, it is very unlikely that a left-hand cut will. Because of the remoteness of the Fermi surface (0) from the region of the left-hand cut, any kind of singularity to the left of $-\epsilon_f$ should have much the same effect as a single pole.

In this approximation, the s-wave equations take the form (3a) and (3b), but with the replacement

$$V\rightarrow V[x_0/(x_0+z)],$$
$$J\rightarrow J[x_0/(x_0+z)],$$

where $-x_0$ is the position of the pole replacing the branch cut ($-x_0<-\epsilon_f$). We have chosen this normalization so that in the limit $x_0\rightarrow\infty$, the results revert to those of Sec. 2. Now that we have a slightly more realistic model, it is worthwhile to also use a more realistic density of single-particle states:

$$\rho(z)=(z+\epsilon_f)^{1/2},$$

with a branch cut from $-\epsilon_f$ to ∞, the square root being counted positive just above the cut and negative below. Following Froissart,[7] we write

$$t(z)=-[1/\pi\rho(z)][R(z)e^{-2i\delta_v(z)}-1]/2i, \tag{20}$$

where

$$R(z)=\exp\left\{\frac{2i}{\pi}\rho(z)\int_{-\epsilon_f}^{\infty}\frac{\delta''(x')}{(x'-z)\rho(x')}dx'\right\}, \tag{21}$$

and where, as before,

$$e^{-4\delta''}=1-a|\tau|^2.$$

(Note that now $\tau\rightarrow 0$ as $z\rightarrow\infty$, and so the unpleasantness about δ'' remaining finite at infinity disappears.) δ_v is the phase angle corresponding to the solution of the elastic problem

$$t_v=V\frac{x_0}{x_0+z}+\int_{-\epsilon_f}^{\infty}\frac{|t_v|^2\rho(x)}{z-x}dx \tag{22}$$

[by $\rho(x)$ we always mean $\rho(x+i0)$].

The solution now proceeds just as in Sec. 2. Equation (11) still holds, [$\delta(x+x_0)/|\tau|^2=0$ since τ has a pole at $-x_0$], with

$$\delta(z)=\frac{1}{\pi}\rho(z)\int_{-\epsilon_f}^{\infty}\frac{\delta''(x')}{(x'-z)\rho(x')}dx'-\delta_v(z)$$
$$=\delta'+i\delta''\text{ just above the real axis.} \tag{23}$$

The solution once again has the form $1/\tau=hu$, where

$$h=\exp\left(-\frac{2}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta'(x)\,dx}{x-z}\right),$$

and u^+-u^- is given by the first of Eqs. (12).

Although we no longer can use the Hilbert transform (10) to simplify the result, it is still true (somewhat remarkably) that

$$\frac{P}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta'(x')}{x'-x}dx'=-\frac{P}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta_v(x')}{x'-x}dx'-\delta''(x),$$

as is verified by direct evaluation of $\int dx'\delta'(x')/(x'-x-i\eta)$ using Eq. (23). Hence, u^+-u^- is still given by

$$u^+-u^-=-4\pi i\rho\tanh\tfrac{1}{2}\beta x\times\exp\left\{-(2/\pi)P\int dx'\,\delta_v(x')/(x'-x)\right\}. \tag{24}$$

We must now solve (22) and (24) subject to t_v and τ

[13] It is perhaps worth noting that the crossing relation proved in Ref. 2 is valid *without* restriction to one-particle intermediate states.

having poles with residues Vx_0 and Jx_0 at $-x_0$. These solutions are standard; for t_v we have:

$$t_v = Vx_0 \Big/ (z+x_0)\left\{1+Vx_0(z+x_0)\int_{-\epsilon_f}^{\infty}\frac{\rho(x')\,dx'}{(x'-x_0)^2(x'-z)}\right\},$$

which is seen to reduce to the previous one in the limit $x_0\to\infty$. To simplify (24), we again need a dispersion relation for $\ln t_v$. This is found to be

$$|t_v| = \frac{A(x_0)}{|x+x_0|}\exp\left[-\frac{P}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta_v(x')}{x'-x}dx'\right\},$$

where

$$A(x_0) = Vx_0\exp\left[\frac{1}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta_v(x')}{x'+x_0}dx'\right].$$

Hence, we have

$$u^+-u^- = -4ni\rho[|x+x_0|\,|t_v(x)|/A(x_0)]\tanh\tfrac{1}{2}\beta x. \quad (24a)$$

Therefore, we obtain

$$\frac{1}{u} = C\Big/(z+x_0)\left[1+\frac{(z+x_0)C}{A(x_0)}\int_{-\epsilon_f}^{\infty}\frac{\tanh\frac{1}{2}\beta x'\rho(x')\,dx'}{|x'+x_0|(x'-z)}\right]$$

where C is a constant. Also

$$h(x+i\eta) = \exp\left[-\frac{2}{\pi}\int_{-\epsilon_f}^{\infty}\frac{\delta'(x')\,dx'}{x'-x-i\eta}\right] = \exp\left[\frac{2}{\pi}P\int\frac{\delta_v(x')\,dx'}{x'-x}\right]\times\exp[-2i\delta(x+i\eta)]$$
$$= [A(x_0)/|x+x_0|\,|t_v|]\exp[-2i\delta(x+i\eta)].$$

Hence

$$\tau = C|x+x_0|\,|t_v(x)|e^{-2i\delta(x+i\eta)}\Big/A(x_0)(z+x_0)\left[1+\frac{(z+x_0)C}{A(x_0)}\int_{-\epsilon_f}^{\infty}\frac{|t_v(x')|\tanh\frac{1}{2}\beta x'\rho(x')\,dx'}{|x'+x_0|(x'-z)}\right];$$

and, since $\lim_{x\to -x_0}|x+x_0|t_v(x) = Vx_0$, we see that

$$C = A(x_0)J/4V,$$

so that, finally,

$$\tau = e^{-2i\delta}F(z),$$

where

$$F(x+i\eta) = \tfrac{1}{4}J\left|\frac{t_v(x+\eta i)(x+x_0)}{V}\right|\Big/(x+x_0)\left[1-\tfrac{1}{4}J(z+x_0)\int_{-\epsilon_f}^{\infty}\frac{|t_v(x')|\tanh\frac{1}{2}\beta x'\rho(x')\,dx'}{|V(x'+x_0)|(x'-x-i\eta)}\right].$$

We now show that complex poles still occur in the limit $V\to 0$, in which $|t_v/V|\to x_0/|x'+x_0|$. Then

$$F(z) = \tfrac{1}{4}J\Big/(z+x_0)\left[1-\tfrac{1}{4}J(z+x_0)x_0\int_{-\epsilon_f}^{\infty}\frac{\rho(x')\tanh\frac{1}{2}\beta x'\,dx'}{(x'+x_0)^2(x'-z)}\right].$$

As $\beta\to\infty$, the integral in the denominator still shows logarithmic behavior for small z, which is the principal requirement for the existence of complex poles at arbitrarily small coupling strength.

5. CONCLUDING REMARKS

The problem remains of accounting for the complex poles in the spin-flip scattering amplitude. We note that omission of multiparticle intermediate states cannot very obviously be held responsible. These will add an additional discontinuity $R(x)$ across the cut, but one that strongly goes to zero at $\epsilon=0$, i.e., just in the critical range. If, in a non-self-consistent way, we simply regard the multiparticle states as providing additonal inelastic channels and apply the method of Froissart,[7] we find that the poles in τ remain unless the continuation of $R(x)$ into the complex plane happens to vanish at these poles.

A possibility that is hard to discount is that the asymptotic scattering states do not exhaust all the states of the system, that there is a band of collective bound states whose wave functions approach zero at an infinite distance from the impurity, and whose energy spectrum covers the critical region around $\epsilon=0$. These states would have to be included as intermediate states and would add a nonvanishing discontinuity across the cut, even at $\epsilon=0$, which might rule out the poles. This would be the equivalent, in the language of scattering theory, of the "instability of the Fermi surface" surmised by Nagaoka.[12] The interesting question arises: can one use scattering theory to *derive* the character of these states? This is not entirely out of the question, since scattering theory has been successful in uncovering ordinary bound-state poles; both in regard to position and residue. It may be that these techniques might be adapted to the present problem.

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